Alg. Lin. 21/10/15

$\boxed{3.2.3}$ $\mathbb{R}[t]$ $p(t) = a_0 + a_1 t + a_2 t^2 + \dots$

- $p(-3) + p''(2) = 0$

$$a_0 - 3a_1 + 9a_2 - 27a_3 + \dots + a_2 + 12a_3 + 48a_4 + \dots = 0$$

ok

- $p(1) \cdot p'''(-1) = 0$

$$(a_0 + a_1 + a_2 \dots) \cdot (6a_3 - 24a_4 + \dots) = 0$$

II grado: no

$0 \in W$; $w \in W, \lambda \in \mathbb{R} \Rightarrow \lambda w \in W$

$$p_1(t) = 1 - t^3 \in W$$

$$p_2(t) = -1 \in W$$

$$p_1(t) + p_2(t) = -t^3 \notin W$$

- $$\underbrace{\underbrace{\sum_{n=0}^{\infty} (-1)^n \cdot t^n \cdot p^{(n)}(t)}_{\text{è un polinomio}} = 0}_{\text{equivale a infinite equazioni nei coeff. } a_j}$$

$$
\begin{array}{l}
a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \dots \\
\quad - a_1 t - 2a_2 t^2 - 3a_3 t^3 - 4a_4 t^4 + \dots \\
\qquad\qquad + 2a_2 t^2 + 6a_3 t^3 + 12a_4 t^4 + \dots \\
\qquad\qquad\qquad \ddots \\
\hline
\end{array} = 0
$$

$$
\begin{cases}
a_0 = 0 \\
a_2 = 0 \\
4a_3 = 0 \\
9a_4 = 0 \\
\vdots
\end{cases}
\implies \text{Sol}: \; W = \operatorname{Span}(t)
$$

- $|p(t)| \leq 1 + e^t \quad \forall t$

<u>No</u>: $\quad p(t) = 1 \in W, \ \lambda = 2, \ 2 \cdot p(t) \notin W$

- $\underbrace{\deg\left(2p'(t) - 3t^2 p''(t)\right) \leq 5}$

significa chiedere che i coeff di $t^6, t^7, t^8, \dots$ in $2p'(t) - 3t^2 p''(t)$ siano 0;

i suoi coeff sono pol. omog. di I grado in $a_0, a_1, \dots$ $\Longrightarrow$ Sì

$\boxed{3.2.4}$ $\mathcal{F}(\{a,b,c\},\mathbb{R})$

<u>Oss</u>: $f \in \mathcal{F}(\{a,b,c\},\mathbb{R})$

$$f \longleftrightarrow \begin{pmatrix} f(a) \\ f(b) \\ f(c) \end{pmatrix} \in \mathbb{R}^3$$

Ho corrisp. biunivoca tra $\mathcal{F}(\{a,b,c\},\mathbb{R})$ e $\mathbb{R}^3$ che rispetta le operazioni.

Dunque ci si riconduce a $\mathbb{R}^3$.

$$\bullet \quad \{ f : \ln(1+|4f(a)-3f(c)|)=0 \}$$

$$\Updownarrow$$

$$\{ x \in \mathbb{R}^3 : \underbrace{\ln(1+|4x_1-3x_3|)=0}_{\text{equivale a } 4x_1-3x_3=0} \}$$

$$\Longrightarrow \pi$$

$\boxed{4.1.1}$ Dire se $v \in \mathrm{Span}(w_1, \ldots, w_k)$ e se l'espr. è unica.

(a) $\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \alpha \cdot \begin{pmatrix} -2 \\ 5 \end{pmatrix}$

$$\begin{cases} -2\alpha = 3 \\ 5\alpha = -4 \end{cases}$$

No

(b) $\begin{pmatrix} \sqrt{6} \\ -2 \end{pmatrix} = \alpha \cdot \begin{pmatrix} -3 \\ \sqrt{6} \end{pmatrix}$ $\alpha = -\frac{\sqrt{3}}{\sqrt{2}}$ Sì, unica

(c) $\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \alpha \cdot \begin{pmatrix} 4 \\ -11 \end{pmatrix} + \beta \cdot \begin{pmatrix} 3 \\ 8 \end{pmatrix}$

$$\begin{cases} 4\alpha + 3\beta = 3 \\ -11\alpha + 8\beta = 7 \end{cases} \qquad \begin{cases} \alpha = 4/55 \\ \beta = 61/55 \end{cases}$$

Sì, unica

(d) $$\begin{pmatrix} -2 \\ 7 \end{pmatrix} = \alpha \cdot \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \beta \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \gamma \cdot \begin{pmatrix} 5 \\ -3 \end{pmatrix}$$

$$\begin{cases} 4\alpha + 2\beta + 5\gamma = -2 \\ \alpha - \beta - 3\gamma = 7 \end{cases}$$

$$\begin{cases} \alpha = \beta + 3\gamma + 7 \\ 6\beta + 17\gamma = -30 \end{cases}$$

infinite soluzioni

$\rightarrow$ Sì, non unica

$\beta = -5, \quad \gamma = 0, \quad \alpha = 2$ due soluz.

$\beta = 0, \quad \gamma = -30/17, \quad \alpha = \dots$ distinte

$\boxed{4.1.2}$

- $\begin{pmatrix} 4 \\ 5 \\ -2 \end{pmatrix} \in \mathrm{Span} \begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix}$? No

- $\begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix} = \alpha \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$

$$\begin{cases} 3\alpha + 4\beta = 2 \\ -2\alpha + \beta = -5 \\ \alpha - 2\beta = 4 \end{cases} \qquad \begin{cases} \alpha = 2\beta + 4 \\ -3\beta = 3 \\ 3\alpha + 4\beta = 2 \end{cases}$$

$$\begin{cases} \beta = -1 \\ \alpha = 2 \\ 0 = 0 \end{cases} \checkmark$$

Sì, unica

- $$\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \alpha \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}$$

$$\begin{cases} 5\alpha - 2\beta = 3 \\ \alpha + 2\beta = 1 \\ 2\alpha + 3\beta = -2 \end{cases} \qquad \begin{cases} \alpha = 2/3 \\ \beta = 1/6 \\ \dfrac{4}{3} + \dfrac{1}{2} = -2 \end{cases}$$

No

$$\bullet \quad \begin{pmatrix} -1 \\ 3 \\ -8 \end{pmatrix} = \alpha \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} + \beta \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} + \gamma \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$$

$$\begin{cases} 2\alpha + 5\beta + 3\gamma = -1 \\ 2\alpha + \beta - \gamma = 3 \\ -3\alpha + 2\beta + 5\gamma = -8 \end{cases} \qquad \begin{cases} \gamma = 2\alpha + \beta - 3 \\ 8\alpha + 8\beta = 8 \\ 7\alpha + 7\beta = 7 \end{cases}$$

$$\begin{cases} \beta = 1 - \alpha \\ \gamma = \alpha - 2 \end{cases}$$

Sì, non unica

$\boxed{4.1.3}$

$$1 - 2t + 3t^2 = \alpha(4 + t - t^2) + \beta \cdot (-3 + 2t + t^2)$$

$$\begin{cases} 4\alpha - 3\beta = 1 \\ \alpha + 2\beta = -2 \\ -\alpha + \beta = 3 \checkmark \end{cases} \qquad \begin{cases} \alpha = \beta - 3 \\ \beta = 1/3 \end{cases} \qquad \begin{cases} \beta = 1/3 \\ \alpha = -8/3 \\ -32/3 - 1 = 1 \end{cases} \quad \underline{\text{No}}$$

<u>Oss</u>: $v, w_1, w_2 \in \mathbb{R}_{\leq 2}[t]$

$$\mathbb{R}_{\leq d}[t] \ni a_0 + a_1 t + \dots + a_d t^d \longleftrightarrow \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_d \end{pmatrix} \in \mathbb{R}^{d+1}$$

corrisp. biunivoca che rispetta operazioni

- $4 - 3t + t^2 \in \mathrm{Span}(2 + t - t^2, 2 - 3t + 5t^2)$?

equivale a $\begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix} \in \mathrm{Span}\left(\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}\right)$?

$\boxed{4.14}$

$$\begin{pmatrix} 2 & 3 \\ -1 & 7 \end{pmatrix} = \alpha \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} + \beta \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \gamma \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{cases} 2\alpha + \beta = 2 \\ \gamma = 3 \checkmark \\ \alpha - \gamma = -1 \checkmark \\ -\alpha + 2\beta = 7 \checkmark \end{cases} \qquad \begin{cases} \gamma = 3 \\ \alpha = 2 \\ \beta = 9/2 \\ \beta = -2 \end{cases}$$

No

Oss:

$$M_{2\times 2}(\mathbb{R}) \ni \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \longleftrightarrow \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix} \in \mathbb{R}^4$$

corrisp. biunivoca che rispetta operaz.

$$\begin{pmatrix} 5 & -1 \\ 2 & 1 \end{pmatrix} \in \mathrm{Span}\left(\begin{pmatrix} 7 & 1 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -6 & 2 \end{pmatrix}\right)?$$

Equivale a $\begin{pmatrix} 5 \\ -1 \\ 2 \\ 1 \end{pmatrix} \in \mathrm{Span}\left(\begin{pmatrix} 7 \\ 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -6 \\ 2 \end{pmatrix}\right)?$

——— o ———

$w_1, w_2, \dots$ generano V? Se sì, l'espr. dei vett. di V come comb. lin. di $w_1, \dots$ è unica?

(Sì se e solo se sono lin. indip.)

$$W = \{x \in \mathbb{R}^3 : x_1 + 3x_2 - 2x_3 = 0\}$$

$$\underset{w_1}{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}}, \underset{w_2}{\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}}, \underset{w_3}{\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}}.$$

Verifica prel: $w_j \in W \quad j = 1, 2, 3$. ✓

Generico el. di W è $\begin{pmatrix} 2x_3 - 3x_2 \\ x_2 \\ x_3 \end{pmatrix}$, $x_2, x_3 \in \mathbb{R}$.

Generano? Equivale a:

$$\begin{pmatrix} 2x_3 - 3x_2 \\ x_2 \\ x_3 \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$

ha sempre soluz. (α, β, γ) al variare $x_2, x_3 \in \mathbb{R}$?

$$\begin{cases} -\alpha + 3\beta = 2x_3 - 3x_2 \\ \alpha - \beta + 2\gamma = x_2 \\ \alpha + 3\gamma = x_3 \end{cases}$$

$$\begin{cases} \beta = \dfrac{2x_3 - 3x_2 + \alpha}{3} \\ \gamma = \dfrac{x_3 - \alpha}{3} \\ 3\alpha - (2x_3 - 3x_2 + \alpha) + 2(x_3 - \alpha) = 3x_2 \end{cases}$$

$$\begin{cases} \beta = \dots \\ \gamma = \dots \\ 0 = 0 \ \checkmark \end{cases}$$

Sì, non unica.

w_1, w_2, w_3 generano W ma non sono base

• $\{x \in \mathbb{R}^3 : 5x_1 + 2x_2 - x_3 = 0\}$ $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ $\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$

$\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \notin W \Longrightarrow$ la domanda non ha senso

• $\{x \in \mathbb{R}^3 : 3x_1 - 2x_2 + x_3 = 0\}$ $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ $\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} x_1 \\ x_2 \\ 2x_2 - 3x_1 \end{pmatrix} = \alpha \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}.$$

$$\begin{cases} \alpha + 3\beta = x_1 \\ \alpha + 5\beta = x_2 \\ -\alpha + \beta = 2x_2 - 3x_1 \end{cases} \qquad \begin{cases} \beta = (x_2 - x_1)/2 \\ \alpha = (5x_1 - 3x_2)/2 \\ -5x_1 + 3x_2 + x_2 - x_1 = 4x_2 - 6x_1 \end{cases}$$

✓

Sì, unico. Dunque

$$\left(\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} \right)$$ sono base di $\overline{W}$.

$\boxed{4.1.6}$

$$\{x \in \mathbb{R}^4 : x_1 + 3x_2 + x_3 - x_4 = 0\} \qquad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 5 \end{pmatrix} \quad \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_1 + 3x_2 + x_3 \end{pmatrix} = \alpha \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 5 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{cases} \alpha + 2\beta = x_1 \\ \alpha - \beta = x_2 \quad \checkmark \\ \alpha + \beta = x_3 \quad \checkmark \\ 5\alpha = x_1 + 3x_2 + x_3 \end{cases} \qquad \begin{cases} \alpha = (x_2 + x_3)/2 \\ \beta = (x_3 - x_2)/2 \\ \text{Non vale la I} \end{cases}$$

<u>No</u>

$$\left\{ x \in \mathbb{R}^4 : \begin{array}{l} x_3 = 2x_1 + x_2 \\ x_4 = x_1 - 3x_2 \end{array} \right\} \qquad \begin{pmatrix} 3 \\ -4 \\ 2 \\ 15 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 0 \\ -7 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ 2x_1 + x_2 \\ x_1 - 3x_2 \end{pmatrix} = \alpha \cdot \begin{pmatrix} 3 \\ -4 \\ 2 \\ 15 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 2 \\ 0 \\ -7 \end{pmatrix}$$

$$\begin{cases} 3\alpha - \beta = x_1 \ ✓ \\ -4\alpha + 2\beta = x_2 \\ 2\alpha = 2x_1 + x_2 \ ✓ \\ 15\alpha - 7\beta = x_1 - 3x_2 \end{cases} \qquad \begin{cases} \alpha = x_1 + \frac{1}{2} x_2 \\ \beta = 2x_1 + \frac{3}{2} x_2 \\ -4x_1 - 2x_2 + 4x_1 + 3x_2 = x_2 \ ✓ \\ 30x_1 + 15x_2 - 28x_1 - 21x_2 = 2x_1 - 6x_2 \ ✓ \end{cases}$$

Sì, unico

$W = \mathbb{R}_{\leq 2}[t]$ $\quad 2 - 7t - 9t^2$

$1 + 4t + 3t^2$

$6 + 9t + 3t^2$

È vero che $\begin{pmatrix} 2 \\ -7 \\ 9 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix}$ generano $\mathbb{R}^3$? Sono base?

Il sistema $\begin{cases} 2\alpha + \beta + 6\gamma = x_1 \\ -7\alpha + 4\beta + 9\gamma = x_2 \\ 9\alpha + 3\beta + 3\gamma = x_3 \end{cases}$

ha soluz. (unica) $\forall x_1, x_2, x_3 \in \mathbb{R}$? [SI]

I vett. dati sono lin. indip.?

$v_1, \dots, v_m$ sono lin. indip. se gli unici $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ t.c. $\alpha_1 v_1 + \dots + \alpha_m v_m = 0$ sono $\alpha_1 = \dots = \alpha_m = 0$, cioè

l'equazione $\alpha_1 v_1 + \dots + \alpha_m v_m = 0$ ha solo la soluzione $\alpha_1 = \dots = \alpha_m = 0$

- $\sqrt{31} \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} \sqrt{\pi} \\ -7e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ <u>Non</u> lin. indip.

- $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ <u>Sì</u> lin. indip.

- $\alpha \cdot \begin{pmatrix} -5/6 \\ 8/3 \end{pmatrix} + \beta \cdot \begin{pmatrix} 1/4 \\ -4/5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{cases} -5/6\,\alpha + \beta/4 = 0 \\ 8/3\,\alpha - 4/5\,\beta = 0 \end{cases} \qquad \begin{cases} -10\alpha + 3\beta = 0 \\ 10\alpha - 3\beta = 0 \end{cases}$$

$\alpha = 3 \quad \beta = 10$ <u>Non</u> lin. indip.

<u>Oss</u>: v_1, v_2 sono lin. dipendenti se e solo se uno dei due è multiplo dell'altro.

- $\begin{pmatrix} 6 \\ -7 \end{pmatrix} \quad \begin{pmatrix} -2 \\ 3 \end{pmatrix} \qquad \begin{cases} 6\alpha - 2\beta = 0 \\ -7\alpha + 3\beta = 0 \end{cases}$

$$\begin{cases} 4\alpha = 0 \\ \end{cases} \Longrightarrow \alpha = 0 \Longrightarrow \beta = 0$$

<u>Lin. indip</u>

- $\alpha \cdot \begin{pmatrix} 1 \\ -3 \end{pmatrix} + \beta \cdot \begin{pmatrix} 2 \\ 5 \end{pmatrix} + \gamma \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{cases} \alpha + 2\beta + 3\gamma = 0 \\ -3\alpha + 5\beta + 2\gamma = 0 \end{cases} \qquad \begin{cases} \alpha = -2\beta - 3\gamma \\ 11\beta + 11\gamma = 0 \end{cases}$$

$\beta = 1 \quad \gamma = -1 \quad \alpha = -5$ è una soluz $\neq 0$

$\Longrightarrow$ lin. dip.

- $\begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 10 \\ 12 \\ -5 \end{pmatrix}$ lin. indip.

- $\begin{pmatrix} 4 \\ 5 \\ -3 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \quad \begin{pmatrix} 7 \\ 12 \\ -10 \end{pmatrix}$

$\underline{\mathrm{III}} = 2 \cdot \mathrm{I} - \mathrm{II}$

$$2 \cdot \begin{pmatrix} \\ \\ \\ \end{pmatrix} - \begin{pmatrix} \\ \\ \\ \end{pmatrix} - \begin{pmatrix} \\ \\ \\ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

lin. dip.

$$\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \quad \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$$

$$\begin{cases} \alpha + 5\beta = 0 \\ -2\alpha + 3\gamma = 0 \\ 4\alpha - \beta + 2\gamma = 0 \end{cases} \qquad \begin{cases} \beta = -\alpha/5 \\ \gamma = 2/3 \cdot \alpha \\ 4\alpha + \frac{\alpha}{5} + \frac{4}{3}\alpha = 0 \end{cases} \qquad \begin{cases} \alpha = 0 \\ \beta = 0 \\ \gamma = 0 \end{cases}$$

l'unica soluz.

$\Longrightarrow$ lin. indip.