Matematica A – 18/5/16

$$f(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

funzione di densità di distrib. gaussiana X

con $\mu(X) = \mu$, $\sigma(X) = \sigma$.

$$F(x) = p(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$$

Fatto: per F non esistono formule.

Caso normalizzato $\mu = 0$, $\sigma = 1$

$$f_0(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}$$

$$F_0(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \, dt .$$

1 . $F(x) = F_0\left(\frac{x-\mu}{\sigma}\right)$ Infatti

$$F(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} dt = \ldots$$

$$s = \frac{t-\mu}{\sigma} \qquad t = \mu + s \cdot \sigma$$

$$t = -\infty \Longleftrightarrow s = -\infty \qquad dt = \sigma \cdot ds$$

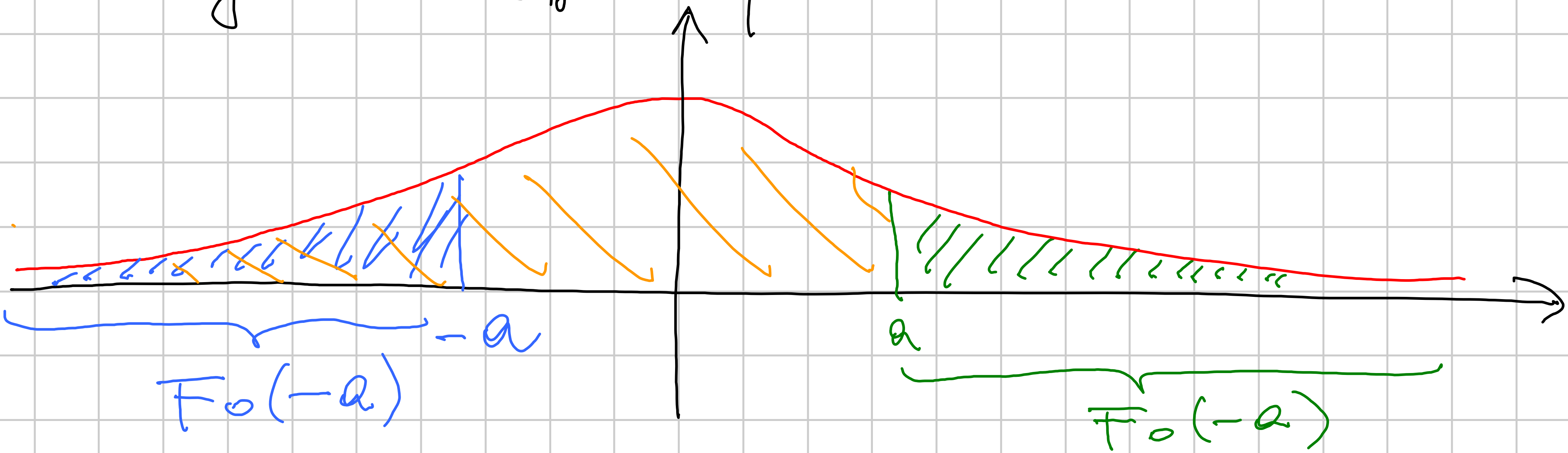
$$t = x \Longleftrightarrow s = \frac{x-\mu}{\sigma}$$

$$\ldots = \frac{1}{\cancel{\sigma} \cdot \sqrt{2\pi}} \int_{-\infty}^{\frac{x-\mu}{\sigma}} e^{-\frac{s^2}{2}} \cancel{\sigma}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-\mu}{\sigma}} e^{-\frac{s^2}{2}}\, ds = F_0\left(\frac{x-\mu}{\sigma}\right).$$

- $F_0(-a) = 1 - F_0(a)$

Ragione: f_0 è pari:



$F_0(a)$

1

blu = verde = rossa − arancione

$$\text{blu} = F_0(-a), \qquad \text{rossa} = 1, \qquad \text{arancione} = F_0(a)$$

$$F(x) = F_0\left(\frac{x-\mu}{\sigma}\right) \qquad F_0(-a) = 1 - F_0(a)$$

se conosco i valori di F_0, conosco quelli di F

se conosco i valori di F_0 su $a>0$ li conosco su tutti

per calcolare $F(x)$ $\quad \forall x \in \mathbb{R}$

bastano $F_0(a)$ $\quad a \geq 0$

(contenuti nelle tabelle).

② $\mu = 100 \qquad \sigma = 64 \qquad \left(F(x) = p(X \leq x) \right)$

$$p(X > 90) = 1 - p(X \leq 90)$$

$$= 1 - F(90) = 1 - F_0\left(\frac{90-100}{64}\right)$$

$$= 1 - F_0(-0.15625)$$

$$= 1 - (1 - F_0(0.15625))$$

$$= F_0(0.15625) \cong 0.562 \cong 56.2\,\%$$

③ $\mu = 0.824 \quad \sigma = 0.042$

- $p(0.784 \leq X \leq 0.934)$

$$= F(0.934) - F(0.784)$$

$$= F_0\left(\frac{0.934-0.824}{0.042}\right) - F_0\left(\frac{0.784-0.824}{0.042}\right)$$

$$= F_0(2.619) - F_0(-2)$$

$$= F_0(2.619) - (1 - F_0(2))$$

$$= F_0(2.619) + F_0(2) - 1$$

$$= 0.9956 + 0.83 - 1 = \dots$$

- Trovare k t.c. $p(|X - 0.824| \leq k) = 0.95$

$$p(0.824 - k \leq X \leq 0.824 + k)$$

$$= F(0.824+k) - F(0.824-k)$$

$$= F_0\left(\frac{k}{\sigma}\right) - F_0\left(-\frac{k}{\sigma}\right)$$

$$= F_0\left(\frac{k}{\sigma}\right) - \left(1 - F_0\left(\frac{k}{\sigma}\right)\right)$$

$$= 2F_0\left(\frac{k}{\sigma}\right) - 1 = 2F_0\left(\frac{k}{0.042}\right) - 1$$

Voglio che faccia 0.95

$$\Rightarrow F_0\left(\frac{k}{0.024}\right) = 0.975$$

$$\Rightarrow \frac{k}{0.024} \cong 1.96$$

$$k \cong 0.08232$$

4 $\mu = 2$, $\sigma = 3$

- $p(-1.5 \leq X \leq 4.2)$

$$= F(4.2) - F(-1.5)$$

$$= F_0\left(\frac{4.2-2}{3}\right) - F_0\left(\frac{-1.5-2}{3}\right)$$

$$= F_0(0.733) - F_0(-1.166)$$

$$= F_0(0.733) - 1 + F_0(1.166)$$

$$= 0.768 - 1 + 0.867 = \dots$$

- Trovare k t.c. $p(X \geq k) = 0.9$

$$\Updownarrow$$

$$1 - p(X \leq k) = 0.9$$

$$p(X \leq k) = 0.1$$

$$F(k) = 0$$

$$F_0\left(\frac{k-2}{3}\right) = 0.1$$

$< 1/2$

дшрие $k-2$

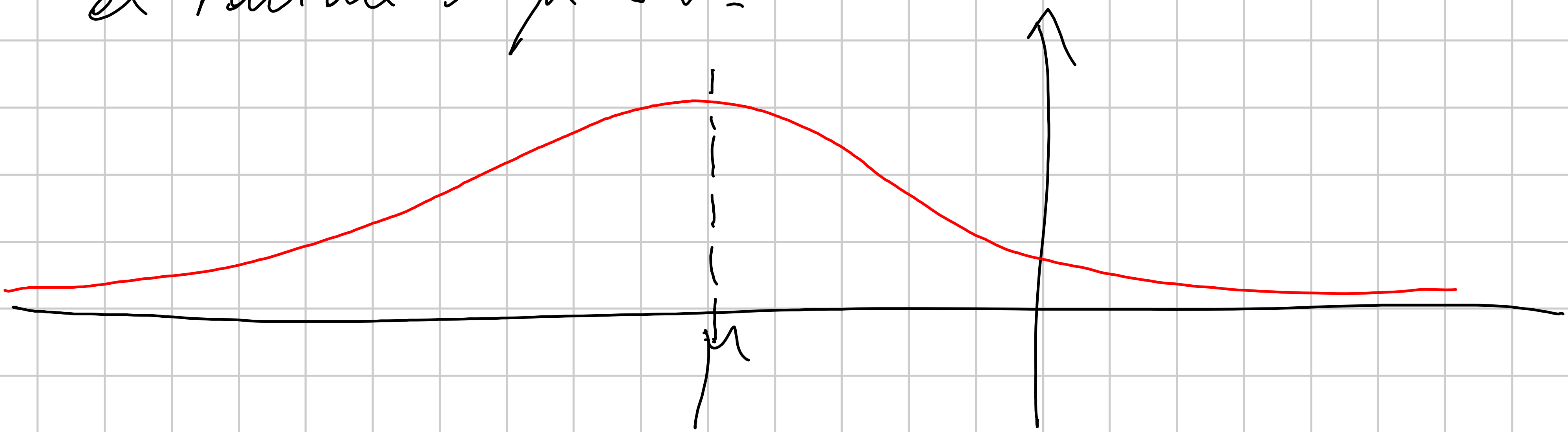
$$1 - F_0\left(\frac{2-k}{3}\right) = 0.1$$

$$F_0\left(\frac{2-k}{3}\right) = 0.9$$

$$\frac{2-k}{3} = 1.28$$

$$K = 2 - 3 \times 1.28 = -1.8\ldots$$

(5) Calcolare il massimo di $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ al variare di μ e σ.



Sempre assunto in $x = \mu$ e vale $\frac{1}{\sigma \cdot \sqrt{2\pi}}$.

⑥ $F(x) = F_0(2x+1)$ μ e σ pu F?

$$F(x) = F_0\left(\frac{x-\mu}{\sigma}\right)$$

$$= F_0\left(\frac{x-(-1/2)}{1/2}\right)$$

$$\mu = -\frac{1}{2} \qquad \sigma = \frac{1}{2}$$

⑦ Ipotesi: $\sigma^2 = 2\pi$ $\quad f(3) = \frac{1}{4\pi}$

Voglio: μ.

$$f(x) = \frac{1}{\sqrt{2\pi} \cdot \sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2 \cdot 2\pi}} = \frac{1}{2\pi} e^{-\frac{(x-\mu)^2}{4\pi}}$$

$$f(3) = \frac{1}{4\pi} \qquad \frac{1}{2\pi} e^{-\frac{(3-\mu)^2}{4\pi}} = \frac{1}{4\pi}$$

$$e^{-\frac{(3-\mu)^2}{4\pi}} = \frac{1}{2} \qquad -\frac{(3-\mu)^2}{4\pi} = -\log 2$$

$$(3-\mu)^2 = 4\pi \log 2 \qquad \mu = 3 \pm \sqrt{4\pi \log 2}$$

⑧ $\sigma = \dfrac{1}{\sqrt{2\pi}} \qquad f(1) = e^{-4\pi} \qquad \mu = ?$

$$f(x) = \frac{1}{\frac{1}{\sqrt{2\pi}} \cdot \sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2 \cdot \frac{1}{2\pi}}} =$$

$$= e^{-\pi \cdot (x-\mu)^2}$$

$$f(1) = e^{-4\pi}$$

$$e^{-\pi(1-\mu)^2} = e^{-4\pi}$$

$$(1-\mu)^2 = 4 \qquad 1-\mu = \pm 2 \qquad \mu = \begin{cases} 3 \\ -1 \end{cases}$$

9 $\sigma = 1 \qquad f(-2) = e^{-8} \qquad \mu = ?$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2}}$$

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{(2+\mu)^2}{2}} = e^{-8}$$

$$-\frac{1}{2}\log(2\pi) - \frac{(2+\mu)^2}{2} = -8$$

$$(2+\mu)^2 = 16 - \log(2\pi)$$

$$\mu = 2 \pm \sqrt{16 - \log(2\pi)}$$

10 $p(♀ \geq 7 \text{ e } ♂ \geq 7)$

$$= p(♀ \geq 7) \cdot p(♂ \geq 7)$$

$$= \big(1 - p(♀ \leq 7)\big) \cdot \big(1 - p(♂ \leq 7)\big)$$

$$= \left(1 - F_0(7-6.2)\right)\left(1 - F_0(7-5)\right)$$

$$= \left(1 - F_0(0.8)\right) \cdot \left(1 - F_0(2)\right)$$

$$= (1 - 0.7881)(1 - 0.9772)$$

$$= 0.00485 \simeq 0.5\%$$

(11) Quale tempo T serve per essere nel 20% più veloce?

$\mu = 15.5$

$\sigma^2 = 1.44$

$\sigma = 1.2$

$$p(X \leq T) = 0.2$$

$$F_0\left(\frac{T-15.5}{1.2}\right)=0.2$$

$$F_0\left(\frac{15.5-T}{1.2}\right)=0.8$$

$$\frac{15.5-T}{1.2}=0.84$$

$$T=15.5-1.2\times 0.84=14.49$$

(12) $\mu = 120 \quad \sigma = 40$

Voglio trovare L (peso soglia) t.c.

$$p(80 \leq X \leq L) = p(X \geq L)$$

$$F(L) - F(80) = 1 - F(L)$$

$$2F(L) = 1 + F(80)$$

$$2 F_0\left(\frac{L - 120}{40}\right) = 1 + F_0\left(\frac{80 - 120}{40}\right)$$

$$= 1 + F_0(-1)$$

$$= 1 + 1 - F_0(1)$$

$$= 2 - 0.8413$$

$$F_0\left(\frac{L-120}{40}\right) = 0.57935$$

$$\frac{L}{40} - 3 = 0.2$$

$$L = 120 + 8 = 128$$