Matematica A - 11/5/16

(1) a) $p(X=2) = 9\, p(X=4)$

$$(1-p)^{\cancel{2}}\,\cancel{p} = 9 \cdot (1-p)^{\cancel{4}\,2} \cdot \cancel{p}$$

$$(1-p)^2 = \frac{1}{9} \qquad 1-p = \frac{1}{3} \qquad p = \frac{2}{3}$$

$$\mu(X) = \frac{3}{2} \qquad \sigma^2(X) = \frac{1-\frac{2}{3}}{\left(\frac{2}{3}\right)^2} = \frac{\frac{1}{3}}{\frac{4}{9}} = \frac{3}{4}$$

X geom $p = \frac{2}{3}$ $\quad\quad$ Y geom $p = \frac{1}{2}$

$$Z = X + Y$$

$$p(Z = k) = \sum_{j=0}^{k} p(X = j) \cdot p(Y = k-j)$$

$$= \sum_{j=0}^{k} \left(\frac{1}{3}\right)^{j} \cdot \cancel{\frac{2}{3}} \cdot \left(\frac{1}{2}\right)^{k-j} \cdot \frac{1}{\cancel{2}}$$

$$= \left(\frac{1}{2}\right)^{k} \cdot \left(\frac{1}{2}\right)^{-j}$$

$$= \frac{1}{3} \cdot \left(\frac{1}{2}\right)^{k} \cdot \sum_{j=0}^{k} \left(\frac{1}{3}\right)^{j} \left(\frac{1}{2}\right)^{-j}$$

$$= \frac{1}{3} \cdot \left(\frac{1}{2}\right)^k \cdot \sum_{i=0}^{k} \left(\frac{2}{3}\right)^i = \ldots$$

(Ricordo: $(x-1)(x^k + x^{k-1} + \ldots + x + 1)$

$$= x^{k+1} + x^k + \ldots + x^2 + x$$
$$- x^k - \ldots - x^2 - x - 1$$

$$= x^{k+1} - 1$$

$$\underbrace{x^k + x^{k-1} + \ldots + x + 1}_{\sum_{i=0}^{k} x^i} = \frac{x^{k+1} - 1}{x - 1}$$

Somma della progressione geometrica di ragione x tra 0 e k

$$= \frac{1}{3} \cdot \left(\frac{1}{2}\right)^k \cdot \frac{\left(\frac{2}{3}\right)^{k+1} - 1}{\frac{2}{3} - 1}$$

$$= \left(\frac{1}{2}\right)^k \cdot \left(1 - \frac{2}{3}\right)^{k+1} = \frac{1}{2^k} - \frac{2}{3^{k+1}}.$$

Perché variabile "geometrica".

Verifico che la somma delle prob. di tutti gli eventi possibili fa 1;

$$\underbrace{\sum_{k=0}^{+\infty}}_{\lim\limits_{N\to+\infty}\sum\limits_{k=0}^{N}} (1-p)^k \cdot p = p \cdot \lim_{N\to\infty} \underbrace{\sum_{k=0}^{N} (1-p)^k}_{\text{somma della progr. geometrica di ragione } 1-p}$$

$$= \cancel{p} \cdot \lim_{N\to\infty} \frac{1-(1-p)^{N+1}}{\cancel{1-(1-p)}} =$$

$$= 1 - \lim_{N\to\infty} (1-p)^N = 1 - 0 = 1$$

(se $p > 0$).

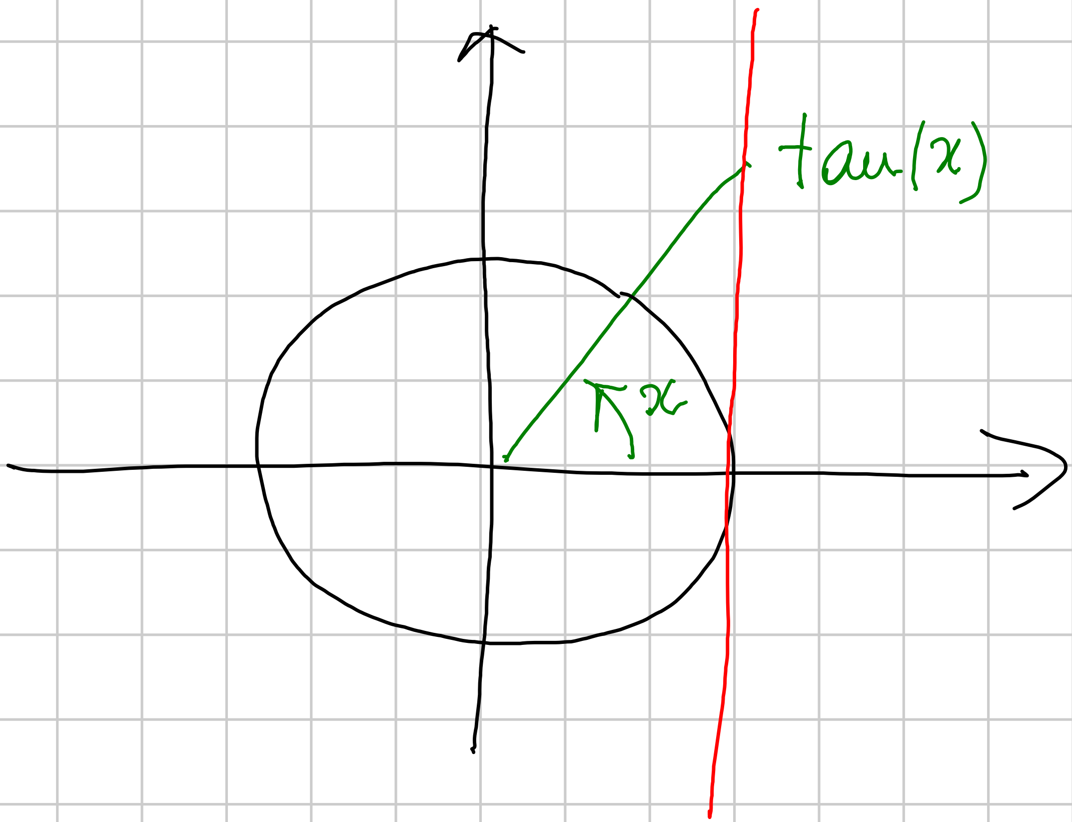
② Verificare che f è densità per X continua e trovare F di ripartizione:

- $f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$ ✓

  - non neg.
  - $\int_{-\infty}^{+\infty} f(x)\,dx = 1$ infatti:

$$\int_{-\infty}^{+\infty} \frac{1}{\pi} \cdot \frac{1}{1+x^2}\, dx = \frac{1}{\pi}\, \operatorname{atan}(x) \Big|_{-\infty}^{+\infty}$$

$$= \frac{1}{\pi} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right) = 1$$



$$F(x) = \int_{-\infty}^{x} f(t)\, dt$$

$$= \frac{1}{\pi} \cdot \operatorname{atan}(t) \Big|_{-\infty}^{x}$$

$$= \frac{1}{\pi} \operatorname{atan}(x) - \frac{1}{\pi} \cdot \left(-\frac{\pi}{2} \right)$$

$$= \frac{1}{2} + \frac{1}{\pi} \operatorname{atan}(x)$$

- $f(x) = \frac{1}{2} \frac{1}{1+2|x|+x^2}$

  - $f(x) \geq 0$ ✓

  - $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

$$\frac{1}{2}\int_{-\infty}^{+\infty} \frac{1}{1+2|x|+x^2}\,dx = \frac{1}{2}\int_{-\infty}^{0} \frac{1}{1-2x+x^2}\,dx + \frac{1}{2}\int_{0}^{+\infty} \frac{1}{1+2x+x^2}\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{0} \frac{1}{(1-x)^2}\,dx + \frac{1}{2}\int_{0}^{+\infty} \frac{1}{(1+x)^2}\,dx$$

$$= \frac{1}{2}\left(\frac{1}{1-x}\right)\Bigg|_{-\infty}^{0} + \frac{1}{2}\left(-\frac{1}{1+x}\right)\Bigg|_{0}^{+\infty} = \dots$$

$$\left(\left(\frac{1}{1-x}\right)' = \left((1-x)^{-1}\right)' = (-1)\cdot(1-x)^{-2}\cdot(-1) = (1-x)^{-2} = \frac{1}{(1-x)^2}\right)$$

$$= \frac{1}{2}(1-0) + \frac{1}{2}(0-(-1)) = 1 \checkmark$$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} \displaystyle\int_{-\infty}^{x} \frac{1}{2} \cdot \frac{1}{(1-t)^2}\,dt & x \leq 0 \\ \displaystyle\int_{-\infty}^{0} \frac{1}{2} \cdot \frac{1}{(1-t)^2}\,dt + \int_{0}^{x} \frac{1}{2} \frac{1}{(1+t)^2}\,dt & x \geq 0 \end{cases}$$

$$= \begin{cases} \displaystyle\frac{1}{2}\left(\frac{1}{1-t}\right)\Bigg|_{t=-\infty}^{t=x} & x \leq 0 \\ \displaystyle\frac{1}{2} + \frac{1}{2}\left(-\frac{1}{1+t}\right)\Bigg|_{t=0}^{t=x} & x \geq 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2} \frac{1}{1-x} & x \leq 0 \\ \frac{1}{2} + \frac{1}{2}\left(-\frac{1}{1+x} - (-1)\right) = 1 - \frac{1}{2}\frac{1}{1+x} & x \geq 0 \end{cases}$$

③ Voglio $\mu(X) = 0$

- $F(0) = \frac{1}{2}$

$$\left.\begin{aligned} F(0) &= F(0) - F(-\infty) \\ &= p(X \leq 0) \\ 1 - F(0) &= p(X \geq 0) \end{aligned}\right\} \text{sempre}$$

$F(0) = \frac{1}{2} \Rightarrow$ c'è la stessa probabilità che X sia negativa o positiva.

Ciò <u>non</u> comporta che $\mu(X) = 0$:

es.:

$\mu(X) = -1$ $F(0) = 1/2$

- $F(x) + F(-x) = 1 \quad \forall x \geq 0$

Derivando otteugo

$$f(x) - f(-x) = 0 \quad \forall x \geq 0$$

$$\Rightarrow f(-x) = f(x)$$

la probabilità di un evento di lunghezza dx vicino a x è la stessa di un evento di lunghezza dx vicino a $-x$

si semplificano calcolando $\mu(X)$

$$\Rightarrow \mu(X) = 0 \, .$$

Fondamentale:

$$\mu(X) = \int_{-\infty}^{+\infty} x \cdot f(x)\,dx = \int_{-\infty}^{0} x \cdot f(x)\,dx + \int_{0}^{+\infty} x \cdot f(x)\,dx$$

$$y = -x$$

$$dx = dy$$

$$(-\infty, 0] \to [0, +\infty)$$

$$\int_{0}^{+\infty} -y \cdot f(-y)\,dy = -\int_{0}^{+\infty} y \cdot f(y) \cdot dy$$

$\Longrightarrow$ fa 0.

Oss: se so che $f(x) - f(-x) = 0$ integrando trovo

$$F(x) + F(-x) = c \qquad \text{costante}$$

$$\lim_{+\infty} \left(F(x) + F(-x) \right) = 1 + 0 = 1$$

$$\lim_{-\infty} \left(F(x) + F(-x) \right) = 0 + 1 = 1$$

dunque per forza $c = 1$.

④ Per quale c la f è densità di prob?

- $f(x) = \begin{cases} 0 & x \leq 0 \\ c \cdot e^{-5x} & x > 0 \end{cases}$

- $f(x) \geq 0$ ✓

- $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

$$\int_{0}^{+\infty} c \cdot e^{-5x}\,dx = 1$$

$$c \cdot \left(-\frac{1}{5}\right) e^{-5x} \Big|_0^{+\infty} = 1$$

$$0 - c \cdot \left(-\frac{1}{5}\right) = 1$$

$$c = 5 .$$

- $$f(x) = \begin{cases} c\,(x^2 + x + 2) & |x| \leq 3 \\ 0 & |x| > 3 . \end{cases}$$

- $f(x) \geq 0$ $\qquad$ $x^2 + x + 2 = \left(x + \frac{1}{2}\right)^2 + \frac{7}{4} > 0$ ✓

$$-\int_{-\infty}^{+\infty} f(x)\,dx = 1$$

$$\int_{-3}^{3} c\,(x^2+x+2)\,dx = 1$$

$$c \cdot \left(\frac{1}{3}x^3+\frac{1}{2}x^2+2x\right)\Bigg|_{-3}^{3} = 1$$

$$c \cdot \left(\left(9+\frac{9}{2}+6\right)-\left(-9+\frac{9}{2}-6\right)\right) = 1$$

$$30 \cdot c = 1 \qquad c = \frac{1}{30}$$

(5) $F(x) = \begin{cases} 0 & x \leq 0 \\ 1 - \dfrac{1}{1+81x^4} & x > 0 \end{cases}$

- non decrescente

$$F'(x) = \begin{cases} 0 & x < 0 \\ \underbrace{-(1+81x^4)^{-2} \cdot (-1) \cdot (81 \cdot 4 \cdot x^3)}_{\text{sempre } \geq 0} & x > 0 \end{cases} \quad \checkmark$$

- $\lim_{-\infty} F(x) = 0 \quad \checkmark$

- $\lim_{+\infty} F(x) = 1$ ✓

m mediana se $F(m) = 1/2$

$$1 - \frac{1}{1+81x^4} = \frac{1}{2}$$

$$\frac{1}{1+81x^4} = \frac{1}{2}$$

$$1+81x^4 = 2$$

$$81x^4 = 1 \qquad x = \frac{1}{3}$$

leggere m invece che x

⑥ Trovare a, b t.c.

$$f(x) = \begin{cases} a + bx^2 & x \in [0,1] \\ 0 & x \notin [0,1] \end{cases}$$

sia densità di X con $\mu(X) = \frac{3}{5}$.

Poi calcolare $\sigma^2(X)$ e $F(x)$.

- $f(x) \geq 0$ (lo verifichiamo dopo)

$$- \int_{-\infty}^{+\infty} f(x)\,dx = 1 \qquad \int_0^1 (a+bx^2)\,dx = 1$$

$$- \int_{-\infty}^{+\infty} x \cdot f(x)\,dx = \frac{3}{5} \qquad \int_0^1 (ax+bx^3)\,dx = \frac{3}{5}$$

$$\begin{cases} a + \frac{b}{3} = 1 \\ \frac{a}{2} + \frac{b}{4} = \frac{3}{5} \end{cases} \qquad \begin{cases} a = 1 - \frac{b}{3} \\ \frac{1}{2} - \frac{b}{6} + \frac{b}{4} = \frac{3}{5} \end{cases} \qquad \begin{cases} a = 1 - \frac{b}{3} \\ \frac{b}{12} = \frac{1}{10} \end{cases}$$

$$\begin{cases} a = 3/5 \\ b = 6/5 \end{cases}$$

$$\sigma^2(X) = \mu(X^2) - \mu(X)^2 = \int_0^1 \left(\frac{3}{5}x^2 + \frac{6}{5}x^4\right)dx - \left(\frac{3}{5}\right)^2$$

$$= \left(\frac{1}{3}\cdot\frac{3}{5} + \frac{1}{5}\cdot\frac{6}{5}\right) - \left(\frac{3}{5}\right)^2 = \ldots$$

$$F(x) = \begin{cases} 0 & x \le 0 \\ \int_0^x \left(\frac{3}{5} + \frac{6}{5}t^2\right)dt & x \in [0,1] \\ 1 & x \geq 1 \end{cases}$$

$$= \begin{cases} 0 & x \le 0 \\ \frac{3}{5}x + \frac{2}{5}x^3 & 0 \le x \le 1 \\ 1 & x \geq 1 \end{cases}$$