Alg. Lin 2/12/15

$\boxed{6.2.17}$

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_m^2 \\ \vdots & & & \\ \lambda_1^{m-1} & \lambda_2^{m-1} & \cdots & \lambda_m^{m-1} \end{pmatrix} = \prod_{1 \le i < j \le m} (\lambda_j - \lambda_i)$$

$\prod$ = produttoria

Es: $n=2$ $\quad \det\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} = \lambda_2 - \lambda_1$ ✓

$n=3$ $\quad \det\begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix} = (\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)$

$$= (\lambda_2-\lambda_1)(\lambda_3^2-\lambda_3\lambda_2 - \lambda_1\lambda_3+\lambda_1\lambda_2)$$

$$= \lambda_2\lambda_3^2 - \lambda_3\lambda_2^2 - \cancel{\lambda_1\lambda_2\lambda_3} + \lambda_1\lambda_2^2 - \lambda_1\lambda_3^2 + \cancel{\lambda_1\lambda_2\lambda_3} + \lambda_3\lambda_1^2 - \lambda_2\lambda_1^2$$

$$= \lambda_2\lambda_3^2 + \lambda_3\lambda_1^2 + \lambda_1\lambda_2^2 - \lambda_2\lambda_1^2 - \lambda_3\lambda_2^2 - \lambda_1\lambda_3^2$$

In generale: per induzione. Base ok. Ind:

$$\det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & & & \vdots \\ \lambda_1^{n-2} & \lambda_2^{n-2} & \cdots & \lambda_n^{n-2} \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & \lambda_2-\lambda_1 & \cdots & \lambda_n-\lambda_1 \\ \vdots & \vdots & & \vdots \\ 0 & \lambda_2^{n-3}(\lambda_2-\lambda_1) & \cdots & \lambda_n^{n-3}(\lambda_n-\lambda_1) \\ 0 & \lambda_2^{n-2}(\lambda_2-\lambda_1) & \cdots & \lambda_n^{n-2}(\lambda_n-\lambda_1) \end{pmatrix}$$

Sottraggo all'ultima riga $\lambda_1 \cdot$ penultima

Sottraggo alla penultima riga $\lambda_1 \cdot$ terzultima ...

... seconda $- \lambda_1 \cdot$ prima

$$= \prod_{j=2}^{n} (\lambda_j - \lambda_1) \det \begin{pmatrix} 1 & \cdots & 1 \\ \lambda_2 & \cdots & \lambda_n \\ \vdots & & \\ \lambda_2^{n-2} & \cdots & \lambda_n^{n-2} \end{pmatrix}$$

$$= \prod_{j=2}^{n} (\lambda_j - \lambda_1) \cdot \prod_{2 \leq i < j \leq n} (\lambda_j - \lambda_i)$$

$$= \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i) .$$

Metodo alternativo: considero $\lambda_1, \dots, \lambda_n$ come indeterminate.

$$\det \begin{pmatrix} 1 & \dots & 1 \\ \lambda_1 & \dots & \lambda_n \\ \vdots & & \\ \lambda_1^{n-1} & \dots & \lambda_n^{n-1} \end{pmatrix} = p(\lambda_1, \dots, \lambda_n)$$

- tutti i coeff sono potenze delle variabili

$$\Longrightarrow p(\lambda_1, \dots, \lambda_n) \in \mathbb{R}[\lambda_1, \dots, \lambda_n]$$

- ogni monomio di $p(\lambda_1, \dots, \lambda_n)$ ha grado

$$0 + 1 + 2 + \dots + n-1 = \frac{n(n-1)}{2}$$

- se $\lambda_i = \lambda_j$ allora p vale 0

$\Rightarrow$ $p(\lambda_1, \dots, \lambda_n)$ è divisibile per ogni $\lambda_j - \lambda_i$

$$\Rightarrow p(\lambda_1, \dots, \lambda_n) = q(\lambda_1, \dots, \lambda_n) \cdot \underbrace{\prod_{1 \le i < j \le n} (\lambda_j - \lambda_i)}_{\binom{n}{2} \cdot 1 = \frac{n(n-1)}{2}}$$

grado $\frac{n(n-1)}{2}$

grado 0

$\Rightarrow$ costante

$$\Rightarrow p(\lambda_1, \dots, \lambda_n) = k \cdot \prod_{1 \le i < j \le n} (\lambda_j - \lambda_i)$$

Resta da vedere che $k = 1$.

Esercizio: confrontare il coeff di

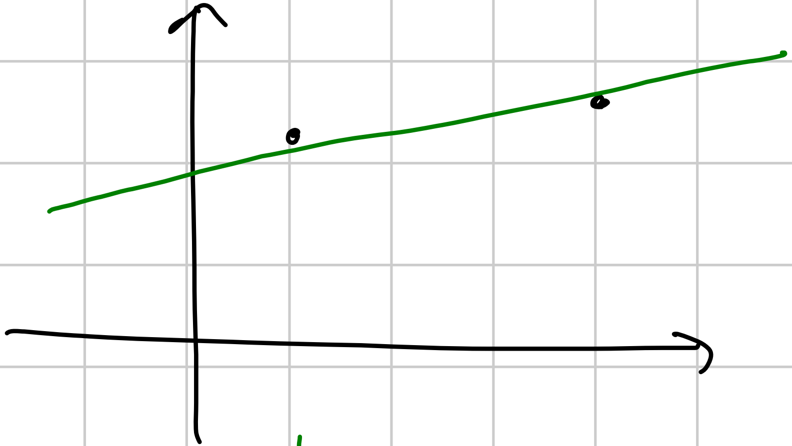
$$\lambda_n^{n-1} \cdot \lambda_{n-1}^{n-2} \cdot \ldots \lambda_3^2 \cdot \lambda_2 \cdot 1 .$$

(determinante di Vandermonde).

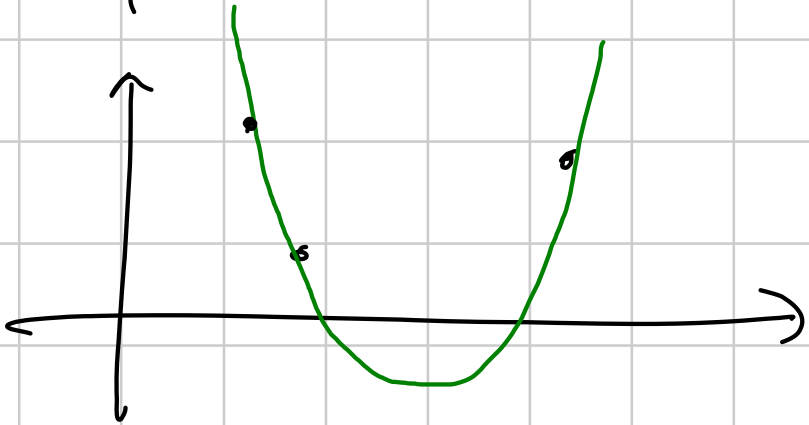
6.2.18 Interpolazione polinomiale:

dati nel piano n pti con ascisse distinte esiste uno e un solo polinomio di grado $\leq n-1$ il cui grafico li contiene.

$n = 2$



$n = 3$



Infatti: dati $(x_1, y_1), \dots, (x_n, y_n)$

cerco $a_0, a_1, \dots, a_{n-1}$ t.c.

$$\begin{cases} a_0 + a_1 x_1 + \dots + a_n x_1^{n-1} = y_1 \\ a_0 + a_1 x_2 + \dots + a_n x_2^{n-1} = y_2 \\ \vdots \qquad\qquad\qquad\qquad\qquad\quad \vdots \\ a_0 + a_1 x_n + \dots + a_n x_n^{n-1} = y_n \end{cases}$$

sist. lin. con incognite $a_0, a_1, \dots, a_{n-1}$ $n \times n$

matrice del sistema:

$$
\begin{pmatrix}
1 & x_1 & \cdots & x_1^{n-1} \\
1 & x_2 & \cdots & x_2^{n-1} \\
\vdots & & & \vdots \\
1 & x_n & \cdots & x_n^{n-1}
\end{pmatrix}
$$

$$\Rightarrow \det(\uparrow) = \prod_{1 \le i < j \le n} (x_j - x_i)$$

$$x_j \neq x_i \quad \forall i \neq j \quad \Rightarrow \det(\nwarrow) \neq 0$$

$\Rightarrow$ soluzione unica

$\boxed{6.3.1\,b}$

$$\begin{pmatrix}-2 & 3\\ 8 & -5\end{pmatrix}^{-1} = \frac{1}{-14}\begin{pmatrix}-5 & -3\\ -8 & -2\end{pmatrix}$$

$$= \frac{1}{14}\begin{pmatrix}5 & 3\\ 8 & 2\end{pmatrix}$$

$\boxed{6.3.1\,d}$

$$\begin{pmatrix}3 & 1 & -1\\ 1 & -2 & 3\\ 4 & 1 & -1\end{pmatrix}^{-1} = \frac{1}{1}\cdot\begin{pmatrix}-1 & 0 & 1\\ \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot\end{pmatrix}$$

$6 + 12 \cancel{-1} - 8 \cancel{+1} - 9$

$\boxed{6.3.1e}$

$$\begin{pmatrix} 1 & 0 & 0 & -2 \\ 2 & 1 & 0 & 1 \\ 3 & 0 & 1 & -1 \\ 0 & 2 & 1 & 1 \end{pmatrix}^{-1} = \frac{1}{-14} \begin{pmatrix} \cdot & \cdot & \cdot & -2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & -2 & \cdot & \cdot \end{pmatrix}$$

$$\det \begin{pmatrix} 1 & 0 & 0 & -2 \\ 2 & 1 & 0 & 1 \\ 3 & 0 & 1 & -1 \\ -3 & 2 & 0 & 2 \end{pmatrix}$$

$$\cancel{2} + 0 - 8 - 6 - 0 - \cancel{2}$$

$\boxed{6.3.4b}$

$$A = \begin{pmatrix} 3 & 2 & 1 & 7 \\ -5 & 5 & 4 & 3 \\ 2 & -3 & -2 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 7 \\ -3 & -1 \end{pmatrix}$$

Calcolare i det delle orlate di B in A.

$$\det \begin{pmatrix} 3 & 2 & 7 \\ -5 & 5 & 3 \\ 2 & -3 & -1 \end{pmatrix} = \dots$$

$$\det \begin{pmatrix} 2 & 1 & 7 \\ 5 & 4 & 3 \\ -3 & -2 & -1 \end{pmatrix} = \dots$$

$\boxed{6.3.4\,c}$ $\qquad A = \ldots 4 \times 4 \qquad\qquad B = \ldots 2 \times 2$

le orlate di B in A sono $2 \cdot 2 = 4$.

$\boxed{6.3.5\,d}$ Calcolare il rango usando orlati di:

$$\begin{pmatrix} -1 & 2 & -3 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 2 & -3 & 2 \\ 0 & 1 & -2 & -1 \\ 2 & -1 & 4 & 0 \end{pmatrix}$$

$\det \blacksquare = 0$

$\det \blacksquare = 11 \neq 0$

$\Rightarrow \text{rank} \geq 3$

Calcolare i det 4x4 delle due orlate di

— se entrambi nulli, rank = 3

— se no, rank = 4.

$\boxed{7.11}$ $\mathbb{R}^2$ $6x-5y=11$ $\dim = 2-1 = 1$

$$\begin{pmatrix}1\\-1\end{pmatrix} + \operatorname{Span}\begin{pmatrix}5\\6\end{pmatrix}$$

oppure ↓ oppure ↓

$$\begin{pmatrix} \sqrt{17}/6 \\ \dfrac{-11+\sqrt{17}}{5} \end{pmatrix} \qquad \begin{pmatrix} 5\pi/2 \\ 3\pi \end{pmatrix}$$

$\boxed{7.1.2}$ $E: \begin{cases} x = -7 + 2t \\ y = 5 + 6t \end{cases}$ significo

$$E = \left\{ \begin{pmatrix} -7+2t \\ 5+6t \end{pmatrix} : t \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} -7 \\ 5 \end{pmatrix} + t \begin{pmatrix} 2 \\ 6 \end{pmatrix} : t \in \mathbb{R} \right\}$$

$$= \begin{pmatrix} -7 \\ 5 \end{pmatrix} + \operatorname{Span} \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} -7 \\ 5 \end{pmatrix} + \operatorname{Span} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$\dim = 1$ Eq. cart:

$3x - y = 0$ (giacitura)

$E: \quad 3x - y = 3 \cdot (-7) \cancel{-} 5$

$3x - y = -26$

$\boxed{7.1.3}$

$$\begin{cases} 2x - 3y + 7z = 6 \\ -5x + y + 6z = 2 \end{cases}$$

$\dim = 3 - 2 = 1$

$\uparrow$ sì perché rank $\begin{pmatrix} 2 & -3 & 7 \\ -5 & 1 & 6 \end{pmatrix} = 2$

Osservo in generale: se ho E definito da k equazioni in $\mathbb{R}^n$ dim attesa $n-k$: è vero solo se $\operatorname{rank}(A) = k$, A = matrice parte omogenea.

Infatti:

Se $\operatorname{rank}(A) = k$, $\operatorname{rank}(A \,\vdots\, b) = k$

con k righe non può essere $> k$

$\Rightarrow$ sistema ha solut

$$x_0 + \underbrace{\operatorname{Ker} A}_{\dim = n-k}$$

Se $\mathrm{rank}(A) < k$

$\mathrm{rank}(A \mid b)$

$\to \ > \mathrm{rank}(A) \implies E = \emptyset$

$\to \ = \mathrm{rank}(A) \implies E = x_0 + \mathrm{Ker}\, A$

$\dim > n - k$

$$\begin{cases} 2x - 3y + 7z = 6 \\ -5x + y + 6z = 2 \end{cases}$$

$$E = x_0 + \mathrm{Span}(v)$$

$$v = -\begin{pmatrix} (-3)\cdot 6 - 1\cdot 7 \\ -\left(2\cdot 6 - (-5)\cdot 7\right) \\ 2\cdot 1 - (-5)\cdot(-3) \end{pmatrix} = \begin{pmatrix} +25 \\ +47 \\ +13 \end{pmatrix} \text{ oppure } \begin{pmatrix} 5\sqrt{3} \\ \frac{47}{5}\sqrt{3} \\ \frac{13}{5}\sqrt{3} \end{pmatrix}$$

$$x_0 = \begin{pmatrix} -12/13 \\ -34/13 \\ 0 \end{pmatrix} \quad \text{oppure}$$

$$\begin{cases} y = 5x - 6z + 2 \\ -13x + 25z = 12 \end{cases}$$

$$\begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

$\boxed{7.1.4}$ $E = \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix} + \operatorname{Span}\begin{pmatrix} -2 \\ 7 \\ 5 \end{pmatrix}$ $\dim = 1$

Giacitura: $\begin{cases} 7x + 2y = 0 \\ 5x + 2z = 0 \end{cases}$

$(5y - 7z = 0)$

$E: \begin{cases} 7x + 2y = 18 \\ 5x + 2z = 26 \end{cases}$

$\boxed{7.1.5}$ $E: 5x - 12y + 10z = 2 \qquad \dim = 3 - 1 = 2$

$$E = \begin{pmatrix} 0 \\ 0 \\ 1/5 \end{pmatrix} + \operatorname{Span}\left(\begin{pmatrix} 12 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \right)$$

oppure

$$\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$$

oppure

$$\operatorname{Span}\left(\begin{pmatrix} 12\sqrt{3} + 2\pi \\ 5\sqrt{3} \\ -\pi \end{pmatrix}, \begin{pmatrix} 10 \\ 5 \\ 1 \end{pmatrix} \right)$$

$\boxed{7.1.6}$ $\quad E: \begin{cases} x = 3+2t-5s \\ y = 6+5t-3s \\ z = -7+4t-7s \end{cases}$ sijunfce

$$E = \left\{ \begin{pmatrix} 3+2t-5s \\ 6+5t-3s \\ -7+4t-7s \end{pmatrix} : t,s \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} 3 \\ 6 \\ -7 \end{pmatrix} + t \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix} + s \begin{pmatrix} -5 \\ -3 \\ -7 \end{pmatrix} : t,s \in \mathbb{R} \right\}$$

$$= \begin{pmatrix} 3 \\ 6 \\ -7 \end{pmatrix} + \mathrm{Span}\left(\begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \\ 7 \end{pmatrix} \right)$$

lin. ubip. $\Rightarrow \dim = 2$

Abbiamo $3-2=1$ equazioni

giacitura: $(5\cdot 7-4\cdot 3)x-(2\cdot 7-4\cdot 5)y+(2\cdot 3-5\cdot 5)z=0$

$$23x+6y-19z=0$$

E: $23x+6y-19z=\underbrace{23\cdot 3+6\cdot 6+19\cdot 7}_{\text{fare conto}}$

$\boxed{7.1.7}$

$$\begin{cases} 2x+3y-4z=1 \\ 3x-2y+5z=-1 \\ \cancel{7x+4y-3z=1} \end{cases}$$

$III = 2\cdot I + II$

$\boxed{7.1.8}$

$$\begin{cases} 5x+3y-2z=1 \\ 3x-2y+4z=-3 \\ x-7y+10z=7 \end{cases}$$

Parti omogenee: $III = -I + 2\cdot II$

Termini noti: No. Sist. impossibile

$E = \emptyset$