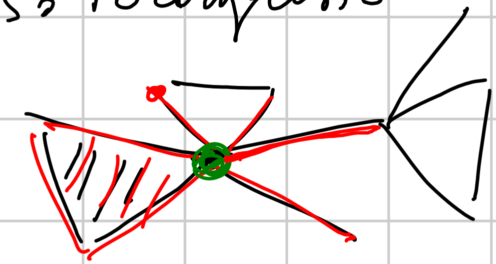
ETA 15/10/15

<u>Teo</u>: date $f: X \to Y$ continua $X = |\mathcal{K}|$, $Y = |\mathcal{L}|$
$\exists\, \mathcal{H} < \mathcal{K}$ e $g: \mathcal{H} \to \mathcal{L}$ simpliciale t.c.
$\forall x \in X\ \exists\, \tau \in \mathcal{L}$ con $f(x), g(x) \in \tau$.

$$St(v, \mathcal{K}) = \left\{ \begin{array}{l} \tau \in \mathcal{K} : \exists \\ \sigma \in \mathcal{K}, \sigma \supset \tau, \sigma \ni v \end{array} \right\} \qquad v \in \mathcal{K}^{[0]}$$

sottocomplesso

$$OSt(v, \mathcal{K}) = \bigcup \{ \operatorname{int}(\sigma) : \sigma \in \mathcal{K}, \sigma \ni v \}$$



<u>Lem</u>: $v_1, \dots, v_n \in K^{[0]}$

$$\bigcap_{j=1}^{n} OSt(v_j, K) \neq \emptyset \iff Conv(v_1, \dots, v_n) \in K$$

<u>Dim</u>: <u>$\Longrightarrow$</u> $x \in OSt(v_j, K) \quad j = 1 \dots n$

$\exists \sigma_j$ t.c. $v_j \in \sigma_j^{[0]}$, $x \in int(\sigma_j)$

ma $|K| = \bigsqcup_{\sigma \in K} int(\sigma)$

$\Rightarrow \sigma_1 = \dots = \sigma_n =: \sigma \Rightarrow v_j \in \sigma^{[0]} \ \forall j$

$\Rightarrow Conv(v_1, \dots, v_n)$ è faccia di $\sigma \Rightarrow \in K$

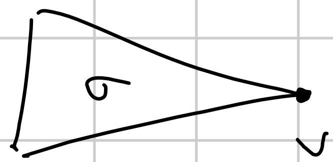
$\Leftarrow$ $\mathrm{Conv}(v_1, \dots, v_m) \in K$

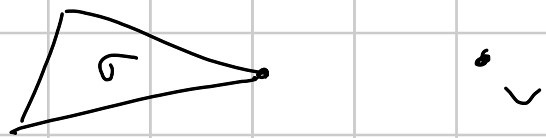
prendo $x \in \mathrm{int}(\mathrm{Conv}(v_1, \dots, v_m))$

$\Rightarrow x \in \mathrm{OSt}(v_j, K) \quad j = 1 \dots m.$ $\boxed{//}$

Lem: $\mathrm{OSt}(v, K)$ è aperto in $|K|$ $\forall v \in K^{[0]}$.

Dim: K sic. chiuso finito $\Rightarrow$ fond. di $|K|$.
Basta vedere che $\mathrm{OSt}(v, K) \cap \sigma$ è aperto in σ $\forall \sigma \in K$. Due casi:





$$\overset{\shortparallel}{\mathrm{Ost}(v,K)} \cap \sigma = \bigcup_{\substack{\tau \subset \sigma \\ \tau \ni v}} \mathrm{int}(\tau) \qquad \emptyset$$

aperto

Dimo (teo): $f : X \to Y$, $X = |K|$, $Y = |L|$

Osservo che $\{ \mathrm{OSt}(w, L) : w \in L^{(0)} \}$ è ric. aperto di Y:

se $y \in Y = \bigsqcup_{\tau \in L} \mathrm{int}(\tau) \Longrightarrow y \in \mathrm{int}(\tau) \Longrightarrow y \in \mathrm{St}(w, L)$

$\forall w \in \tau^{[0]}$

$$\Rightarrow \left\{ f^{-1}\left(\mathrm{OSt}(w,\mathcal{L})\right) : w \in \mathcal{L}^{[0]} \right\}$$
r.a. di X

Sia ε numero di Lebesgue per esso, cioè t.c. se $C \subset X$, $\mathrm{diam}(C) < \varepsilon$ $\exists\, w \in \mathcal{L}^{[0]}$ t.c.
$$f(C) \subset \mathrm{OSt}(w,\mathcal{L}).$$

(X cpt, metrico, $\mathcal{A}$ ric. aperto $\exists\, \varepsilon > 0$ t.c.
$\forall\, C \subset X$, $\mathrm{diam}(C) < \varepsilon$ $\exists\, A \in \mathcal{A}$ t.c. $C \subset A$:

P.a. $\exists\, C_n$ con $\mathrm{diam}(C_n) \to 0$

wlog $C_n \to x_\infty$; $\exists\, A \in \mathcal{A}$ t.c. $A \supset B(x_\infty, \varepsilon)$.

Prendo $\mathcal{H} < K$ t.c. $\max \mathrm{diam}(\mathcal{H}) < \varepsilon/2$

$\forall v \in \mathcal{H}^{[0]}$ so che $\exists\, w \in \mathcal{L}^{[0]}$ t.c. $f(B_{\varepsilon/2}(v, X)) \subset \mathrm{Ost}(w, \mathcal{L})$

Scelgo un tale w a caso e lo chiamo $g_0(w)$. Affermo:

- g_0 si estende a g simpliciale
- $\forall x \in X$ $\exists\, \tau \in \mathcal{L}$ t.c. $f(x), g(x) \in \tau$.

- Sia $\sigma = \mathrm{Conv}(v_1, \dots, v_m) \in \mathcal{H}$; $x \in \mathrm{int}(\sigma)$
  Poiché $\mathrm{maxdiam}(\mathcal{H}) < \varepsilon/2$ ho $x \in B_{\varepsilon/2}(v_j, X)$
  $\Longrightarrow f(x) \in OSt(g_0(v_j), \mathcal{L}) \quad \forall j$
  $\Longrightarrow \bigcap_{j=1\dots m} OSt(g_0(v_j), \mathcal{L}) \neq \emptyset \Longrightarrow$
  $\mathrm{Conv}(g_0(v_1), \dots, g_0(v_m)) \in \mathcal{L}$. ✓

- $x \in X$; $\exists\, \sigma \in \mathcal{H}$ t.c. $x \in \mathrm{int}(\sigma)$
  Stesso argomento qui sopra $\Longrightarrow f(x) \in \mathrm{int}(g(\sigma))$
  Inoltre g è lineare su $\mathrm{Conv}(v_1, \dots, v_m)$

$\Longrightarrow g(x) \in \mathrm{int}\,(g(\sigma))$ _ $\blacksquare$

(Riempio buco: $f : K \to L$ simpliciale

$f_n : C_n(K) \to C_n(L)$

K, L orientati

$$f_n(\sigma) = \begin{cases} \delta(\sigma) \cdot f(\sigma) & \text{se } \dim f(\sigma) = n \\ 0 & \text{al}\_\_\_\_ \end{cases}$$

$\delta(\sigma) = \pm 1$ l'ordinamento pos. di $f(\sigma)^{[0]}$

$= f$ (ordinamento pos di $\sigma^{[0]}$).)

Versione relativa di TAS:

$A \subset X \qquad X = |K|, \ A = |\mathcal{M}| \qquad \mathcal{M} \subset K$

$B \subset Y \qquad Y = |\mathcal{L}|, \ B = |\mathcal{N}| \qquad \mathcal{N} \subset \mathcal{L}$

$f : (X, A) \to (Y, B)$ cont.

$\Longrightarrow \exists \mathcal{H} < K$ e $g : \mathcal{H} \to \mathcal{L}$ simpliciale t.c.

- $\forall x \in A \quad \exists \eta \in \mathcal{N}$ t.c. $f(x), g(x) \in \eta$
- $\forall x \in X \quad \exists \tau \in \mathcal{L}$ t.c. $f(x), g(x) \in \tau$

Dimo (idea): prima applico la stessa tecnica a $f|_A : A \to B$

e poi a tutta f senza cambiare la def. di g_0 sui vertici in A. $\square$

Scopo: provare che $f : X \to Y$ continua induce $f_* : H(X) \to H(Y)$ che dipende solo da $f/\simeq$.

$X = |K|$
$Y = |L|$

Passi:

1) Prendo $\mathcal{H} < K$ $\quad g: \mathcal{H} \to \mathcal{L}$ approx simpl. di f e definisco

$$\begin{tikzcd} H(K) \arrow[r, "\gamma", "\text{suddivisione}"'] \arrow[dr, "f_*"'] & H(\mathcal{H}) \arrow[d, "g_*"] \\ & H(\mathcal{L}) \end{tikzcd}$$

2) Buona def: $\mathcal{H}_0, \mathcal{H}_1 < K$ $\quad g_j : \mathcal{H}_j \to \mathcal{L}$ simpliciale

con $g_0 \underset{\text{CONT}}{\simeq} g_1 \implies$

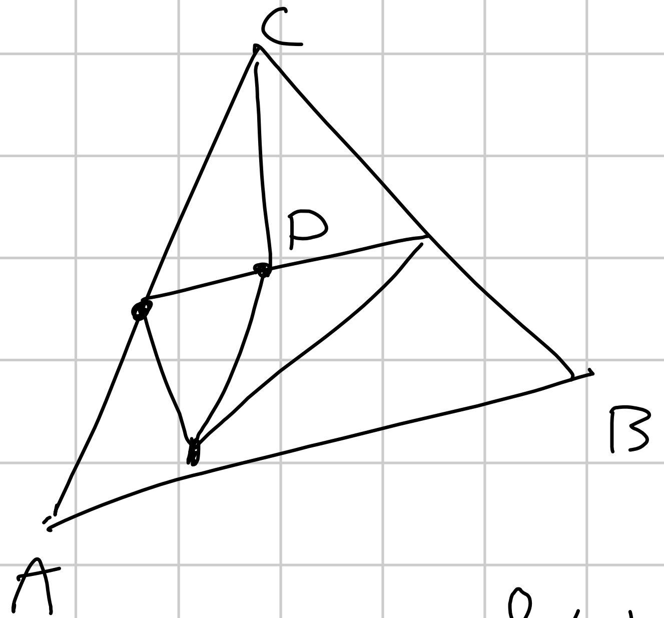
$$\begin{tikzcd}
 & H(\mathcal{H}_0) \arrow[dr, "g_{0*}"] & \\
H(\mathcal{K}) \arrow[ur, "f_0"] \arrow[dr, "f_1"'] & & H(\mathcal{L}) \\
 & H(\mathcal{H}_1) \arrow[ur, "g_{1*}"'] &
\end{tikzcd}$$

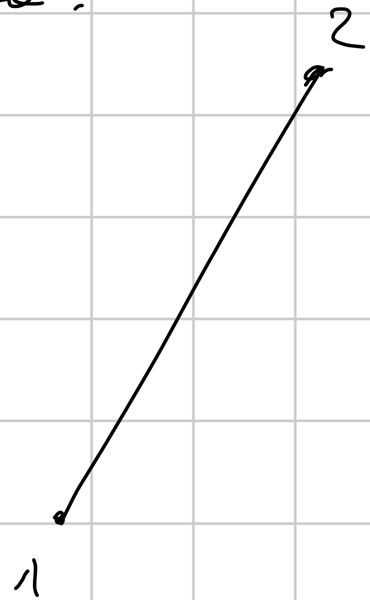
commuta.

<u>Oss</u>: (2) implica anche che $f_0 \simeq f_1 \Rightarrow f_{0*} \simeq f_{1*}$

<u>Oss</u>: per provare (2) non è utile passare
a una suddivisione comune di $\mathcal{H}_0$ e $\mathcal{H}_1$ poiché

una mappa simpliciale non lo è più passando a una suddivisione in partenza:





$f(A) = f(B) = 1$

$f(C) = 2$

$f(D) \notin \{1,2\}$

__Lem__: $f : \mathcal{K} \to \mathcal{L}$ simpliciale $\quad \mathcal{H} < \mathcal{K}$

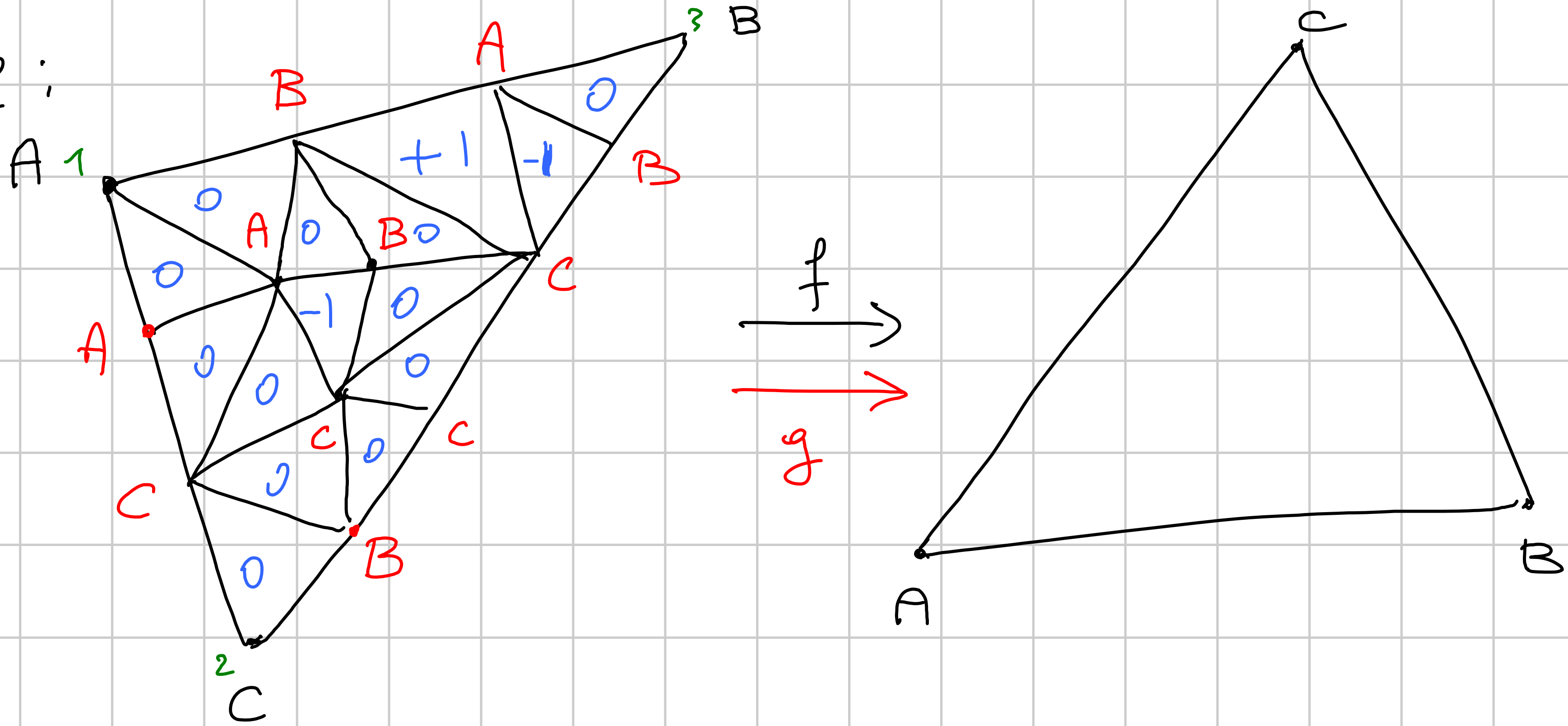
$g : \mathcal{H} \to \mathcal{L}$ simpliciale t.c. $g(v) \in f(\sigma)^{[0]}$

$\forall v \in \mathcal{H}^{[0]}$, $v \in \mathrm{int}(\sigma)$, $\sigma \in \mathcal{K}$.

$$\Longrightarrow \quad \begin{tikzcd} C_n(\mathcal{K}) \arrow[r, "J"] \arrow[dr, "f_n"'] & C_n(\mathcal{H}) \arrow[d, "g_n"] \\ & C_n(\mathcal{L}) \end{tikzcd}$$

commuta.

Esempio:



$$f_2(123) = -ABC$$

$$g_2(g(123)) = \dots = -ABC$$

"Dimo". Basta vederlo su $\sigma = \mathrm{Conv}(v_1, \dots, v_n) \in K^{[n]}$ con $f(\sigma) \in \mathcal{L}^{[n]}$ (altrimenti $0 = 0$).

Chiamo $w_j = f(v_j)$. Affermo:

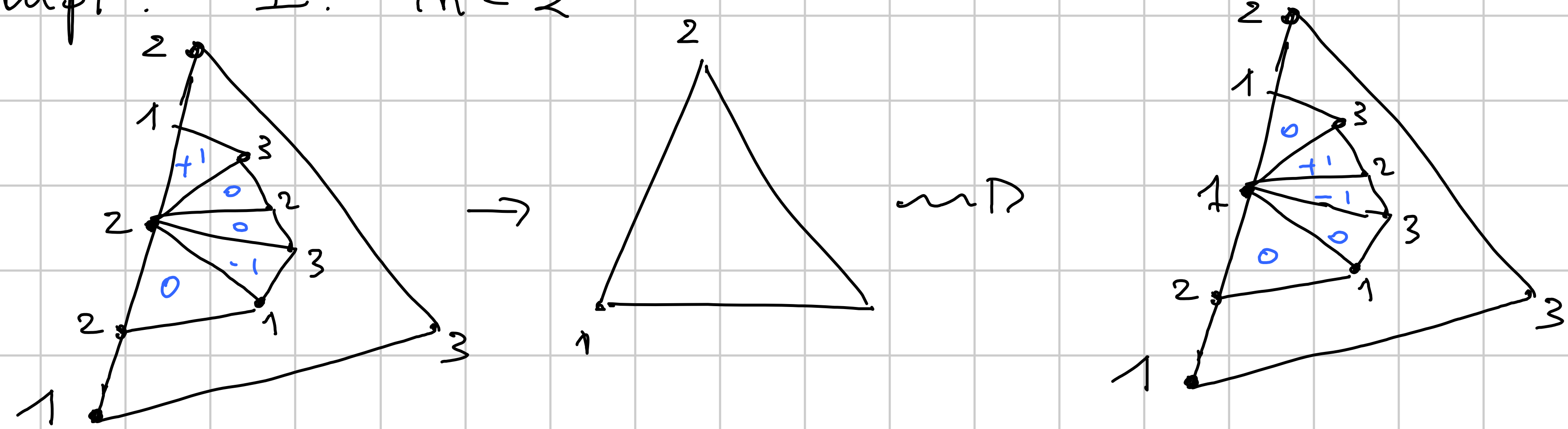
I. Dato $v \in \mathcal{H}^{[0]}$, $v \in \sigma$, prendo $\eta \subset \sigma$ faccia t.c. $v \in \mathrm{int}(\eta)$; avrò $\eta = \mathrm{Conv}(v_{i_1}, \dots, v_{i_p})$; $i_1 < i_2 < \dots < i_p$

provo che $g(\mathcal{A}(\sigma))$ non cambia se ridefinisco $g(v)$ come w_{i_1} (prima era uno dei $w_{i_1}, \dots, w_{i_p}$).

II. Fatto I per tutti i v si ha $f(\sigma) = g(\mathcal{A}(\sigma))$.

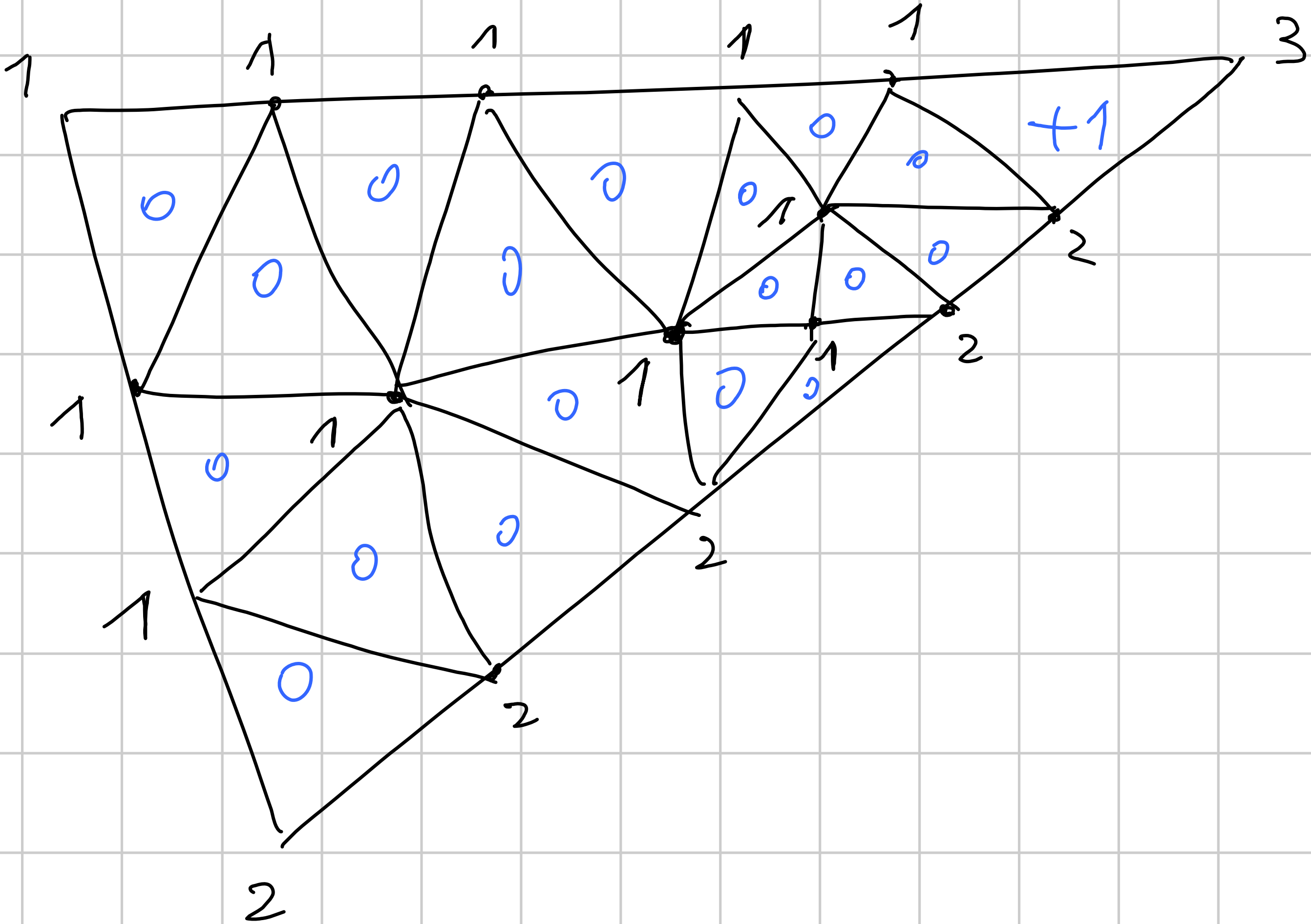
Esempi: I. $n = 2$

$p = 1$



$p = 2$

II.



Thm: $\mathcal{H}_0, \mathcal{H}_1 < K$, $g_j : \mathcal{H}_j \to \mathcal{L}$ simpliciale $g_0 \underset{\text{CONT}}{\simeq} g_1 \Rightarrow$

$$\begin{tikzcd}
 & H(\mathcal{H}_0) \arrow[dr, "g_{0*}"] & \\
H(K) \arrow[ur, "\partial_0"] \arrow[dr, "\partial_1"'] & & H(\mathcal{L}) \\
 & H(\mathcal{H}_1) \arrow[ur, "g_{1*}"'] &
\end{tikzcd}$$

commuta.

Dim: $\exists\, G : X \times [0,1] \to Y$ continua $g_0 = G(\cdot, 0)$
$g_1 = G(\cdot, 1)$.

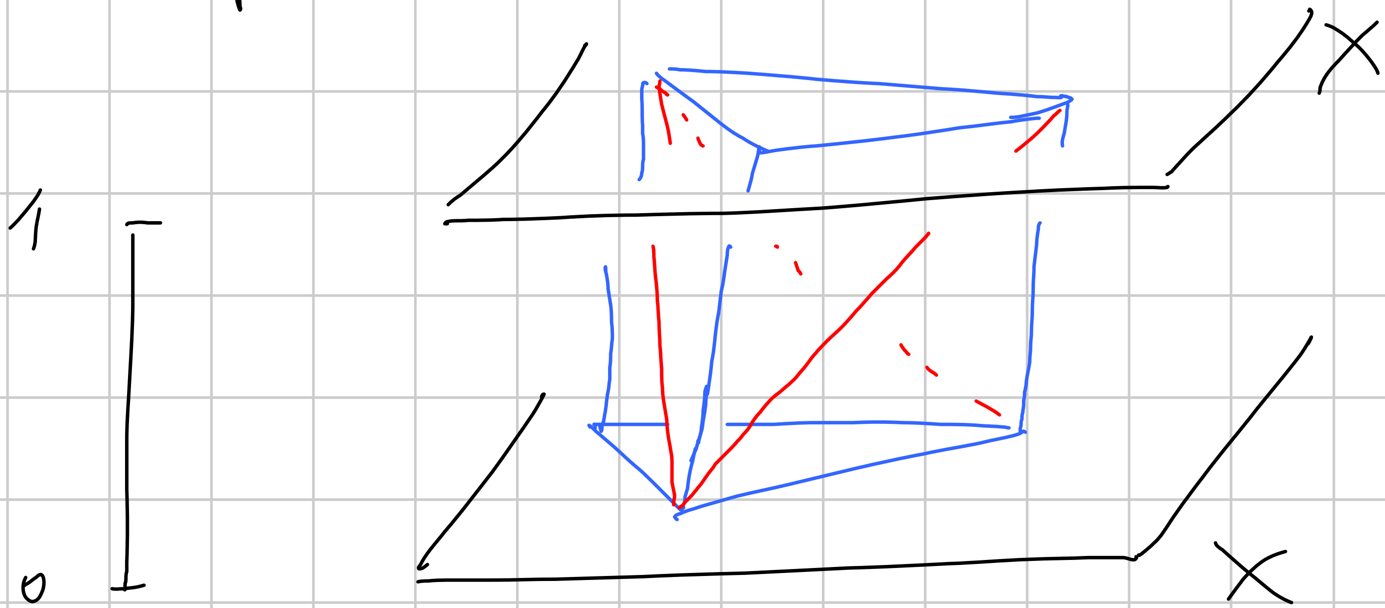
Prendo $\mathcal{H} < \mathcal{H}_0, \mathcal{H}_1$ una $\mathcal{M}$ triang. di $X \times [0,1]$ t.c.

$\mathcal{M} \cap (X \times \{0\}) = \mathcal{H} \times \{0\}$

$\mathcal{M} \cap (X \times \{1\}) = \mathcal{H} \times \{1\}$

$\sigma \times [0,1]$ sottocomplesso di $\mathcal{M}$ $\forall \sigma \in \mathcal{H}$.

Idea: prendo $\{\sigma \times [0,1] : \sigma \in \mathcal{H}\}$ e li suddivido.



TAS $\Longrightarrow \exists\, M < \mathcal{M}$, $F : M \to \mathcal{L}$ simpliciale t.c.

$\forall (x,t) \in X \times [0,1]$ $\exists\, \tau \in \mathcal{L}$ t.c.

$$F(a,t), G(a,t) \in \mathcal{I}.$$

So che $\mathcal{M} \cap (X \times \{j\}) = \mathcal{H}$, $\mathcal{N} < \mathcal{M}$

$\Longrightarrow \mathcal{N}_j := \mathcal{M} \cap (X \times \{j\})$ è suddiv. di $\mathcal{H}$.

Inoltre ho

$$g_j : \mathcal{H}_j \to \mathcal{L} \text{ simpliciale}$$

$$f_j = F\big|_{X \times \{j\}} : \mathcal{N}_j \to \mathcal{L} \text{ simpliciale}$$

Affermo che si applica il lemma di pag. 14:

sia $v \in \mathcal{N}_j^{[0]}$, $\sigma \in \mathcal{H}_j$ $\quad v \in \mathrm{int}(\sigma)$

sia $\eta \in \mathcal{L}$ l'unico simplesso di $\mathcal{L}$ t.c. $g_j(v) \in \mathrm{int}(\eta)$

allora ho $\quad g_j(\sigma) = \eta \qquad$ (ovvio)

e $\quad f_j(v) \in \eta \qquad$ (per la TAS)

$$\Longrightarrow f_j(v) = \eta^{[0]} = g_j(\sigma)^{[0]} .$$ $\underline{\text{OK}}$

Lemma dice:

$$
\begin{tikzcd}
C_n(\mathcal{H}_j) \arrow[r, "\text{sudd.}"'] \arrow[dr, "g_{j_n}"'] & C_n(\mathcal{M}_j) \arrow[d, "f_{j_n}"] \arrow[dl, phantom, "\circlearrowright"] \\
 & C_n(\mathcal{L})
\end{tikzcd}
\qquad j=0,1
$$

Ora suddivido $\mathcal{M}$ a una $\mathcal{T}$ (di $X\times[0,1]$) in modo che

$$\mathcal{T}\cap(X\times\{0\})=\mathcal{T}\cap(X\times\{1\})=:\mathcal{R}$$

e $\sigma\times[0,1]$ sottocomplesso di $\mathcal{T}$ $\forall\sigma\in\mathcal{R}$

(prendo suddivisione comune di M_0, M_1
estendo a $X \times [0,1]$, interseco con $\mathcal{N}$
e risuddivido senza toccare $X \times \{0,1\}$) _

Ora ridefinisco $F : \mathcal{N} \longrightarrow \mathcal{L}$ simpliciale
in modo che valga la condizione del Lemma
(avevo $F : \mathcal{M} \to \mathcal{L}$ simpliciale, $\mathcal{N} < \mathcal{M}$) _ Dunque:

$$\begin{tikzcd} C_n(\mathcal{H}_j) \arrow[r, "\text{sudd.}"] \arrow[dr, "g_{j_n}"'] & C_n(\mathcal{N}) \arrow[d, "f_{j_n}"] \\ & C_n(\mathcal{L}) \end{tikzcd}$$

Ora:

$$
\begin{tikzcd}
 & H(\mathcal{H}_0) \arrow[dr, "j_0'"] \arrow[rrrd, bend left, "g_{0*}"] & & & \\
H(K) \arrow[ur, "j_0"] \arrow[dr, "j_1"'] \arrow[rrrr, bend left=70] \arrow[rrrr, bend right=70] & & H(R) \arrow[rr, bend left, "f_{0*}"] \arrow[rr, bend right, "f_{1*}"'] & & H(L) \\
 & H(\mathcal{H}_1) \arrow[ur, "j_1'"'] \arrow[rrru, bend right, "g_{1*}"'] & & &
\end{tikzcd}
$$

Tesi

naturalità

Lemma pag. 14

Si conclude trovando che $f_{0*} = f_{1*}$ —

Ricordo $\quad F : \mathcal{T} \longrightarrow \mathcal{L}$ simpliciale

$|\mathcal{T}| = X \times [0,1]$

$|\mathcal{L}| = Y$

$\mathcal{T} \cap (X \times \{j\}) = \mathcal{R} \times \{j\}$

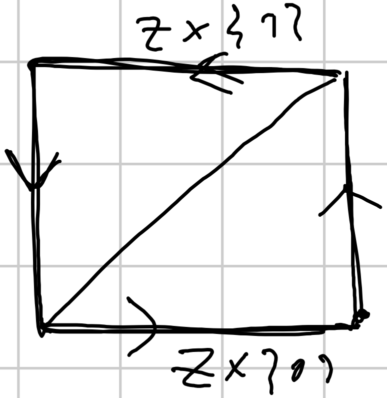
$\sigma \times [0,1]$ sottocomplesso di $\mathcal{T}$ $\forall \sigma \in \mathcal{R}$

$f_j = F(\cdot, j) \qquad f_{j_*} : H(\mathcal{R}) \longrightarrow H(\mathcal{L})$

Prendo $z \in Z_m(\mathcal{R}) \Longrightarrow z \times [0,1] \in C_{m+1}(\mathcal{T})$

(devo esprimerlo come somme di simplessi di $\mathcal{T}$)

$$\partial^{\Delta}_{n+1}\left(z\times[0,1]\right) = (\underset{\substack{\parallel\\ 0}}{\partial z})\times[0,1] + z\times\{1\} - z\times\{0\}$$



$$f_{0_n}(z) - f_{1_n}(z) = F_{n+1}\left(\partial^{\Delta}_{n+1}\left(z\times[0,1]\right)\right)$$

$$= \partial^{Z}_{n+1}\left(F_{n+1}\left(z\times[0,1]\right)\right)$$

So già $F \longmapsto (F_n)_{n=0}^{+\infty}$ è mappa di complessi di catene $\blacksquare$

$$\Longrightarrow f_{0*}([z]) = f_{1*}([z]).$$