Alg. Lin. 11/11/15

5.1.2. (d) $f : \mathbb{R}^3 \to \mathbb{R}^4$ $f = f_A$ $A \in \mathcal{M}_{4\times 3}$

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 4 & 5 & -1 \\ 5 & 7 & 1 \end{pmatrix}$$

$\mathrm{Ker}\, f = \{ x : A \cdot x = 0 \} = \text{soluz}$ $\begin{cases} x_1 + x_2 - x_3 = 0 \\ \_\_\_\_ \\ \_\_\_\_ \\ \_\_\_\_ \end{cases}$

$$\begin{cases} x_3 = x_1 + x_2 \\ 3x_1 + 4x_2 = 0 \\ \cancel{3x_1 + 4x_2 = 0} \\ \cancel{6x_1 + 8x_2 = 0} \end{cases} \qquad \operatorname{Ker}(f) = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$$

$$\operatorname{Im} f = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 2 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 5 \\ 7 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right)$$

✓ ✓ ✗ $-4 \cdot \mathrm{I} + 3 \cdot \mathrm{II}$

$$\dim \mathbb{R}^3 = \dim \operatorname{Ker} f + \dim \operatorname{Im} f$$

$$3 = 1 + 2 .$$

(h) $f : M_{n\times n} \to M_{n\times n}$ $\quad f(A) = A + {}^tA$

$$\mathrm{Ker}\, f = \{A : A + {}^tA = 0\} = \mathcal{A}_n(\mathbb{R})$$

$$\dim = \frac{n(n-1)}{2}$$

$$\mathrm{Im}\, f = \{A + {}^tA : A \in M_{n\times n}\} \subset \mathcal{S}_n(\mathbb{R})$$

$${}^t(A + {}^tA) = {}^tA + A = A + {}^tA$$

$$\Rightarrow A + {}^tA \in \mathcal{S}_n(\mathbb{R})$$

B simmetrica, ${}^tB = B$, $\quad B = A + {}^tA$ con $A = \frac{1}{2}B$.

$$\begin{pmatrix} 1 & 3 \\ 3 & 7 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 3 \\ 3 & 7 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 3 \\ 3 & 7 \end{pmatrix}$$

$$\Rightarrow \operatorname{Im} f = S_n(\mathbb{R}) \qquad \dim = \frac{n(n+1)}{2}$$

$$\dim M_{n\times n} = \dim \operatorname{Ker} f + \dim \operatorname{Im} f$$

$$n^2 = \frac{n(n-1)}{2} + \frac{n(n+1)}{2} \qquad \checkmark$$

(m) $f : \mathbb{R}_{\leq 3}[t] \longrightarrow \mathbb{R}^2$

$$f(p(t)) = \begin{pmatrix} p(2) - p'(-1) \\ p(1) + 2p''(2) \end{pmatrix}$$

$$f(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = \begin{pmatrix} a_0 + 2a_1 + 4a_2 + 8a_3 - (a_1 - 2a_2 + a_3) \\ a_0 + a_1 + a_2 + a_3 + 2(2a_2 + 12a_3) \end{pmatrix}$$

$$= \begin{pmatrix} a_0 + a_1 + 6a_2 + 7a_3 \\ a_0 + a_1 + 5a_2 + 25a_3 \end{pmatrix}$$

$$\operatorname{Ker} f = \left\{ a_0 + a_1 t + a_2 t^2 + a_3 t^3 : \begin{array}{l} a_0 + a_1 + 6a_2 + 7a_3 = 0 \\ a_0 + a_1 + 5a_2 + 25a_3 = 0 \end{array} \right\}$$

$$\dim \operatorname{Ker} f = 2$$

$$\Rightarrow \dim \operatorname{Im} f = \dim \mathbb{R}_{\leq 3}[t] - \dim \operatorname{Ker} f$$

$$= 4 - 2 = 2$$

$\Rightarrow$ f suriettiva, infatti

$$\operatorname{Im} f = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 6 \\ 5 \end{pmatrix}, \begin{pmatrix} 7 \\ 25 \end{pmatrix} \right)$$

$\boxed{5.1.3}$ $f : \mathbb{R}^5 \to \mathbb{R}^9$

$*$ $f(e_1 + 2e_4 - 7e_5) = f(3e_2 + e_3 + e_4) = 8e_1 - e_9$

$\dim \operatorname{Im} f = ?$

Senza $*$: $0 \leq \dim \operatorname{Im} f \leq 5$

$\dim \operatorname{Ker} f = 5$ $\qquad$ $\dim \operatorname{Ker} f = 0$

Con $*$: $1 \leq \dim \operatorname{Im} f \leq 4$

$8e_1 - e_9 \in \operatorname{Im} f$ $\qquad$ $(e_1 + e_4 - 7e_5) - (3e_2 + e_3 + e_4) \in \operatorname{Ker} f$

$\boxed{5.1.4}$

$f : \mathbb{R}^5 \to \mathbb{R}^8$

$\dim \operatorname{Ker} f = 2$

$Z \subset \mathbb{R}^8$, $Z \cap \operatorname{Im} f = \{0\}$

$\dim Z = \underset{\sim}{?}$

$0 \leq \ldots \leq 5$

$\boxed{5.1.5}$

$f : \mathbb{R}^{11} \to \mathbb{R}^7$ lin

$\dim \operatorname{Im} f = 5$

$X \subset \mathbb{R}^{11}$ $X + \operatorname{Ker} f = \mathbb{R}^{11}$

$\dim X = ?$

$5 \leq \ldots < 11$

$\boxed{5.2.1}$ $f : \mathbb{R}^7 \to \mathbb{R}^6$

$X \subset \mathbb{R}^7$, $Y \subset \mathbb{R}^6$, $\dim Y = 2$

$\mathbb{R}^7 = X \oplus \operatorname{Ker} f$ $\qquad$ $\mathbb{R}^6 = Y \oplus \operatorname{Im} f$

$7 = 4 + 3$ $\qquad$ $6 = 2 + 4$

$\dim X = ?$

4

$\boxed{5.2.2}$

$X, Y \subset \mathbb{R}^9$

$Z \subset \mathbb{R}^8$

$f : X \to \mathbb{R}^8$ iniettiva $\qquad \dim \operatorname{Ker} f = 0$

$\mathbb{R}^9 = X \oplus Y$

$\mathbb{R}^8 = Z \oplus \operatorname{Im} f$

$8 = 3 + 5$

$\dim Z = 3$

$\dim X = 5$

$\Downarrow$

4

$\dim Y ?$

$\boxed{5.23}$

$X, Y \subset \mathbb{R}^7$

$Z \subset \mathbb{R}^9$

$f : \mathbb{R}^9 \to X$ surjective

$\mathbb{R}^7 = X \oplus Y$

$7 = 5+2$

$\mathbb{R}^9 = Z \oplus \ker f$

$\dim Y = 2$

$\dim Z = ?$

$\dim \ker f = 4$

$\Downarrow$

$\dim Z = 5$

Provare che $V = W \oplus Z$ e trovare p, q

$\boxed{5.2.4}$ $\mathbb{R}^2$ $\quad \text{Span}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{Span}\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ $\quad \cap = \{0\}$

$$2 = 1 + 1$$

(Fatto generale: se $\dim W + \dim Z = \dim V$
e $W \cap Z = \{0\}$ allora $W + Z = V$.)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = p\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + q\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\in W$ $\qquad \in Z$

cioè $t \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\qquad$ cioè $s \cdot \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

$$\begin{cases} 2t - 3s = x_1 \\ t + 2s = x_2 \end{cases} \qquad t = \frac{1}{7}(2x_1 + 3x_2), \quad s = \frac{1}{7}(-x_1 + 2x_2)$$

$$p(x) = \frac{1}{7} \cdot (2x_1 + 3x_2) \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 4x_1 + 6x_2 \\ 2x_1 + 3x_2 \end{pmatrix}$$

$$= \underbrace{\frac{1}{7} \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}}_{P} \cdot x$$

$$q(x) = \frac{1}{7}(-x_1 + 2x_2) \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \underbrace{\frac{1}{7} \begin{pmatrix} 3 & -6 \\ -2 & 4 \end{pmatrix}}_{Q} \cdot x$$

Verificare $p \circ p = p$, $q \circ q = q$, $p \circ q = q \circ p = 0$, $p + q = \mathbb{1}$
equivale a $\underline{P} \cdot \underline{P} = \underline{P}$, $Q \cdot Q = Q$, $\underline{P} \cdot Q = Q \cdot \underline{P} = 0$, $P + Q = \underline{I}_2$:

$$P \cdot P = \frac{1}{49} \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix} = \frac{1}{49} \begin{pmatrix} 28 & 42 \\ 14 & 21 \end{pmatrix}$$

$$= \frac{1}{7} \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix} = \underline{P}$$

$$P \cdot Q = \frac{1}{49} \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & -6 \\ -2 & 4 \end{pmatrix} = \frac{1}{49} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

$$P + Q = \frac{1}{7} \left(\cdot \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} 3 & -6 \\ -2 & 4 \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underline{I}_2 .$$

$\boxed{5.2.5}$ $V = \mathbb{R}^2$ $W = \text{Span}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ $Z: 5x_1 + 2x_2 = 0$

2 1 1

$\cap = \{0\}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = p\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + q\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$p\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in W$, cioè è $t \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$q\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in Z$, cioè soddisfa l'equaz. di Z

OVVERO: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - t \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ soddisfa l'equaz. di Z cioè $\quad 5\cdot(x_1-3t)+2(x_2+t)=0$

$$\Longrightarrow \; t=\frac{1}{13}(5x_1+2x_2)$$

$$p(x)=\frac{1}{13}(5x_1+2x_2)\cdot\begin{pmatrix}3\\-1\end{pmatrix}=\underbrace{\frac{1}{13}\begin{pmatrix}15 & 6\\-5 & -2\end{pmatrix}}_{P}\cdot x$$

$$q(x)=x-p(x)=\underset{\displaystyle I_2-P=\frac{1}{13}\begin{pmatrix}-2 & -6\\5 & 15\end{pmatrix}}{\underset{\shortparallel}{Q}}\cdot x$$

verifiche ...

$\boxed{5.7.7}$ $V = \mathbb{R}^3$ $W = \left(\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} \right)$ $Z = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

3 2 1

$\cap = \{0\}$

$$x = \underset{\substack{\cap \\ W}}{p(x)} + \underset{\substack{\cap \\ Z}}{q(x)}$$

cioè $t_1 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$ cioè $s \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$$\begin{cases} 2t_1 - t_2 = x_1 \checkmark \\ t_1 + s = x_2 \checkmark \\ 3t_2 - s = x_3 \checkmark \end{cases} \qquad \begin{cases} s = 3t_2 - x_3 \\ t_1 = \frac{1}{2}(x_1 + t_2) \\ \frac{1}{2}(x_1 + t_2) + 3t_2 - x_3 = x_2 \end{cases}$$

$$\begin{cases} x_1 + t_2 + 6t_2 - 2x_3 = 2x_2 \\ \text{—} \\ \text{—} \end{cases} \qquad \begin{cases} t_2 = \frac{1}{7}(-x_1 + 2x_2 + 2x_3) \\ t_1 = \frac{1}{14}(6x_1 + 2x_2 + 2x_3) \\ s = \frac{1}{7}(-3x_1 + 6x_2 - x_3) \end{cases}$$

$$p(x) = \frac{1}{7}(3x_1 + x_2 + x_3)\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{7}(-x_1 + 2x_2 + 2x_3)\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$$

$$q(x) = \frac{1}{7}(-3x_1 + 6x_2 - x_3) \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$\leadsto$ P, Q + verifiche

$\boxed{5.2.8}$ $V = \mathbb{R}^3$ $\quad W: 5x_1 - 3x_2 + 2x_3 = 0 \quad Z = \text{Span}\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$

3 $\qquad$ 2 $\qquad$ 1

$\cap = \{0\}$

Per proiez: $\quad W = \text{Span}\begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$ + come prima.

Invece:

$$x = p(x) + q(x)$$

$p(x) \in W$ cioè soddisfa. eq. di W

$q(x) \in Z$ cioè è $s \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$

ovvero: $x - s \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ soddisfa eq. di W:

$$5(x_1 - 3s) - 3(x_2 - s) + 2(x_3 + 2s) = 0$$

$$s = \frac{5x_1 - 3x_2 + 2x_3}{8}$$

$$q(x) = \frac{5x_1 - 3x_2 + 2x_3}{8} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \underbrace{\frac{1}{8}\begin{pmatrix} 15 & -9 & 6 \\ 5 & -3 & 2 \\ -10 & 6 & -4 \end{pmatrix}}_{Q} \cdot x$$

$$p(x) = x - q(x) = P \cdot x$$

$$P = I_3 - Q = \frac{1}{8}\begin{pmatrix} -7 & 9 & -6 \\ -5 & 11 & -2 \\ -10 & -6 & 12 \end{pmatrix}$$

$$\underline{P}\cdot\underline{P} = \frac{1}{8\cdot 8}\begin{pmatrix} -7 & 9 & -6 \\ -5 & 11 & -2 \\ 10 & -6 & 12 \end{pmatrix}\begin{pmatrix} -7 & 9 & -6 \\ -5 & 11 & -2 \\ 10 & -6 & 12 \end{pmatrix}$$

$$= \frac{1}{8\cdot 8}\begin{pmatrix} -56 & 72 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} = \frac{1}{8}\begin{pmatrix} -7 & 9 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

$$= \underline{P}$$

$$\underline{P}\cdot Q = \frac{1}{8\cdot 8}\begin{pmatrix} -7 & 9 & -6 \\ -5 & 11 & -2 \\ \cdot & \cdot & \cdot \end{pmatrix}\begin{pmatrix} 15 & \cdot & \cdot \\ 5 & \cdot & \cdot \\ -10 & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} 0 & \cdot & \cdot \\ 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

$-105+45+60$

$-75+55+20$

$\boxed{5.2.11}$ $V = \{x \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0\}$ 3

$W = \begin{pmatrix} 2 \\ 3 \\ -9 \\ 4 \end{pmatrix}$ $Z : 3x_1 + 2x_2 + 5x_3 + 7x_4 = 0$

1 2 $\cap = \{0\} \ldots$

$x = p(x) + q(x)$ — soddisfa le eq. di Z

$p(x) = t \cdot \begin{pmatrix} 2 \\ 3 \\ -9 \\ 4 \end{pmatrix}$ (dove x soddisfa l'eq. di V)

$$\begin{cases} (x_1 + 2t) + (x_2 + 3t) + (x_3 - 9t) + (x_4 + 4t) = 0 \\ 3(x_1 + 2t) + 2(x_2 + 3t) + 5(x_3 - 9t) + 7(x_4 + 4t) = 0 \end{cases}$$ ✓

$$t = -\frac{3x_1 + 2x_2 + 5x_3 + 7x_4}{5}$$

$$6 + 6 - 45 + 28$$

$$q(x) = -\frac{3x_1 + 2x_2 + 5x_3 + 7x_4}{5}\begin{pmatrix} 2 \\ 3 \\ -9 \\ 4 \end{pmatrix}$$

$$p(x) = x - q(x) = \frac{1}{5}\begin{pmatrix} 5x_1 + 6x_1 + 4x_2 + 10x_3 + 14x_4 \\ 5x_2 + 9x_1 + 6x_2 + 15x_3 + 21x_4 \\ 5x_3 \\ 5x_4 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 11x_1 + 4x_2 + 10x_3 + 14x_4 \\ - \;\; - \;\; - \\ - \;\; - \;\; . \\ - \;\; - \end{pmatrix}$$

$= \underline{P} \cdot x$

posso scriverlo come

$\underline{P} \in M_{4\times 4}$

Ma $\underline{\underline{\text{NON}}}$ è vero che $\underline{P}$ è la matrice di p:

$$p : V \to V \qquad (V \subset \mathbb{R}^4,\ \dim V = 3)$$

$$\underline{P} : \mathbb{R}^4 \to \mathbb{R}^4$$

(e infatti $\underline{\text{non}}$ è vero che $\underline{P} \cdot \underline{P} = \underline{P}$).

Invece: scelgo $\mathcal{B} = \left(\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right)$

e scrivo $M = [p]^{\mathcal{B}}_{\mathcal{B}}$ allora ho

$M \cdot M = M$.