AL 18/11/15

5.3.2

$$A = \begin{pmatrix} 2 & 3 \\ -1 & 4 \\ -2 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & 1 & 2 \\ -3 & 4 & -1 \end{pmatrix} \qquad x = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$$

$$f_{A\cdot B}(x) = (f_A \circ f_B)(x)$$

$$= f_A(f_B(x))$$

$$= A\cdot(B\cdot x)$$

$$f_{A\cdot B}(x) = (A\cdot B)\cdot x$$

$$\left(\begin{pmatrix} 2 & 3 \\ -1 & 4 \\ -2 & 5 \end{pmatrix} \cdot \begin{pmatrix} 5 & 1 & 2 \\ -3 & 4 & -1 \end{pmatrix}\right) \cdot \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 14 & 7 \\ -17 & 15 & -6 \\ -25 & 18 & -9 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 33 \\ 85 \\ 115 \end{pmatrix}$$

$$\left(\begin{pmatrix} 2 & 3 \\ -1 & 4 \\ -2 & 5 \end{pmatrix} \cdot \begin{pmatrix} 5 & 1 & 2 \\ -3 & 4 & -1 \end{pmatrix}\right) \cdot \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} = \ldots\ldots$$

5.3.3 Esibire base di nucleo e immagine di

$$A = \begin{pmatrix} 2 & -3 & 1 & -4 \\ -3 & 5 & -1 & 7 \\ -1 & 3 & 1 & 5 \end{pmatrix}$$

$f_A : \mathbb{R}^4 \to \mathbb{R}^3$ $\quad (\Rightarrow \dim \ker f \geq 1)$

$\mathrm{Im}(f_A) = \mathrm{Span}(\text{colonne di } A)$

si vede che ha $\dim \geq 2$

$\Rightarrow \quad \underset{}{4} = \underset{K}{1} + \underset{I}{3} \quad$ o $\quad 4 = \underset{K}{2} + \underset{I}{2}$

Estrapolo delle col. base di Im f:

$$\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \begin{pmatrix} -3 \\ 5 \\ 3 \end{pmatrix} \cancel{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} \cancel{\begin{pmatrix} -4 \\ 7 \\ 5 \end{pmatrix}}$$

$2 \cdot I + II$ $\qquad$ $I + 2 \cdot II$

$$\Rightarrow 4 = 2 + 2$$

Base di Ker f:

$$\begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}$$

<u>Oss</u>: $(v_1, \dots, v_n)\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix} = \alpha_1 v_1 + \dots + \alpha_n v_n$

$v_3 = 2v_1 + v_2 \implies (v_1, v_2, v_3, v_4)\begin{pmatrix}2\\1\\-1\\0\end{pmatrix} = 0$

$v_4 = v_1 + 2v_2 \implies (v_1, v_2, v_3, v_4)\begin{pmatrix}1\\2\\0\\-1\end{pmatrix} = 0.$

$\boxed{5.3.4}$ $A \in M_{m \times n}$ $B \in M_{k \times h}$

$$f : M_{n \times k} \longrightarrow M_{m \times h}$$
$$X \longmapsto A \cdot X \cdot B$$

$m \times n \cdot n \times k \cdot k \times h$

$m \times n$

Lineare $f(\lambda_1 X_1 + \lambda_2 X_2) = A \cdot (\lambda_1 X_1 + \lambda_2 X_2) \cdot B$

$$= \lambda_1 A X_1 B + \lambda_2 A X_2 B = \lambda_1 f(X_1) + \lambda_2 f(X_2)$$

$$\operatorname{Ker} f = \{ X : A \cdot X \cdot B = 0 \}$$

$$= \{ X : A \cdot X \cdot B \cdot v = 0 \ \forall v \in \mathbb{R}^n \}$$

$= \{ X : X(w) \in \operatorname{Ker} A \quad \forall\, w \in \operatorname{Im} B \}$

$= \{ X : X(\operatorname{Im} B) \subset \operatorname{Ker} A \}$

$\boxed{5.3.5}$ $X = \{ x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0 \}$

$$A = \begin{pmatrix} 4 & -5 & 3 \\ -7 & 1 & -9 \\ 1 & 2 & 4 \end{pmatrix}$$

Provare che la formula $g(x) = A \cdot x$ dà $g : X \to X$ lineare.

DA NON FARE:

$$g(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 g(x_1) + \lambda_2 g(x_2)$$

INVECE:

So che $f_A : \mathbb{R}^3 \to \mathbb{R}^3$
$$x \mapsto A \cdot x$$ è lineare.

Posso considerare $f_A|_X$ che è lineare.

Se dimostro che $\mathrm{Im}(f_A|_X) = f_A(X)$ è contenuta in X posso definire

$$g = f_A|_X^X$$

Morale: l'unica verifica da fare è che
$f_A(X) \subset X$.

<u>I Modo</u>: Ricordo che $X = \{x : x_1 + x_2 + x_3 = 0\}$
$= \{x : \omega \cdot x = 0\}$ dove $\omega = (1,1,1)$.

Devo provare che $f_A(X) \subset X$ cioè
$\omega \cdot y = 0 \quad \forall y \in f_A(X)$ cioè
$(\omega \cdot A) \cdot x = 0 \quad \forall x \in X$

$$w \cdot A = (1,1,1) \cdot \begin{pmatrix} 4 & -5 & 3 \\ -7 & 1 & -9 \\ 1 & 2 & 4 \end{pmatrix} = (-2,-2,-2)$$
$$= -2w$$

$$(w \cdot A) \cdot x = -2 \underbrace{w \cdot x}_{0 \text{ poiché } x \in X}$$

II MODO : Prendo la base $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ di X.

Basta provare che

$$A \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad e \quad A \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

appartengono a X.

$$\begin{pmatrix} 6 & -5 & 3 \\ -7 & 1 & -3 \\ 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ -8 \\ -1 \end{pmatrix} \checkmark, \quad \begin{pmatrix} 6 & -5 & 3 \\ -7 & 1 & -3 \\ 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \checkmark$$

$\boxed{5.3.6}$ $A = \begin{pmatrix} 2 & 1 \\ 3 & -4 \end{pmatrix}$

$$X = \{ B \in M_{2\times 2} : A \cdot B = B \cdot A \}.$$

$$\begin{pmatrix} 2 & 1 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & -4 \end{pmatrix}$$

$$\begin{cases} 2\cancel{b_{11}} + b_{21} = \cancel{2b_{11}} + 3b_{12} \\ 2b_{12} + b_{22} = b_{11} - 4b_{12} \\ 3b_{11} - 4b_{21} = 2b_{21} + 3b_{22} \\ 3b_{12} \cancel{-4b_{22}} = b_{21} \cancel{-4b_{22}} \end{cases} \checkmark$$

$$\begin{cases} b_{21} = 3b_{12} \\ 6b_{12} + b_{22} - b_{11} = 0 \\ 18b_{12} + 3b_{22} - 3b_{11} = 0 \end{cases} \checkmark \qquad \begin{cases} b_{21} = 3b_{12} \\ b_{11} = 6b_{12} + b_{22} \end{cases}$$

$$\dim = 2 \; ; \; \text{base} \begin{pmatrix} 6 & \boxed{1} \\ 3 & \boxed{0} \end{pmatrix} \begin{pmatrix} 1 & \boxed{0} \\ 0 & \boxed{1} \end{pmatrix} -$$

Fissiamo $A \in M_{2\times 2}$.

$$X = \{ B \in M_{2\times 2} : A \cdot B = B \cdot A \}$$

sicuramente contiene : I_2 e A

due casi : lin dip $\Rightarrow A = \lambda \cdot I_2$

$\Rightarrow X = M_{2\times 2}$

lin. indip $\Rightarrow \dim X \geq 2$

Verifichiamo che $\begin{pmatrix} 6 & 1 \\ 3 & 0 \end{pmatrix} \in \mathrm{Span}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 3 & -4 \end{pmatrix} \right)$

$$\begin{pmatrix} 6 & 1 \\ 3 & 0 \end{pmatrix} = 4 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 2 & 1 \\ 3 & -4 \end{pmatrix}.$$

(dim X non può venire 3 ...)

———— o ————

$\boxed{5.4.1}$ $\underset{n}{\dim(V)} \geq \underset{m}{\dim(W)}$

- esistono $f: V \to W$ surj. :

prendo $v_1, \dots, v_n$ base di V

$w_1, \dots, w_m$ base di W. Definisco :

$$f(v_j) = \begin{cases} w_j & \text{per } j = 1, \dots, m \\ \text{a caso} & \text{per } j = m+1, \dots, n \end{cases}$$

- data $f : V \to W$ surg. $\exists\, g : W \to V$ lin. t.c. $f \circ g = id_W$

(Oss : 1. $X \underset{g}{\overset{f}{\rightleftarrows}} Y$, $g \circ f = id_X$

$\Rightarrow$ f iniettiva , g surgettiva

(dim^e^: g è inversa sin. di f

f è inversa sin. di g)

2. Se $f: X \to Y$ è suriettiva esiste $g: Y \to X$ t.c. $f \circ g = id_Y$:

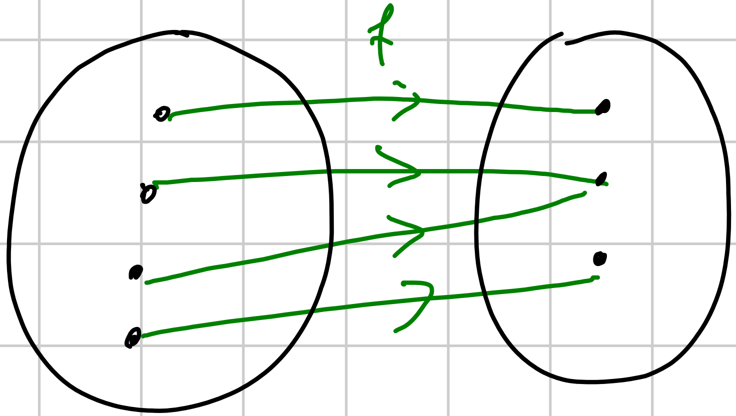
dato $y \in Y$ so che esiste $x \in X$ con $f(x) = y$;
ne prendo uno e lo chiamo $g(y)$

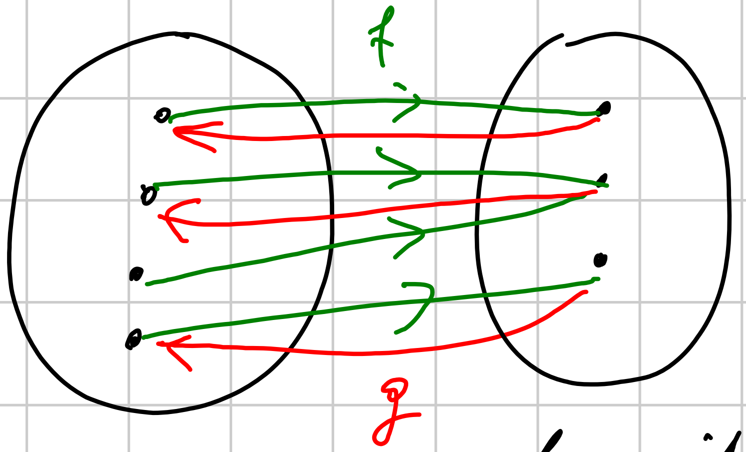
3. Se $f: X \to Y$ è iniettiva esiste $g: Y \to X$ t.c. $g \circ f = id_X$:

dato $y \in Y$ ho due casi:

- $y \in Im f \Rightarrow y = f(x)$ x unico
  chiamo tale $x = g(y)$
- $y \notin Im f \Rightarrow$ scelgo $g(y)$ a caso.

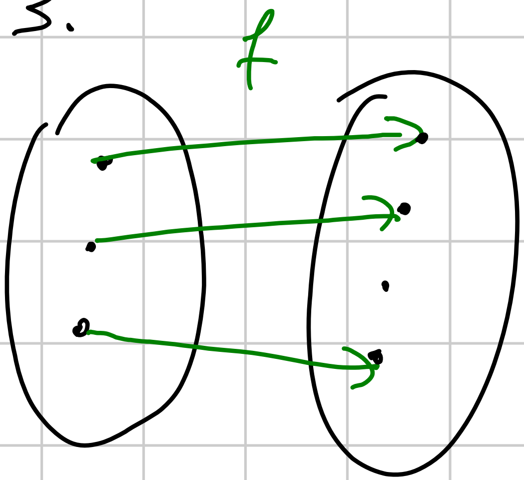
2.

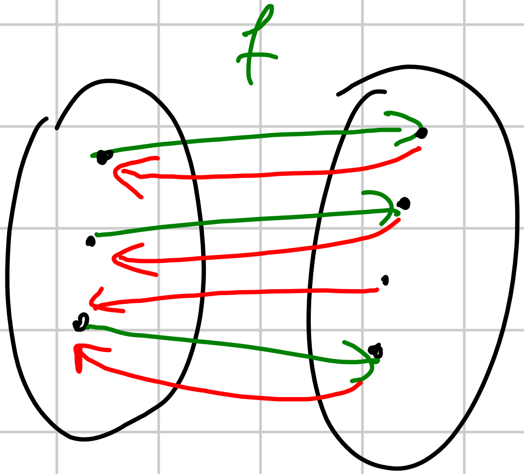




$f \circ g = id_Y$

3.





$g \circ f = id_X$

- $\underset{n}{V} \xrightarrow{f} \underset{m}{W}$ $\quad n \geq m \quad$ f suriettiva

cerchiamo $g : W \to V$ t.c. $f \circ g = id_W$:

prendo $w_1, \dots, w_m$ base di W ; so che esistono $v_1, \dots, v_m$ con $w_j = f(v_j)$. Definisco

$$g(w_j) = v_j \,.$$

$\boxed{5.4.2}$ $\underset{n}{V}$ $\underset{m}{W}$ $n \leq m$

- esistono $f: V \to W$ iniettive

  Prendo basi $v_1, \dots, v_n$ $w_1, \dots, w_m$ e pongo

$$f(v_j) = w_j \qquad j = 1, \dots, n$$

- se $f: V \to W$ è iniettiva esiste $g: W \to V$ t.c. $g \circ f = id_V$.

Scelgo una base $w_1, \dots, w_n$ di $\operatorname{Im} f$

e la completo a base $w_1, \dots, w_m$ di W.

Per $j = 1, \dots, n$ esiste unico v_j t.c.

$f(v_j) = w_j$; ora

$$g(w_j) = \begin{cases} v_j & j = 1 \dots n \\ \text{a caso} & j = n+1 \dots m \end{cases}$$

$\boxed{5.4.3}$ Trovare matrici associate —

$$f : \mathbb{R}^2 \to \mathbb{R}^3 \qquad f(x) = \begin{pmatrix} x_1 + 2x_2 \\ 3x_1 - x_2 \\ 5x_1 + 4x_2 \end{pmatrix} \qquad f = f_A , \ A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \\ 5 & 4 \end{pmatrix}$$

$$\mathcal{B} = \left(\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \end{pmatrix} \right) \qquad \mathcal{C} = \left(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right)$$

$$M = [f]_{\mathcal{B}}^{\mathcal{C}}$$

($\mathcal{B}$ base di $\mathbb{R}^2$ ✓
$\mathcal{C}$ base di $\mathbb{R}^3$ ✓)

(**ITALIANO**: "X è associato a Y"

= prima c'era Y e X si origina da lui).

$f = f_A$: f è l'applicazione associata ad A

$\left(A \in M_{m \times n} ;\ \text{le associo } f_A : \mathbb{R}^n \to \mathbb{R}^m \right)$

cerchiamo $M = [f]_{\mathcal{B}}^{\mathcal{C}}$

che è la matrice associata a f
rispetto a $\mathcal{B}$ in partenza e $\mathcal{C}$ in arrivo _

<u>Oss</u>: $[f_A]_{\mathcal{E}^{(n)}}^{\mathcal{E}^{(m)}} = A$ _

$[f]^{\mathcal{E}}_{\mathcal{B}}$ = matrice in cui la j-ma colonna contiene le coord. risp. a $\mathcal{E}$ di f(j-mo vett. di $\mathcal{B}$) _

$$\begin{pmatrix} 1 & 2 \\ 3 & -1 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ 6 \end{pmatrix} = 5 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & -1 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 13 \\ 4 \\ 35 \end{pmatrix} = 40 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - 5 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - 18 \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

$$\begin{cases} \beta - \gamma = 0 \\ \alpha + 2\gamma = 7 \\ \alpha + \beta = 6 \end{cases} \qquad \begin{array}{l} \alpha = 5 \\ \beta = 1 \\ \gamma = 1 \end{array}$$

$$M = \begin{pmatrix} 5 & 40 \\ 1 & -5 \\ 1 & -18 \end{pmatrix}$$

$$\begin{cases} \beta - \gamma = 13 \\ \alpha + 2\gamma = 4 \\ \alpha + \beta = 35 \end{cases} \qquad \begin{cases} \gamma = \beta - 13 \\ \alpha + 2\beta = 30 \\ \alpha + \beta = 35 \end{cases} \qquad \begin{cases} \beta = -5 \\ \alpha = 40 \\ \gamma = -18 \end{cases}$$

(d) $f : \mathbb{R}^2 \longrightarrow A_3$

$$f(x) = \begin{pmatrix} 0 & -x_1 & 2x_1 - x_2 \\ x_1 & 0 & x_1 - 3x_2 \\ x_2 - 2x_1 & 3x_2 - x_1 & 0 \end{pmatrix}$$

$$\mathcal{B} = \left(\begin{pmatrix} 3 \\ 7 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right) \qquad \mathcal{C} = \left(\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 3 \\ 1 & -3 & 0 \end{pmatrix} \right)$$

$$f\begin{pmatrix}3\\7\end{pmatrix} = \begin{pmatrix} 0 & -3 & -1 \\ * & 0 & -18 \\ * & * & 0 \end{pmatrix} = -17\begin{pmatrix} 0 & 1 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{pmatrix} + 7\begin{pmatrix} 0 & 2 & -1 \\ * & 0 & 0 \\ * & * & 0 \end{pmatrix} - 6\begin{pmatrix} 0 & 0 & -1 \\ * & 0 & 3 \\ * & * & 0 \end{pmatrix}$$

$$f\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix} 0 & -2 & 1 \\ * & 0 & -7 \\ * & * & 0 \end{pmatrix} = u\begin{pmatrix} 0 & 1 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{pmatrix} + v\begin{pmatrix} 0 & 2 & -1 \\ * & 0 & 0 \\ * & * & 0 \end{pmatrix} - \frac{7}{3}\begin{pmatrix} 0 & 0 & -1 \\ * & 0 & 3 \\ * & * & 0 \end{pmatrix}$$

$$\Longrightarrow [f]^{e}_{\mathcal{B}} = \begin{pmatrix} -17 & u \\ 7 & v \\ -6 & -7/3 \end{pmatrix}$$

(e) $f: \mathbb{R}^2 \longrightarrow \{y : y_1 + y_2 + y_3 = 0\}$

$$f(x) = \begin{pmatrix} 3x_1 - x_2 \\ -x_1 + 2x_2 \\ -2x_1 - x_2 \end{pmatrix}$$

$$\big((3x_1 - x_2) + (-x_1 + 2x_2) + (-2x_1 - x_2) = 0\big) \checkmark$$

$$\mathcal{B} = \left(\begin{pmatrix} 3 \\ -7 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right)$$

$$\mathcal{C} = \left(\begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ -7 \end{pmatrix} \right)$$

$$\begin{pmatrix} 3 & -1 \\ -1 & 2 \\ -2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -7 \end{pmatrix} = \begin{pmatrix} 16 \\ -17 \\ 1 \end{pmatrix} = \frac{117}{13} \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} - \frac{51}{13} \begin{pmatrix} 5 \\ 2 \\ -7 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -1 \\ -1 & 2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -7 \\ 4 \\ 3 \end{pmatrix} = u \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} + v \begin{pmatrix} 5 \\ 2 \\ -7 \end{pmatrix}$$

$$\Rightarrow [f]^{\mathcal{E}}_{\mathcal{B}} = \begin{pmatrix} 117/13 & u \\ -51/13 & v \end{pmatrix}$$

$\boxed{5.4.4}$ $\underset{n}{V} \supset \underset{k}{X}$ $\underset{m}{W} \supset \underset{h}{Y}$

$$Z = \{ f \in \mathcal{L}(V, W) : f(X) \subset Y \}$$

Provare che è sottospazio

$f_1, f_2 \in Z$, $\lambda_1, \lambda_2 \in \mathbb{R}$ $\overset{?}{\Rightarrow}$ $\lambda_1 f_1 + \lambda_2 f_2 \in Z$

cioè | Sì se

$f_1(x), f_2(x) \in Y \quad \forall x \in X$ $\Big|$ $\underbrace{(\lambda_1 f_1 + \lambda_2 f_2)(x)}_{\lambda_1 f_1(x) + \lambda_2 f_2(x)} \in Y \quad \forall x \in X$

$$\underbrace{\underset{Y}{\underset{\cup}{\lambda_1 f_1(x)}} + \underset{Y}{\underset{\cup}{\lambda_2 f_2(x)}}}_{\underset{Y}{\cap}}$$

$\underline{OK}$

$$Z = \{ f : \underset{n}{V} \longrightarrow \underset{m}{W} \; : \; f(\underset{k}{X}) \subset \underset{h}{Y} \}$$

Prendo basi $\underbrace{v_1, \dots, v_k}, v_{k+1}, \dots, v_n = \mathcal{B}$

$$\underbrace{\overbrace{\qquad\qquad\qquad\qquad}^{\dim X}}_{\dim V}$$

$$\underbrace{\underbrace{w_1, \dots, w_k}_{\dim Y}, w_{k+1}, \dots, w_m}_{\dim W} = \mathcal{C}$$

$$f \in Z \implies [f]^{\mathcal{C}}_{\mathcal{B}} = \begin{array}{cc} & \begin{array}{cc} k & m-k \end{array} \\ \begin{array}{c} k \\ m-k \end{array} & \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) \end{array} \begin{array}{l} \}\, Y \\ \\ \end{array} \Big\}\, W$$

$$\underbrace{\underbrace{\qquad\quad}_{X}\qquad\quad}_{V}$$

$$(f(X) \subset Y)$$

$$\Rightarrow \dim Z = n \cdot m - k(m-h)$$
$$= k \cdot h + m \cdot (n-k) \text{ .}$$