# ETA 9/12/15

**Lem**: (1) $G \otimes_{\mathbb{Z}} (\mathbb{Z}/m) \cong G/m \cdot G$

(2) $\mathbb{Z}/k \otimes \mathbb{Z}/h \cong \mathbb{Z}/n$ $\qquad n = GCD(k, h)$

**Dim**: (1) $\varphi : G \otimes (\mathbb{Z}/m) \to G/m \cdot G$

$g \otimes [p] \longmapsto [p \cdot g] = p \cdot g + m \cdot G$ $\qquad$ estesa per $\mathbb{Z}$-lin.

- ben def $\quad (p + m \cdot q) \cdot g + m \cdot G = p \cdot g + m \cdot G$

- surj : $g \otimes 1 \longmapsto [g] = g + m \cdot G$

- inj : $\sum_i g_i \otimes [p_i] \in \operatorname{Ker} \varphi$

$\Longrightarrow \sum p_i g_i \in m \cdot G$

$\Longrightarrow \sum g_i \otimes [p_i] = \sum \big( (p_i g_i) \otimes [1] \big)$

$= \left( \sum p_i g_i \right) \otimes [1] = (m \cdot g) \otimes [1] = g \otimes [m] = 0$

(2) $\psi : \mathbb{Z}/k \big/ h \cdot (\mathbb{Z}/k) \longrightarrow \mathbb{Z}/m$

$\big[\!\big[ [a]_k \big]\!\big] \longmapsto [a]_m$

affermo: isomorfismo

$K = \tilde{K} \cdot m \qquad h = H \cdot m$

$m = u \cdot k + v \cdot h$

- ben def: $\overline{\overline{[a]_k}} = \overline{\overline{[b]_k}}$

$\Longrightarrow \quad [a]_k - [b]_k = h \cdot [c]_k$

$a - b = h \cdot c + k \cdot d = H \cdot m \cdot c + K \cdot m \cdot d$

$= m ( \quad )$

- surgettiva : ovvio

• inj: $[a]_n = 0 \implies a = p \cdot n = p \cdot (uk + vh)$

$\implies a - (pu) \cdot k = h \cdot (pv)$

$\implies [a]_k \in h \cdot (\mathbb{Z}/k)$

$\implies [\![ [a]_k ]\!] = 0$ $\blacksquare$

<u>Def</u>: chiamo risoluz. libera di $A$ abeliano una

$$0 \longrightarrow K \xrightarrow{i} F \xrightarrow{p} A \longrightarrow 0$$

esatta con $K, F$ liberi $(\cong \mathbb{Z}^A)$.

Oss: data tale risoluz. $\exists$ indotta

$$0 \dashrightarrow K \otimes G \xrightarrow{i \otimes id_G} F \otimes G \xrightarrow{p \otimes id_G} A \otimes G \to 0$$

e $p \otimes id_G$ è surgettiva ma $i \otimes id_G$ può non essere iniettiva:

- dato $a \in A$, $g \in G$ $\exists f \in F$ t.c. $a = p(f)$
$\Rightarrow a \otimes g = (p \otimes id_G)(f, g)$

- $0 \longrightarrow \mathbb{Z} \xrightarrow[m\cdot]{i} \mathbb{Z} \xrightarrow{p} \mathbb{Z}/m \longrightarrow 0$

$$\begin{tikzcd}
0 \arrow[r, dashed] & \mathbb{Z}\otimes\mathbb{Z}/m \arrow[r, "i\otimes id_{\mathbb{Z}}", "(m\cdot)\otimes id_{\mathbb{Z}}"'] \arrow[d, phantom, "\cong" rotate=-90] & \mathbb{Z}\otimes\mathbb{Z}/m \arrow[r, dashed] \arrow[d, phantom, "\cong" rotate=-90] & {} \\
& \mathbb{Z}/m \arrow[r, "0"] & \mathbb{Z}/m &
\end{tikzcd}$$

Def: $\mathrm{Tor}(A,G) = \mathrm{Ker}(i\otimes id_G)$

per $0 \to K \xrightarrow{i} F \xrightarrow{p} A \to 0$ ris. libera.

**Prop:** (1) E' ben def

(2) $\mathrm{Tor}(A,B) \cong \mathrm{Tor}(B,A)$

(3) $\mathrm{Tor}(A_1 \oplus A_2, B) \cong \mathrm{Tor}(A_1,B) \oplus \mathrm{Tor}(A_2,B)$

(4) $\mathrm{Tor}(A,\mathbb{Z}) \cong 0$

(5) $\mathrm{Tor}(A,\mathbb{Z}/p) \cong \{a \in A : p \cdot a = 0\}$

(6) $\mathrm{Tor}(\mathbb{Z}/p,\mathbb{Z}/q) \cong \mathbb{Z}/_{GCD(p,q)}$ .

**Dim:** (2) $\Rightarrow$ (1) .

**Oss:** $C \xrightarrow{f} D \xrightarrow{g} E$ esatta , $L$ libero

$$C\otimes L \xrightarrow{f\otimes id_L} D\otimes L \xrightarrow{g\otimes id_L} E\otimes L \quad \text{esatta}$$

Sia $\mathcal{L}$ insieme di gen liberi di $L$, cioè $L = \bigoplus_{l\in\mathcal{L}} \mathbb{Z}\langle l\rangle$.

$$\Rightarrow \quad C = \bigoplus_{l\in\mathcal{L}} C^{(l)} \qquad D = \dots \qquad E = \dots$$

$$f\otimes id_L = \bigoplus_{l\in\mathcal{L}} f^{(l)} \qquad g = \dots \quad \Rightarrow \text{OK}.$$

$$
\begin{tikzcd}
 & 0 \arrow[r] & K \arrow[r, "i"] & F \arrow[r, "p"] & A \arrow[r] & 0 \\
 & & & & 0 \arrow[d] & \\
0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & {}_{z}\mathrm{Tor}(B,A) \arrow[d] & \\
H \arrow[d, "j"'] & 0 \arrow[r] & {}_{n}K\otimes H \arrow[r, "i\otimes id_H"] \arrow[d, "id_K\otimes j"] & {}_{w}F\otimes H \arrow[r, "p\otimes id_H"] \arrow[d, "id_F\otimes j"] & {}_{z}A\otimes H \arrow[r] \arrow[d, "id_A\otimes j"] & 0 \\
G \arrow[d, "q"'] & 0 \arrow[r] & {}_{x}K\otimes G \arrow[r, "i\otimes id_G"] \arrow[d, "id_K\otimes q"] & {}_{u}F\otimes G \arrow[r, "p\otimes id_G"] \arrow[d, "id_F\otimes q"] & A\otimes G \arrow[r] & 0 \\
B \arrow[d] & 0 \to \mathrm{Tor}(A,B) \arrow[r] & {}_{y}K\otimes B \arrow[r, "i\otimes id_B"] \arrow[d] & F\otimes B & & \\
0 & & 0 & & &
\end{tikzcd}
$$

$u \in \mathrm{Ker}(p\otimes id_G)$
$\quad\quad \shortparallel$
$\quad\ \mathrm{Im}(i\otimes id_G)$

$y \in \mathrm{Ker}(i\otimes id_B) = \mathrm{Tor}(A,B)$

Se provo che $z \longmapsto y$ è ben def. ho finito

perché $y \longmapsto z$ è definito dagli stessi passaggi $\Rightarrow$ è l'inverso.

Unica scelta: $w$. Se uso $w' = w + t$ $\quad t \in \mathrm{Ker}\, p \otimes id_H$

$$\qquad\qquad\qquad\qquad \overset{\shortparallel}{\mathrm{Im}\,(i \otimes id_H)}$$

$$t = i \otimes id_H(r)$$

trovo $x' = x + (id_K \otimes j)(r)$

$\Rightarrow y' = y$.

(3) Tor commuta con $\oplus$ ovvio

(4) $\mathrm{Tor}(A, \mathbb{Z}) = 0$

$$0 \longrightarrow 0 \xrightarrow{i} \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

$$0 \otimes A \xrightarrow{i \otimes id_A} \mathbb{Z} \otimes A$$

$$\mathrm{Tor}(A, \mathbb{Z}) < 0 \otimes A$$

(5) $\mathrm{Tor}(A, \mathbb{Z}/p) = \{ a \in A : p \cdot a = 0 \}$

$$0 \longrightarrow \mathbb{Z} \xrightarrow[p\cdot]{i} \mathbb{Z} \longrightarrow \mathbb{Z}/p \longrightarrow 0$$

$$\mathrm{Tor}(A, \mathbb{Z}/p) = \mathrm{Ker}\left(\begin{array}{ccc} \mathbb{Z} \otimes A & \xrightarrow[(p\cdot)\otimes \mathrm{id}_A]{1 \otimes \mathrm{id}_A} & \mathbb{Z} \otimes A) \\ \cong & & \cong \\ A & \xrightarrow{p\cdot} & A \end{array}\right)$$

(6) Visto: $\mathrm{Tor}(\mathbb{Z}/p, \mathbb{Z}/q) = \{ [1] \in \mathbb{Z}/q : p \cdot [1] = 0 \} =: S$

Sia $m = GCD(p, q)$ $\quad p = a \cdot m \quad q = b \cdot m$

Affermo che $S = \langle [b] \rangle$. $\quad$ ($\Rightarrow$ OK)

$[b] \in S$ infatti $\quad p \cdot [b] = [a \cdot m \cdot b] = [q \cdot a] = 0$

$[s] \in S \Longrightarrow p \cdot [s] = 0 \qquad p \cdot s = q \cdot u$

$\Longrightarrow a \cdot m \cdot s = b \cdot m \cdot u \Longrightarrow b \cdot u = a \cdot r \Longrightarrow b \mid r$. $\blacksquare$

<u>Cor</u>: $\operatorname{Tor}\left(\mathbb{Z}^n \oplus \mathbb{Z}/_{p_1} \oplus \dots \oplus \mathbb{Z}/_{p_h}, \mathbb{Z}^m \oplus \mathbb{Z}/_{q_1} \oplus \dots \oplus \mathbb{Z}/_{q_k}\right)$

$$\cong \bigoplus_{\substack{i=1\dots h \\ j=1\dots k}} \mathbb{Z}/_{GCD(p_i, q_j)}$$

**Prop/def**: data
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "p"] \arrow[l, bend left, dashed, "q"] & C \arrow[r] \arrow[l, bend left, dashed, "s"] & 0 \end{tikzcd}$$
esatta sono fatti equiv:

(1) $\exists\, s : C \to B$ omomorf. t.c. $p \circ s = id_C$ (sezione)

(2) $\exists\, q : B \to A$ omomorf. t.c. $q \circ i = id_A$ (proiezione)

In tal caso diciamo che la successione splitta e abbiamo che sia $q$ sia $s$ inducono isomorfismi di $B$ con $A \oplus C$.

**Dimo**: Strategia: usare $s/q$ per ottenere l'isomorfismo e usarlo per definire $q/s$.

$(1) \Rightarrow (2)$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "p"] \arrow[l, bend left, dashed, "q"] & C \arrow[r] \arrow[l, bend left, "s"] & 0 \end{tikzcd}$$

Definisco $\Phi : A \oplus C \to B$

$$(a, c) \mapsto i(a) + s(c)$$

- omomorf.: $\phi((a_1, c_1) + (a_2, c_2)) = i(a_1 + a_2) + s(c_1 + c_2)$

$$\phi(a_1, c_1) + \phi(a_2, c_2) = i(a_1) + s(c_1) + i(a_2) + s(c_2) \checkmark$$

- iniettivo: $i(a) + s(c) = 0 \Rightarrow p(\underbrace{i(a)}_{=0} + \underbrace{s(c)}_{=c}) = 0$

$$\Rightarrow c = 0 \Rightarrow a = 0$$

• suriettivo $\quad b \in B \Longrightarrow p(b - s(p(b))) = p(b) - p(b) = 0$

$$\Longrightarrow b - s(p(b)) = i(a)$$

$$\Longrightarrow b = \Phi(a, p(b)).$$

Definisco $\quad q = \pi_A \circ \Phi^{-1}$.

(2) $\Longrightarrow$ (1)

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "p"] \arrow[l, bend left, "q"] & C \arrow[r] \arrow[l, bend left, dashed, "s"] & 0 \end{tikzcd}$$

Definisco $\psi : B \longrightarrow A \oplus C$

$\psi(b) = (a, p)$ è isomorfismo (esercizio)

Ora $f = \psi^{-1} \circ \left( G \xrightarrow{\cong} 0 \oplus G \subset A \oplus G \right)$. ///

<u>Oss</u>: (classif. gruppi ab) $\Longrightarrow$

A libero $\Longleftrightarrow$ privo di torsione

<u>Cor</u>: A libero, $B < A$ $\Longrightarrow$ B libero.

<u>Oss:</u> Se $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$ è esatta

e $C$ è libero allora splitta

(si definisca $s: C \to B$ sui gen. liberi di $C$).

UCT — teorema dei coeff universali (in omologia)

<u>Thm</u>: sia $\mathcal{C} = (C_n, \partial_n)_{n=0}^{+\infty}$ complesso, $R$ anello

$$\mathcal{C}^R = (C_n^R = C_n \otimes R, \partial_n^R = \partial_n \otimes id_R)$$

con omologie $H_*$ e $H_*^R$. Allora esiste succ. esatta canonica

$$0 \longrightarrow H_n \otimes R \xrightarrow{g_n} H_n^R \xrightarrow{f_n} \mathrm{Tor}(H_{n-1}, R) \to 0$$

che splitta ma non canonicamente.

<u>Cor</u>: $H_n^R \cong (H_n \otimes R) \oplus \mathrm{Tor}(H_{n-1}, R)$ non canonicamente

<u>Cor</u>: se $h : (X, A) \longrightarrow (Y, B)$ mappe continue

$$\begin{tikzcd}
0 \arrow[r] & H_n(X,A)\otimes R \arrow[r] \arrow[d, "h_{*_n}\otimes id_R"] & H_n(X,A;R) \arrow[r] \arrow[d, "h_{*_n}^R"] & \mathrm{Tor}(H_{n-1}(X,A),R) \arrow[r] \arrow[d, "*"] & 0 \\
0 \arrow[r] & H_n(Y,B)\otimes R \arrow[r] & H_n(Y,B;R) \arrow[r] & \mathrm{Tor}(H_{n-1}(Y,B),R) \arrow[r] & 0
\end{tikzcd}$$

* naturalmente indotta da $h_{*_{n-1}} \otimes id_R$

commuta

Es: $X = \mathbb{P}^2(\mathbb{R})$

$$
\begin{array}{llcccccccccc}
 & & 3 & & 2 & & 1 & & 0 & & -1 \\
\mathbb{Z} & G & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2\cdot} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \longrightarrow & 0 \\
 & H & & & 0 & & \mathbb{Z}/2 & & \mathbb{Z} & & 0 \\
\mathbb{Z}/2 & C^{\mathbb{Z}/2} & 0 & \longrightarrow & \mathbb{Z}/2 & \xrightarrow{0} & \mathbb{Z}/2 & \xrightarrow{0} & \mathbb{Z}/2 & \longrightarrow & 0 \\
 & & & & \mathbb{Z}/2 & & \mathbb{Z}/2 & & \mathbb{Z}/2 & &
\end{array}
$$

$$H_2(\mathbb{P}^2, \mathbb{Z}/2) \cong \underbrace{H_2(\mathbb{P}^2, \mathbb{Z})}_{0} \otimes \mathbb{Z}/2 \oplus \underbrace{\mathrm{Tor}(H_1(\mathbb{P}^2, \mathbb{Z}), \mathbb{Z}/2)}_{\mathbb{Z}/2}$$

$$H_1(\mathbb{P}^2;\mathbb{Z}/2) \cong \underbrace{\underbrace{H_1(\mathbb{P}^2;\mathbb{Z})}_{\mathbb{Z}/2} \otimes \mathbb{Z}/2}_{\mathbb{Z}/2} \oplus \underbrace{\overset{\mathbb{Z}/2}{\mathrm{Tor}}(\underset{\mathbb{Z}}{H_0(\mathbb{P}^2;\mathbb{Z})};\mathbb{Z}/2)}_{0}$$

Lem: Sia $\theta : A \to B$ ; esso induce $\tilde{\theta} : \mathrm{Tor}(A,C) \to \mathrm{Tor}(B,C)$.

Dim: Sia $0 \to K \xrightarrow{i} F \xrightarrow{p} C \to 0$

ris. libera di $C$

$$\Rightarrow \quad \begin{tikzcd} A \otimes K \arrow[r, "id_A \otimes i"] \arrow[d, "\theta \otimes id_K"'] \arrow[dr, phantom, "\circlearrowright"] & A \otimes F \arrow[d, "\theta \otimes id_F"] \\ B \otimes K \arrow[r, "id_B \otimes i"] & B \otimes F \end{tikzcd}$$

$$\Rightarrow (\theta \otimes id_K)\big(\underbrace{\operatorname{Ker}(id_A \otimes i)}_{= \operatorname{Tor}(A,C)}\big) \subset \underbrace{\operatorname{Ker}(id_B \otimes i)}_{= \operatorname{Tor}(B,C)}$$

$\Rightarrow \tilde{\theta}$ è definita. $\square$

<u>Dim</u> (UCT) : Uso $Z_n$, $B_n$, $Z_n^R$, $B_n^R$ – Oss:

1) $z \in Z_n$, $\lambda \in R$ $\qquad \partial_n^R (z \otimes \lambda) = (\partial_n \otimes id_R)(z \otimes \lambda)$

$$= (\partial_n z) \otimes \lambda = 0 \otimes \lambda = 0$$

$$\Longrightarrow \text{"} Z_n \otimes R \subset Z_n^R \text{"}$$

Formalmente $\quad \mathrm{Im}\begin{pmatrix} (Z_n \hookrightarrow C_n) \otimes id_R \\ Z_n \otimes R \longrightarrow C_n^R \end{pmatrix} \subset Z_n^R$ –

2) $z \in B_n, \lambda \in R \Longrightarrow z = \partial_{n+1} w$

$$z \otimes \lambda = (\partial_{n+1} w) \otimes \lambda = (\partial_{n+1} \otimes id_R)(w \otimes \lambda)$$

$$= \partial^R_{n+1}(w \otimes \lambda)$$

$\Longrightarrow$ "$B_n \otimes R \subset B_n^R$" ___

Notazione:

$$0 \longrightarrow B_n \xrightarrow{j_n} Z_n \xrightarrow{p_n} H_n \longrightarrow 0$$

(è risoluz. libera di $H_n$)

$$0 \longrightarrow B_n^R \xrightarrow{j_n^R} Z_n^R \xrightarrow{p_n^R} H_n^R \longrightarrow 0$$

Vogliamo:

$$0 \longrightarrow H_n \otimes R \xrightarrow{g_n} H_n^R \xrightarrow{f_n} \mathrm{Tor}(H_{n-1}, R) \longrightarrow 0$$

$\shortparallel$

$$\mathrm{Ker}\left(B_{n-1}\otimes R \xrightarrow{j_{n-1}\otimes \mathrm{id}_R} Z_{n-1}\otimes R\right)$$

può essere $\neq 0$.