Alg Lin 4/11/15

V sp. vett. di dim. n

- $v_1, \dots, v_k$ lin. indip. $\longrightarrow$ posso completare a base
- $v_1, \dots, v_h$ generatori $\longrightarrow$ posso estrarre base

$\boxed{4.4.2}$ (a) $\mathbb{R}^3$ $\begin{pmatrix} 4 \\ 6 \\ -10 \end{pmatrix}, \begin{pmatrix} -6 \\ -9 \\ 15 \end{pmatrix}, \begin{pmatrix} 5 \\ -7 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 13 \\ -12 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} \pi \\ \sqrt{17} \\ e \end{pmatrix}$

✓ ✗ ✓ ✗ ✓ ✗

è I - II

$$\begin{cases} 4\alpha + 5\beta = 3 \\ 6\alpha - 7\beta = 1 \\ -10\alpha + 2\beta = -2 \end{cases} \qquad \begin{cases} \alpha = 26/58 = 13/29 \\ \beta = 7/29 \\ \dfrac{-130+14}{29} = -2 \end{cases}$$

NO

(b) $V = \{x \in \mathbb{R}^4 : x_1 + x_2 = x_3 + x_4\}$ $\qquad$ $\dim = 3$

$$\begin{pmatrix} 3 \\ -7 \\ 1 \\ -5 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \\ 9 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 32 \\ 25 \\ 7 \end{pmatrix} \begin{pmatrix} 6 \\ -1 \\ 1 \\ 4 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \\ -5 \\ 20 \end{pmatrix}$$

✓ ✓ ✗ ✓ ✗

$\bar{e}$ $- 3\mathrm{I} + 2\mathrm{II}$

$$\begin{cases} 3\alpha + 2\beta = 6 \\ -7\alpha + 6\beta = -1 \\ \alpha + 9\beta = 1 \end{cases} \qquad \begin{cases} \alpha = 52/25 \\ \beta = -9/25 \\ -7 \cdot 52 + 6 \cdot (-9) = -1 \cdot 25 \end{cases}$$

NO

<u>Oss</u>. Se ho $v_1, \dots, v_h \in V$ e applico l'estrazione senza sapere se generano V ottengo una base $v_{i_1}, \dots, v_{i_p}$ di $\mathrm{Span}(v_1, \dots, v_h)$; dunque trovo che generano V se e solo se $p = \dim V$.

$\boxed{4.4.1}$ Completamento

Osservo: se ho $v_1, \dots, v_k$ lin. indip. e ho $w_1, \dots, w_n$ una base di V allora

$v_1, \dots, v_k, w_1, \dots, w_n$ generano V;

se applico estrazione a $v_1, \dots, v_k, w_1, \dots, w_n$ ottengo

$$v_1, \dots, v_k, w_{i_1}, \dots, w_{i_{n-k}} \quad \text{base}$$

$\Longrightarrow$ è completamento di V.

(a) $\mathbb{R}^4$ $\begin{pmatrix} 4 \\ 5 \\ -6 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \\ 1 \\ 7 \end{pmatrix}, \underset{\text{OK}}{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}, \underset{\text{OK}}{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}$

$$\begin{cases} 4\alpha - 3\beta + \gamma = 0 \\ 5\alpha + 4\beta = 1 \\ -6\alpha + \beta = 0 \\ \alpha + 7\beta = 0 \end{cases} \Longrightarrow \alpha = \beta = 0 \qquad \underline{\text{No}}$$

(b) $\begin{pmatrix}3\\-1\\2\\5\end{pmatrix}\begin{pmatrix}4\\2\\-1\\3\end{pmatrix}\begin{pmatrix}2\\-4\\5\\7\end{pmatrix}\begin{pmatrix}-\sqrt{131}\\\bar{\pi}\\e\\-\log 7\end{pmatrix}$ oppure $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

d 99.9% OK

ok ?

$$\begin{cases}3\alpha+4\beta+2\gamma=1\\-\alpha+2\beta-4\gamma=0\ \checkmark\\2\alpha-\beta+5\gamma=0\ \checkmark\\5\alpha+3\beta+7\gamma=0\ \checkmark\end{cases}\quad\begin{cases}\alpha=2\beta-4\gamma\\3\beta-3\gamma=0\\13\beta-13\gamma=0\ \checkmark\end{cases}\quad\begin{cases}\gamma=\beta\\\alpha=-2\beta\\0=1\\\text{no solut}\end{cases}$$

$\Longrightarrow$ OK

(c) $V = \{ x \in \mathbb{R}^4 : 4x_1 + 3x_2 + 5x_3 - 2x_4 = 0 \}$ $\dim = 3$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 6 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 0 \\ 3 \end{pmatrix} \cancel{\begin{pmatrix} 0 \\ 0 \\ 2 \\ 5 \end{pmatrix}}$$

(f) $V = \left\{ x \in \mathbb{R}^4 : \begin{array}{l} x_1 + x_2 + x_3 - x_4 = 0 \\ 2x_1 + 5x_2 - x_3 + x_4 = 0 \end{array} \right\}$ $\dim = 2$

$$\begin{pmatrix} 4 \\ -2 \\ 1 \\ 3 \end{pmatrix}$$

devo aggiungere un solo vettore.
Se trovo vett. di V con $x_2 = 0$
sicuramente va bene.

$$\begin{cases} x_1 + x_3 - x_4 = 0 \\ 2x_1 - x_3 + x_4 = 0 \end{cases} \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \qquad \underline{\text{OK}}$$

(g) $\mathcal{S}_2(\mathbb{R}) = \{ A \in \mathcal{M}_{2\times 2}(\mathbb{R}) : {}^tA = A \}$

$\dim = 3$

$$\begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix} \quad \begin{pmatrix} -1 & 1 \\ 1 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$\underline{\text{OK}}$

Grassmann $\quad V \,,\quad W, Z \subset V$ stsp

$$\dim V + \dim W = \dim(V+W) + \dim(V \cap W)$$

note

si ricava dalle altre tre

facile

(a) $\mathbb{R}^4$ $\quad W = \operatorname{Span}\left(\begin{pmatrix} -1 \\ 4 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -1 \\ 3 \end{pmatrix} \right)$

$Z = \operatorname{Span}\left(\begin{pmatrix} 3 \\ 2 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} -9 \\ 3 \\ 0 \\ 8 \end{pmatrix} \right)$

Possibilità:

$2+2 = 4+0$ se $W \cap Z = \{0\}$

$3+1$ se $W \cap Z$ è retta

$2+2$ se $W = Z$.

Cerchiamo $W \cap Z$:

$x \in \mathbb{R}^4$ è in $W \cap Z$ se è sia del tipo

$$\alpha\begin{pmatrix}-1\\4\\2\\1\end{pmatrix}+\beta\begin{pmatrix}2\\1\\-1\\3\end{pmatrix}$$ sia del tipo $$\gamma\begin{pmatrix}3\\2\\1\\-1\end{pmatrix}+\delta\begin{pmatrix}-9\\3\\0\\8\end{pmatrix}.$$

Cerco le soluz. del sistema

$$\begin{cases}-\alpha+2\beta = 3\gamma-9\delta \checkmark\\ 4\alpha+\beta = 2\gamma+3\delta\\ 2\alpha-\beta = \gamma \checkmark\\ \alpha+3\beta = -\gamma+8\delta \checkmark\end{cases}$$

$$\begin{cases}\gamma = 2\alpha-\beta\\ \delta = \dfrac{3\alpha+2\beta}{8}\\ -8\alpha+16\beta = 48\alpha-24\beta\\ \qquad\qquad\qquad -27\alpha-18\beta\end{cases}$$

$$\cancel{32}\alpha+8\beta = \cancel{32}\alpha-16\beta+9\alpha+6\beta$$

$$\begin{cases} 29\alpha - 58\beta = 0 \\ 9\alpha - 18\beta = 0 \\ \gamma = 2\alpha - \beta \\ \delta = (3\alpha + 2\beta)/8 \end{cases} \qquad \begin{cases} \alpha = 2\beta \\ \gamma = 3\beta \\ \delta = \beta \end{cases} \quad \Rightarrow \dim W \cap Z = 1$$

$$W \cap Z = \operatorname{Span}\left(\underbrace{2 \cdot \begin{pmatrix} -1 \\ 4 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ -1 \\ 3 \end{pmatrix}}_{\shortparallel} \right)$$

$$3 \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -9 \\ 3 \\ 0 \\ 8 \end{pmatrix}$$

Oss. generale: se $w_1, \dots, w_k$ sono base di W
$z_1, \dots, z_h$ sono base di Z $\Longrightarrow$

$$W + Z = \mathrm{Span}\Big(\underbrace{w_1, \dots, w_k, z_1, \dots, z_h}_{\text{possono non essere lin. indip.}}\Big)$$

(lo sono solo se $W \cap Z = \{0\}$)

Per noi: $$W + Z = \mathrm{Span}\left(\begin{pmatrix}-1\\4\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\-1\\3\end{pmatrix}, \begin{pmatrix}3\\2\\1\\-1\end{pmatrix}, \begin{pmatrix}-9\\3\\0\\8\end{pmatrix}\right)$$

è $9 \cdot I + II - 3 \cdot III$

$$\Longrightarrow \quad 2+2 = 3+1$$

$\boxed{4.5.2}$

$$W = \mathrm{Span}\begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix}\begin{pmatrix} 3 \\ -4 \\ 5 \\ 1 \end{pmatrix}\begin{pmatrix} 7 \\ 2 \\ -1 \\ -1 \end{pmatrix} \qquad \dim W = 3$$

$$Z: \begin{cases} 2x_1 + x_2 - 3x_3 + 5x_4 = 0 \\ 3x_1 - x_2 + 2x_3 + x_4 = 0 \end{cases} \qquad \dim Z = 2$$

$$\begin{cases} \alpha + 3\beta = 7 \checkmark \\ 2\alpha - 4\beta = 2 \\ -\alpha + 5\beta = -1 \checkmark \\ 3\alpha + \beta = -1 \end{cases} \quad \begin{cases} \alpha = 7 - 3\beta \\ 7 - 3\beta = 5\beta + 1 \\ = \end{cases} \quad \begin{cases} \beta = 3 \\ \alpha = -2 \\ -4 - 12 = 2 \end{cases} \quad \underline{\text{No}}$$

	W	Z	=	W+Z		W∩Z	
<u>Possibilità</u>:	3	+ 2	=	5	+	0	NO
				4	+	1	intersez = retta somme = tutto
				3	+	2	se $Z \subset W$
				2	+	3	NO
				1	+	4	N
				0	+	5	NO

Cerco $W \cap Z$: cerco i vettori che sono

sia del tipo $\alpha \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ -4 \\ 5 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 7 \\ 2 \\ -1 \\ -1 \end{pmatrix}$

sia soluz di $\begin{cases} 2x_1 + x_2 - 3x_3 + 5x_4 = 0 \\ 3x_1 - x_2 + 2x_3 + x_4 = 0 \end{cases}$.

Cerco le soluz di:

$$\begin{cases} 2(\alpha + 3\beta + 7\gamma) + (2\alpha - 4\beta + 2\gamma) - 3(-\alpha + 5\beta - \gamma) \\ \qquad + 5(3\alpha + \beta - \gamma) = 0 \\ 3(\alpha + 3\beta + 7\gamma) - (2\alpha - 4\beta + 2\gamma) + 2(-\alpha + 5\beta - \gamma) \\ \qquad + (3\alpha + \beta - \gamma) = 0 \end{cases}$$

$$\begin{cases} \overset{11}{\cancel{22}}\alpha - \overset{-4}{\cancel{8}}\beta + \overset{7}{\cancel{14}}\gamma = 0 \\ \underset{1}{\cancel{2}}\alpha + \underset{12}{\cancel{24}}\beta - \underset{-8}{\cancel{16}}\gamma = 0 \end{cases}$$

$$\Longrightarrow \dim W \cap Z = 1 \Longrightarrow 3+2=4+1$$

$\boxed{5.1.1}$ Composiz. di lineari è lineare:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & W \arrow[r, "g"] & Z \end{tikzcd}$$

$$(g\circ f)(\lambda_1 v_1+\lambda_2 v_2) = g(f(\lambda_1 v_1+\lambda_2 v_2)) = g(\lambda_1 f(v_1)+\lambda_2 f(v_2))$$

$$= \lambda_1 g(f(v_1)) + \lambda_2 g(f(v_2))$$

$$= \lambda_1 \cdot (g\circ f)(v_1) + \lambda_2 \cdot (g\circ f)(v_2)$$

$\boxed{5.1.2}$ $f : \mathbb{R}^3 \to \mathbb{R}^2$, $f(x) = \begin{pmatrix} 3x_1 - 4x_2 + 5x_3 \\ -2x_1 + 5x_2 + 7x_3 \end{pmatrix}$

lineare : $f = f_A$ con $A = \begin{pmatrix} 3 & -4 & 5 \\ -2 & 5 & 7 \end{pmatrix}$ –

nucleo : $\operatorname{Ker} f = \{x \in \mathbb{R}^3 : f(x) = 0\}$

$$= \left\{ x : \begin{cases} 3x_1 - 4x_2 + 5x_3 = 0 \\ -2x_1 + 5x_2 + 7x_3 = 0 \end{cases} \right\}$$

$$= \left\{ x : \begin{array}{l} x_1 = -\frac{53}{7} x_3 \\ x_2 = -\frac{31}{7} x_3 \end{array} \right\} = \mathrm{Span} \begin{pmatrix} 53 \\ 31 \\ -7 \end{pmatrix}$$

$\dim \mathrm{Ker} f = 1$

<u>Oss</u>: se $f: V \to W$ è lineare e $v_1, \dots, v_m$ è base di V allora $f(v_1), \dots, f(v_m)$ sono generatori di $\mathrm{Im} f$

non sempre base
(solo se $\mathrm{Ker}\, f = 0$)

Per $f = f_A : \mathbb{R}^n \to \mathbb{R}^m$ sono generatori di $\mathrm{Im} f$

$$f(e_1), \dots, f(e_n)$$

$$\parallel$$

$$A \cdot e_1 = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}$$

$f(e_j) =$ j-esima colonna di A

$\Longrightarrow$ $\mathrm{Im}(f_A)$ è generata dalle colonne di A.

Per noi: $\mathrm{Im}(f)$ è generata da

$$\begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad \begin{pmatrix} -4 \\ 5 \end{pmatrix} \quad \begin{pmatrix} 5 \\ 7 \end{pmatrix} \qquad \dim(\mathrm{Im} f) = 2$$

$$\Longrightarrow 3 = 2 + 1$$