Geometria 27/3/15

Esercizi (I foglio) –

(20) (c) Calcolare $H(f)(0)$:

$$f(x,y) = \sin(1+x-2y) \cdot e^{x^2-y+2xy}$$

$$\frac{\partial f}{\partial x} = \left(\cos(1+x-2y) + \sin(1+x-2y)(2x+2y)\right) \cdot e^{x^2-y+2xy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \Big(-\sin(\dots)\cdot(-2) + \cos(\dots)(-2)(2x+2y)$$
$$+\sin(\dots)\cdot 2 + \big(\cos(\dots) + \sin(\dots)(2x+2y)\big)\cdot(-1+2x)\Big) e^{\dots}$$

$$(\dots)$$

(21) Calcolare proiez. ortog. su W.

(c) $\mathbb{R}^3$, $\langle \cdot | \cdot \rangle$, $W = \{ x : x_1 - 2x_2 + 5x_3 = 0 \} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}^{\perp}$

$$P_W(x) = x - P_{W^\perp}(x) = x - \frac{x_1 - 2x_2 + 5x_3}{1+4+25} \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$$

$$= \frac{1}{30} \underbrace{\begin{pmatrix} 29 & 2 & -5 \\ 2 & 26 & 10 \\ -5 & 10 & 5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$${}^tA = A = A^2.$$

(d) $\mathbb{R}^3, \langle \cdot | \cdot \rangle, W = \{ x : 3x_1 + 2x_2 = 3x_2 - 5x_3 = 0 \}$

$$= \left(\left\{ \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ -5 \end{pmatrix} \right\} \right)^{\perp}$$

$$= \text{Span}\left(\begin{pmatrix}3\\2\\0\end{pmatrix} \times \begin{pmatrix}0\\3\\-5\end{pmatrix}\right)$$

$$= \text{Span}\begin{pmatrix}-10\\15\\9\end{pmatrix}$$

$$\Rightarrow P_W(x) = \frac{-10x_1 + 15x_2 + 9x_3}{100 + 225 + 81}\begin{pmatrix}-10\\15\\9\end{pmatrix}$$

$$= \dots$$

(e) $\mathbb{R}^3, \langle\cdot|\cdot\rangle_A \qquad A = \begin{pmatrix}1 & 2 & 0\\ 2 & 5 & -1\\ 0 & -1 & 3\end{pmatrix}$.

$W = \{x : x_1 - 2x_2 + 5x_3 = 0\}$ (come in (c))

Verifica che A è def. pos:

$\det(A_1) = 1 > 0$

$\det(A_2) = 1\cdot 5 - 2\cdot 2 > 0$

$\det(A_3) = \det(A) = 15 - 12 - 1 > 0$ ✓

Anche qui: $p_W^{(A)}(x) = x - p_{W^{\perp_A}}^{(A)}(x)$:

$W^{\perp_A} = \{ y : \langle x | y \rangle_A = 0 \ \forall x \in W \}$

$= \{ y : {}^t y \cdot A \cdot x = 0 \ \forall x \in W \}$

$$W = \{ x : (1,-2,5) \cdot x = 0 \}$$

$$y \in W^{\perp A} \iff {}^t y \cdot A \in \mathrm{Span}(1,-2,5)$$

$$\iff A \cdot y \in \mathrm{Span}\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$$

$$\iff y \in \mathrm{Span}\left(A^{-1} \cdot \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \right)$$

Dinepue $W^{\perp A} = \mathrm{Span}\left(A^{-1} \cdot \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \right)$

$$\frac{1}{2}\begin{pmatrix} 14 & -6 & -2 \\ -6 & 3 & 1 \\ -2 & 1 & 1 \end{pmatrix}$$

$$\Longrightarrow \quad W^{\perp_A} = \operatorname{Span}\begin{pmatrix} 16 \\ -7 \\ 1 \end{pmatrix} .$$

$$P_W^{(A)}(x) = x - p^{(A)}_{W^{\perp_A}}(x) = x - \frac{\left\langle x \,\middle|\, \begin{pmatrix} 16 \\ -7 \\ 1 \end{pmatrix} \right\rangle_A}{\left\| \begin{pmatrix} 16 \\ -7 \\ 1 \end{pmatrix} \right\|_A^2} \cdot \begin{pmatrix} 16 \\ -7 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \frac{\begin{pmatrix} 16 & -7 & 1 \end{pmatrix} \cdot A \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}{\begin{pmatrix} 16 & -7 & 1 \end{pmatrix} \cdot A \cdot \begin{pmatrix} 16 \\ -7 \\ 1 \end{pmatrix}} \cdot \begin{pmatrix} 16 \\ -7 \\ 1 \end{pmatrix}$$

$$= \dots = \underbrace{\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}}_{\underline{P}} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$\underline{P}_{proiez} \Longrightarrow \underline{P}^2 = \underline{P}$;

$\underline{P}$ autoaggiunta rispetto $\langle \cdot | \cdot \rangle_A$

~~$\Longrightarrow$ simmetrico~~

$$\Longrightarrow \langle x | P y \rangle_A = \langle P x | y \rangle_A \quad \forall x, y$$

$${}^t y \cdot {}^t P \cdot A \cdot x = {}^t y \cdot A P \cdot x \quad \forall x, y$$

$$\Longrightarrow {}^t P \cdot A = A P$$

autoaggiunzione di P rispetto $\langle \cdot | \cdot \rangle_A$

(Oss: se $A = I_n$, $\langle \cdot | \cdot \rangle_A = \langle \cdot | \cdot \rangle_{\mathbb{R}^n}$ ritrovo

$^tP = P$, cioè simmetrica.)

(22) (a) Equaz. param. di retta $\perp$ al piano

$$\mathrm{Span}\left(\begin{pmatrix}-2\\4\\-3\end{pmatrix}, \begin{pmatrix}-2\\1\\5\end{pmatrix}\right)$$

Soluz.: $\mathrm{Span}\left(\begin{pmatrix}-2\\4\\-3\end{pmatrix} \wedge \begin{pmatrix}-2\\1\\5\end{pmatrix}\right)$

$$= \mathrm{Span}\begin{pmatrix}23\\16\\6\end{pmatrix}.$$

(23) (b) Trovare eq. cart. di $\ell^\perp$

$$\ell : \begin{cases} 3x - 4y + 2z = 0 \\ 7x + 3y + 2z = 0 \end{cases}$$

$$\ell = \left(\operatorname{Span}\left(\begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}, \begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix} \right) \right)^\perp$$

$$\Longrightarrow \ell^\perp = \left(\begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} \wedge \begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix} \right)^\perp$$

$$= \begin{pmatrix} -14 \\ 8 \\ 37 \end{pmatrix}^\perp$$

$\Longrightarrow \ell^{\perp}: -14x + 8y + 37z = 0$.

(24) Trovare $X \in \mathcal{M}_{n\times n}$ autoagg. rispetto a $\langle \cdot | \cdot \rangle_A$
+ altre condiz.

$\langle Xv|w\rangle_A = \langle v|Xw\rangle_A$

$$\boxed{A \cdot X = {}^tX \cdot A}$$

${}^tw \cdot A \cdot X \cdot v = {}^tw\, {}^tX A v$

(a) $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ $\qquad X = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

$3\beta = 2\gamma$

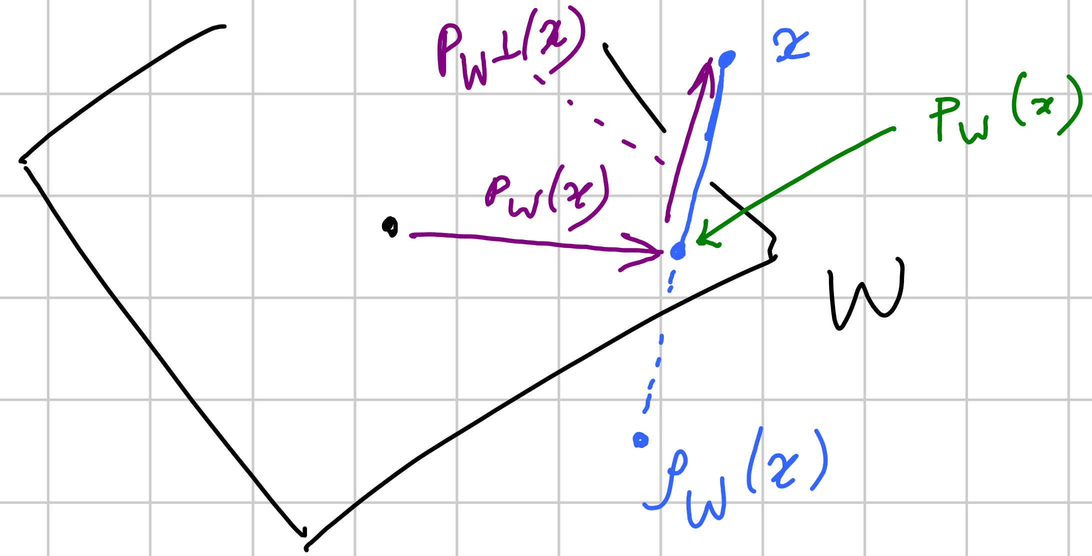
$$X \in \operatorname{Span}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix} \right)$$

(b) $A = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$; $X \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$

$$\begin{cases} \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \\ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \end{cases}$$

$$\begin{cases} \emptyset \\ 2\beta - \delta = -\alpha + 3\gamma \\ \quad '' \qquad\qquad '' \\ \emptyset \\ 2\alpha - \beta = 5 \\ 2\gamma - \delta = 0 \end{cases} \quad \Rightarrow \ \ldots \ \dim = 1$$

(25) (a) Trovare la matrice della riflessione in $\mathbb{R}^3$ rispetto a $W: 2x - 3y + 4z = 0$ _



$$\rho_W(x) = P_W(x) - P_{W^\perp}(x)$$

$$\left(x = P_W(x) + P_{W^\perp}(x) \right)$$

$$\Longrightarrow \rho_W(x) = x - 2P_{W^\perp}(x)$$

$$\rho_W(x) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - 2 \cdot \frac{2x - 3y + 4z}{4 + 9 + 16} \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$$

$$= \underbrace{\frac{1}{29} \begin{pmatrix} 21 & 12 & -16 \\ 12 & 11 & 24 \\ -16 & 24 & -3 \end{pmatrix}}_{F} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Sappiamo:

- f isometria $\Rightarrow$ F ortogonale
  cioè $\ {}^tF = F^{-1}$

- $f \circ f = id \Rightarrow F \cdot F = I_3$
  cioè $F^{-1} = F$

$$\Rightarrow {}^tF = F$$

come risulta dal calcolo.