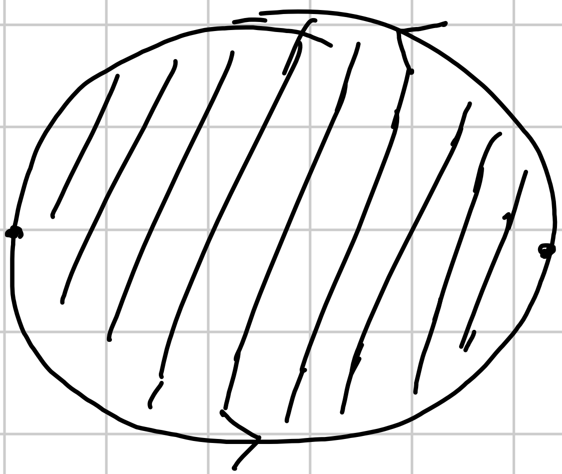
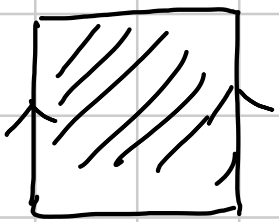
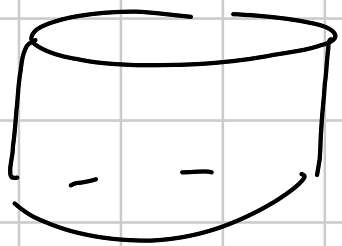
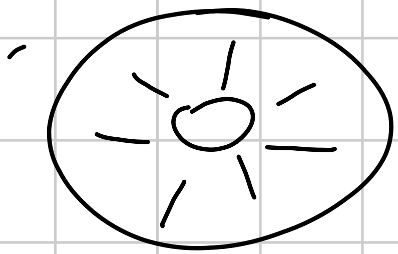
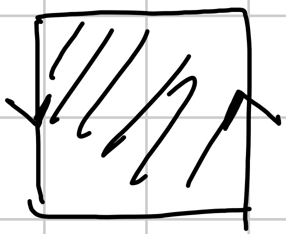
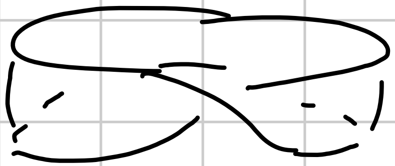
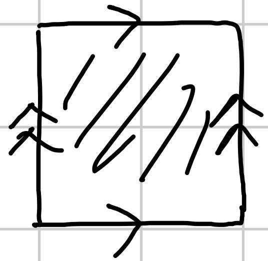
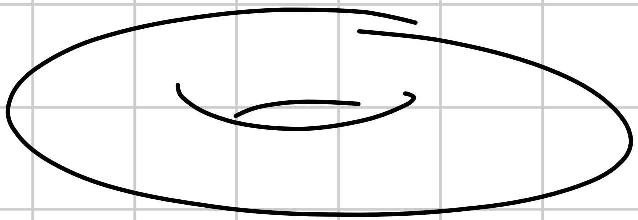
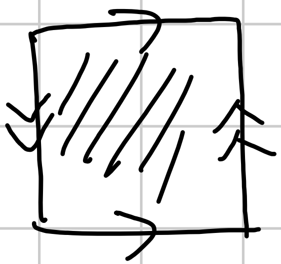
Geometrie 22/4/15

$\mathbb{P}^2(\mathbb{R}) =$ 

 $=$  $=$ 

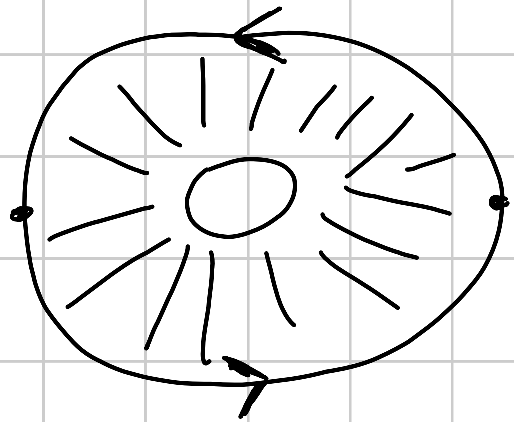
 = 

 = 

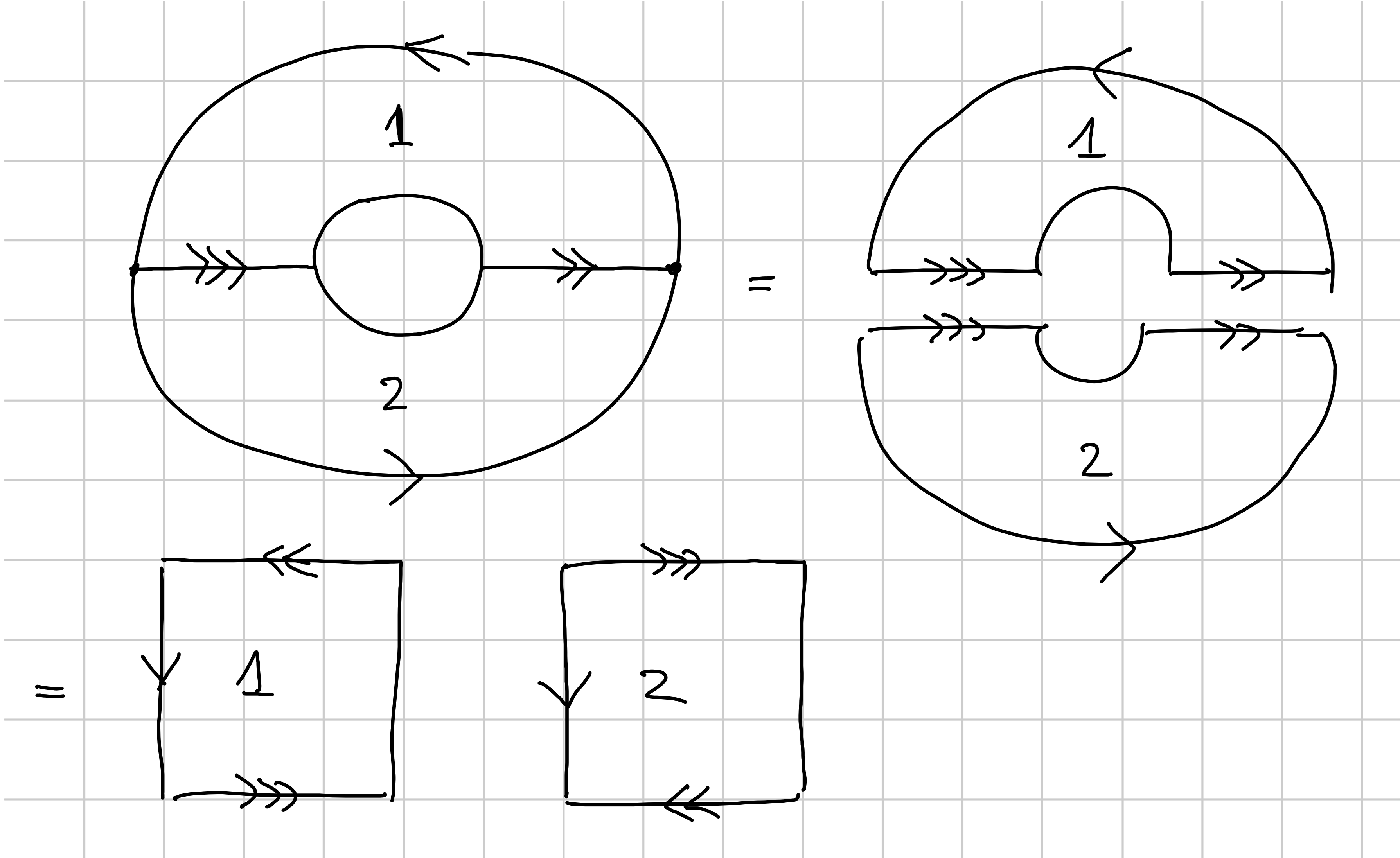
 = bottiglia di Klein.

Esercizi :

①



Eseguo dei tagli ricordando che devo tornare a incollare dove ho tagliato.
(Questo non modifica l'oggetto.)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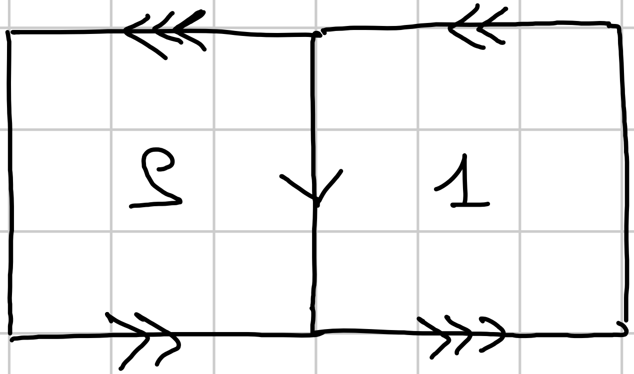

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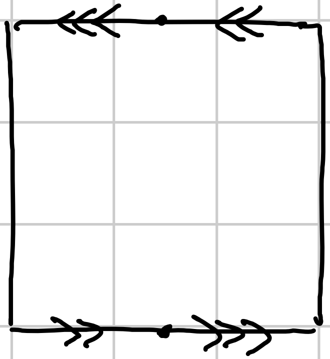

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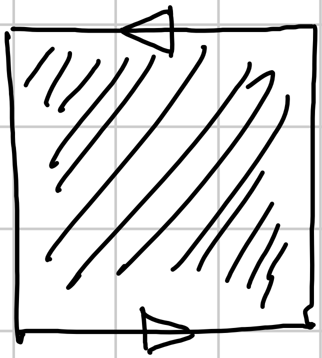


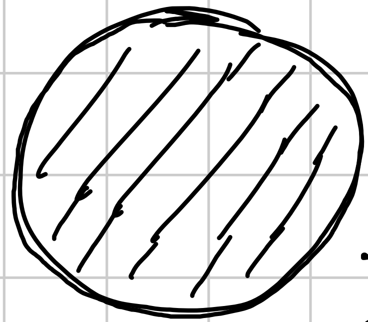
=

Nastro di Möbius

(Fatto: $\mathbb{P}^2(\mathbb{R})$ non si può realizzare in $\mathbb{R}^3$

ma:

$\cup$



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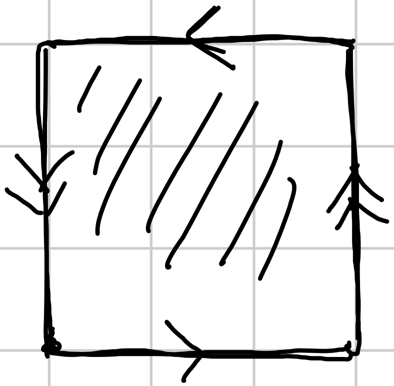
②



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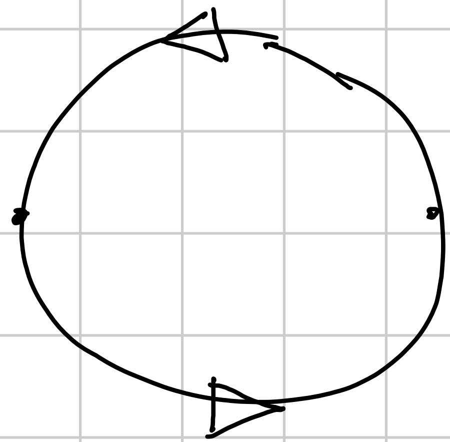
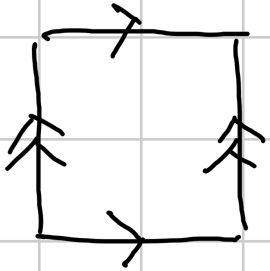
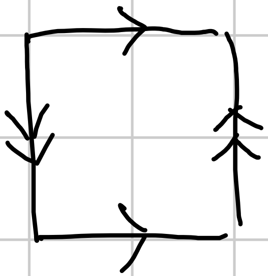


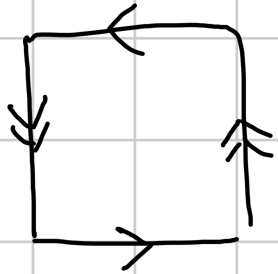
$= \mathbb{P}^2(\mathbb{R})$



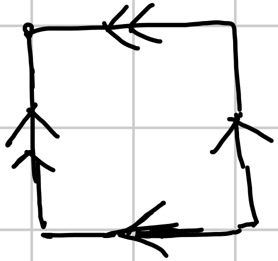
toro



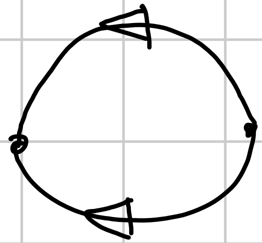
Klein



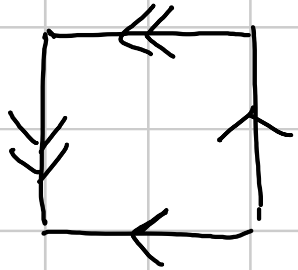
$\mathbb{P}^2$



$\parallel$



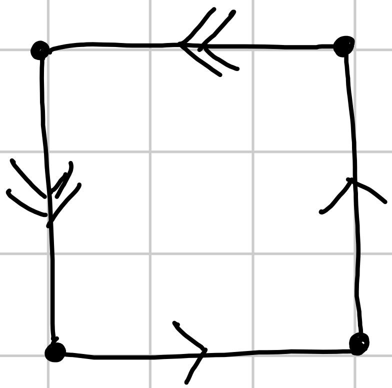
sfera



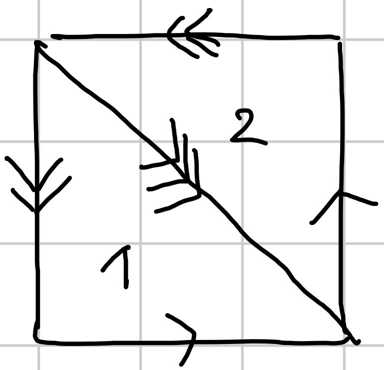
$\parallel$

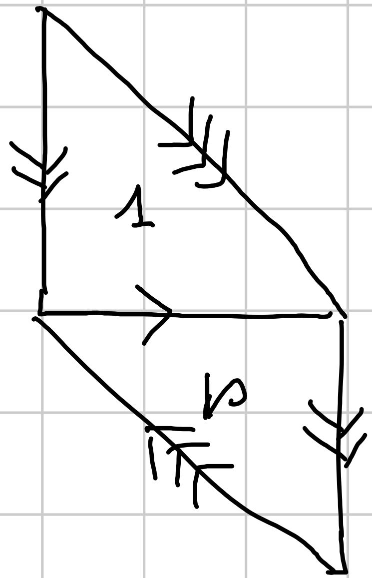
$\mathbb{P}^2$

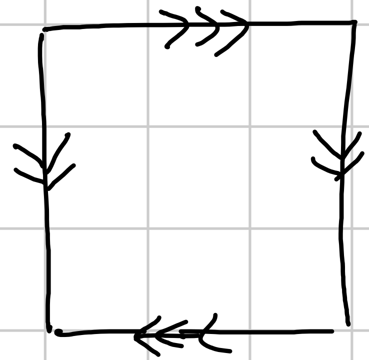
2′



Taglio e ricordo:

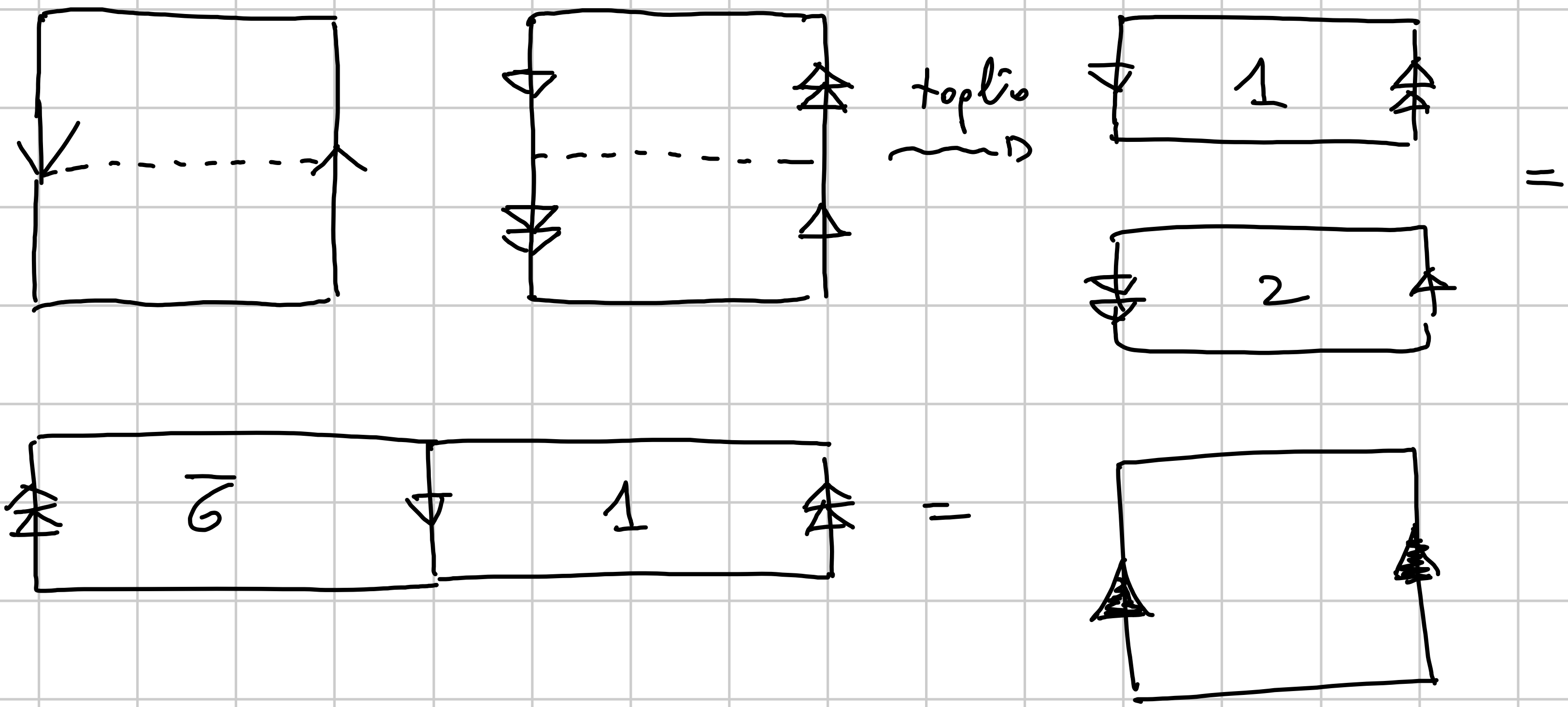






Klein

3. Cosa si trova tagliando a metà (longitudinalmente) un nastro di Möbius?



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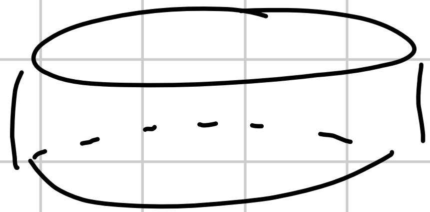

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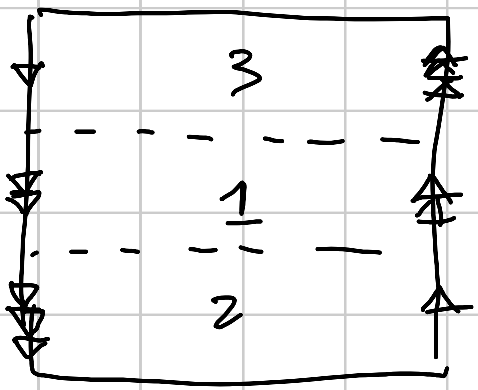


1 = Möbius

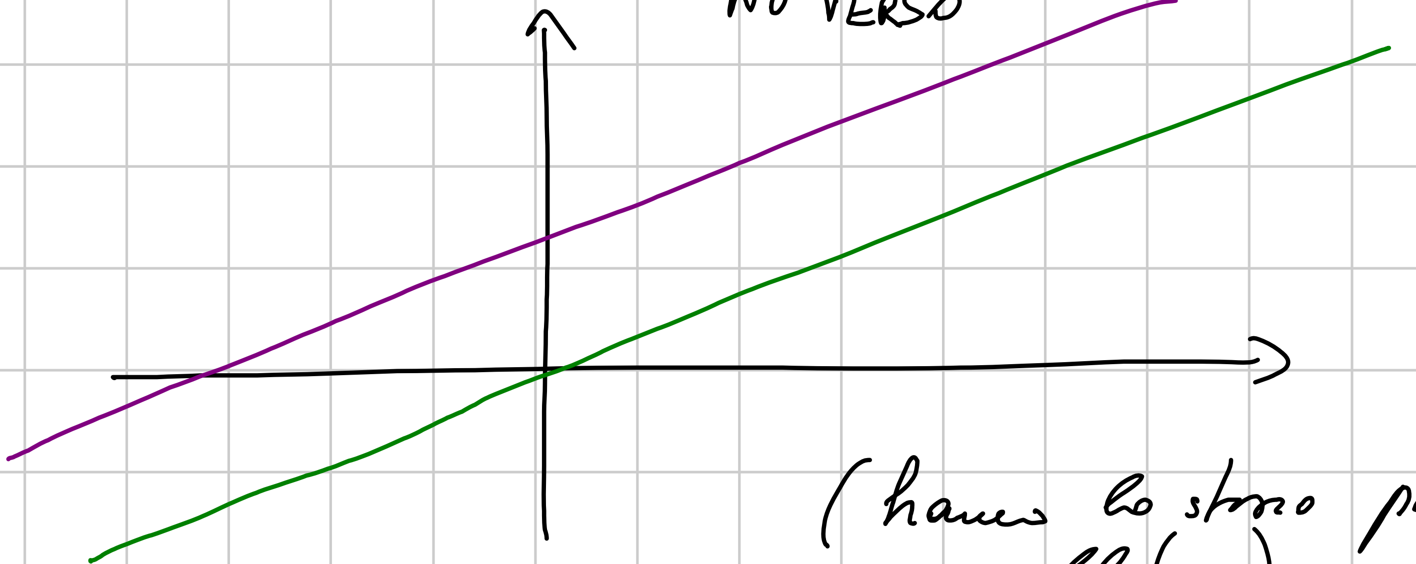
2 + 3 = cilindro

Estrinsecamente vengono allacciati fra loro

$$\mathbb{P}^2(\mathbb{R}) = \mathbb{R}^2 \cup \underbrace{\text{i suoi punti all'}\infty}_{\mathbb{P}^1(\mathbb{R})}$$

(le direzioni delle rette affini)

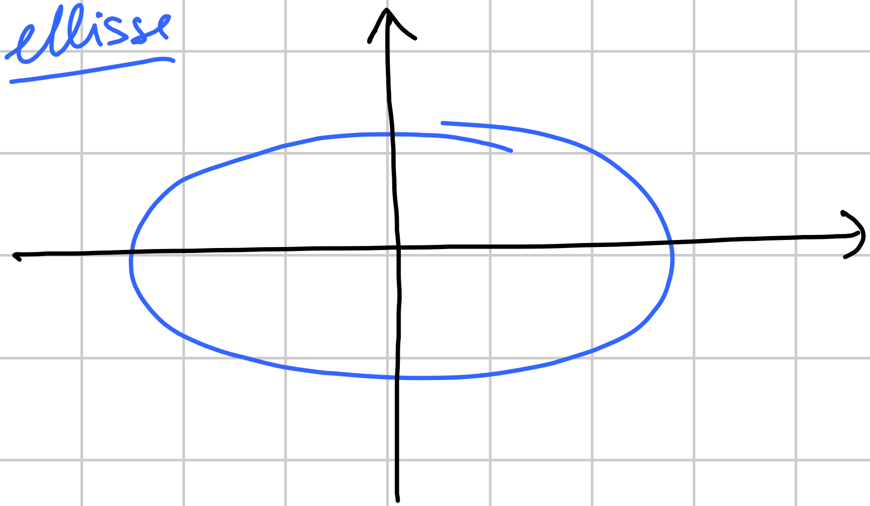
NO VERSO



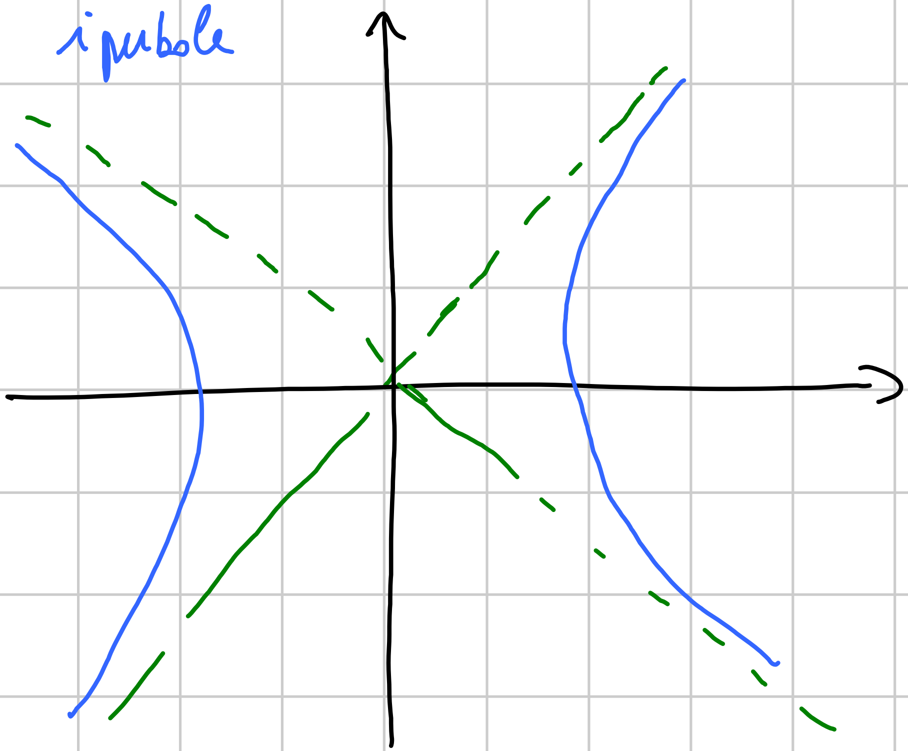
(hanno lo stesso punto all'∞)

Quali sono i punti all'∞ delle coniche?

<u>ellisse</u>

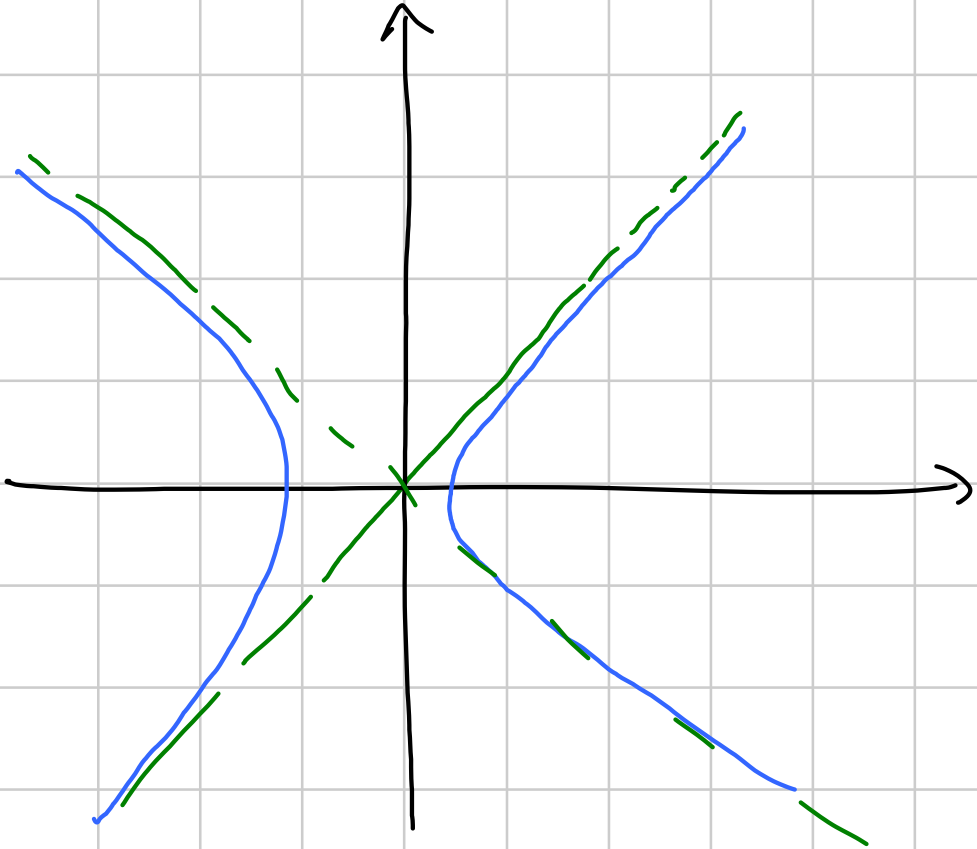


all'∞ : nessuno

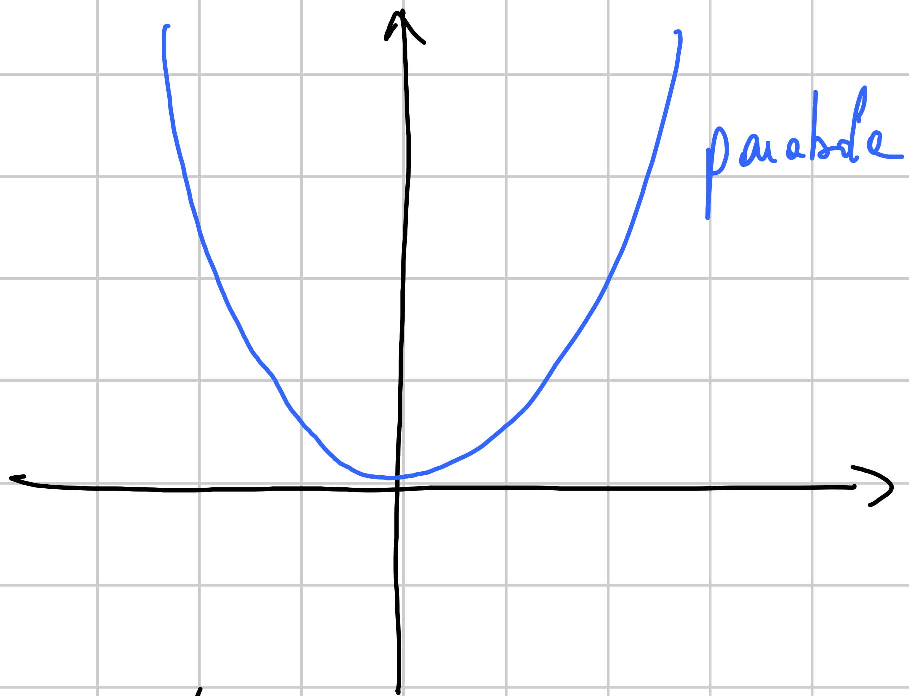


Punti all'∞: le direzioni degli asintoti (2).

zoom out



l'iperbole tende a giacere sugli asintoti._



all'∞ tende a giacere sul suo semiasse

$\Rightarrow$ ha un solo pto all'∞ .

Def: chiamo coordinate omogenee di un punto $[x] \in \mathbb{P}^n(\mathbb{R})$

(ovvero $x \in \mathbb{R}^{n+1}$, $x \neq 0$

$[x]$ = la sua classe di equivalenza)

la scrittura

$$[x_1 : x_2 : \ldots : x_n : x_{n+1}]$$

(Idea: contano solo i rapporti tra le x_i;

$$[-6 : 10 : 14] = [9 : -15 : -21]$$

$$\in \mathbb{P}^2(\mathbb{R})$$

infatti $\begin{pmatrix} 9 \\ -15 \\ -21 \end{pmatrix} \sim \begin{pmatrix} -6 \\ 10 \\ 14 \end{pmatrix}$ poiché

$$\begin{pmatrix} 9 \\ -15 \\ -21 \end{pmatrix} = -\frac{3}{2} \begin{pmatrix} -6 \\ 10 \\ 14 \end{pmatrix}.)$$

Con queste scritture:

$$\mathbb{P}^n(\mathbb{R}) = \{ [x_1 : x_2 : \dots : x_{n+1}] : x \in \mathbb{R}^{n+1} \setminus \{0\} \}$$

$$\mathbb{P}^n(\mathbb{R}) \supset \mathbb{R}^n = \left\{ [x_1 : \dots : x_n : 1] \ : x \in \mathbb{R}^n \right\}$$

$$\mathbb{P}^n(\mathbb{R}) = \mathbb{P}^{n-1}(\mathbb{R}) = \left\{ [x_1 : \dots : x_n : 0] \ : x \in \mathbb{R}^n \setminus \{0\} \right\}.$$

Ora se $X = \left\{ x \in \mathbb{R}^n : \sum_{i,j=1}^{n} q_{ij} x_i x_j + \sum_{j=1}^{n} l_j x_j + c = 0 \right\}.$

(**Oss**: se $f(x)$ è un polinomio omogeneo di grado k in $x \in \mathbb{R}^{n+1}$ l'equazione $f(x) = 0$

ha senso in $[x] \in \mathbb{P}^n(\mathbb{R})$: ovvero
cambiando il rappresentante non varia il
fatto che l'equazione sia risolta o meno ;
infatti se $y \in [x]$ ho $y = \lambda \cdot x$, $\lambda \neq 0$

$$\Rightarrow f(y) = f(\lambda \cdot x) = \lambda^k \cdot f(x)$$

dunque $f(y) = 0 \Leftrightarrow f(x) = 0$.

Invece le equaz. non omogenee <u>non</u> hanno senso .)

$$X = \left\{ x \in \mathbb{R}^n : \sum_{i,j=1}^{n} a_{ij} x_i x_j + \sum_{j=1}^{n} l_j x_j + c = 0 \right\}.$$

Poupo $$\overline{X} = \left\{ [x] \in \mathbb{P}^n(\mathbb{R}) : \sum_{i,j=1}^{n} a_{ij} x_i x_j + \sum_{j=1}^{n} l_j \cdot x_j x_{n+1} + c \cdot x_{n+1}^2 = 0 \right\}$$

ottenendo un luogo in $\mathbb{P}^n(\mathbb{R})$ t.c.

$$\overline{X} \cap \mathbb{R}^n = \overline{X} \cap \{x_{n+1} = 1\} = X.$$

(Chiusura proiettiva di X.)

Es: $X = \{x \in \mathbb{R}^2 : 7x_1^2 + 2x_1x_2 + 5x_2^2 + 9x_1 - 4x_2 + 11 = 0\}$

$\overline{X} = \{[x] \in \mathbb{P}^2(\mathbb{R}) : 7x_1^2 + 2x_1x_2 + 5x_2^2 + 9x_1x_3 - 4x_2x_3 + 11x_3^2 = 0\}$

Def: Chiamo $X_\infty = \overline{X} \setminus X$ = i punti all'∞ di X.

Es: (1) $\mathcal{E} = \{ x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1 \}$

$$\overline{\mathcal{E}} = \{ [x] \in \mathbb{P}^2(\mathbb{R}) : x_1^2 + x_2^2 = x_3^2 \}$$

$$\mathcal{E}_\infty = \overline{\mathcal{E}} \cap \{ x_3 = 0 \} = \{ [x : 0] : [x] \in \mathbb{P}^1(\mathbb{R}),\ x_1^2 + x_2^2 = 0 \}$$

$= \emptyset$ ($[0:0]$ non è un punto di $\mathbb{P}^1(\mathbb{R})$.)

② $\mathcal{Y} = \{ x \in \mathbb{R}^2 : x_1^2 - x_2^2 = 1 \}$

$$\overline{\mathcal{Y}} = \{ [x] \in \mathbb{P}^2(\mathbb{R}) : x_1^2 - x_2^2 = x_3^2 \}$$

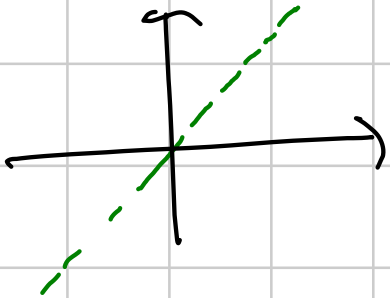
$$\mathcal{Y}_\infty = \overline{\mathcal{Y}} \cap \{ x_3 = 0 \} = \{ [x:0] : [x] \in \mathbb{P}^1(\mathbb{R}),\ x_1^2 - x_2^2 = 0 \}$$

$$= \{ [1:1], [1:-1] \}$$

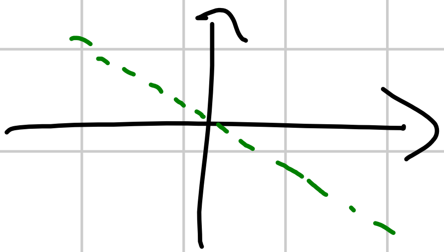
due punti

(sono esattamente le direzioni degli asintoti;

$[1:1]$:



$[1:-1]$:



) —

$$\mathcal{P} = \{ x \in \mathbb{R}^2 : x_2 = x_1^2 \}$$

$$\overline{\mathcal{P}} = \left\{ [x] \in \mathbb{P}^2(\mathbb{R}) : x_2 x_3 = x_1^2 \right\}$$

$$\mathcal{P}_\infty = \overline{\mathcal{P}} \cap \{ x_3 = 0 \}$$

$$= \left\{ [x:0] : [x] \in \mathbb{P}^1(\mathbb{R}), \quad x_1^2 = 0 \right\}$$

$$= \{ [0:1] \}$$ un punto all'∞

(è quello che rappresenta l'asse

[0:1] .

) –

DISCUTERE DIAGO

(9j) $\begin{pmatrix} 2-k & 3k-2 \\ 1 & 2k-1 \end{pmatrix}$

$tr = \lambda_1 + \lambda_2 = k+1$

$det = \lambda_1 \cdot \lambda_2 = -2k^2 + 4k - 2$
$+k$
$-3k+2 = -2k^2 + 2k$
$= 2k(1-k)$

$\Rightarrow \lambda_1 = 2k \, , \quad \lambda_2 = 1-k$

A è certamente diago. tranne se $\lambda_1 = \lambda_2$ ovvero tranne se $k = 1/3$. Invece per $k = 1/3$ ha il solo autovalore $2/3$ (doppio) ma <u>non</u> è già diagonale $\Longrightarrow$ <u>non</u> è diagonalizzabile.

(9k) $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & k \\ -2 & 2 & 1 \end{pmatrix}$

$$p_A(t) = \det \begin{pmatrix} t-1 & 0 & -1 \\ 0 & t-1 & -k \\ 2 & -2 & t-1 \end{pmatrix}$$

$$= (t-1)^3 + 2(t-1) - 2k(t-1)$$
$$= (t-1)(t^2-2t+1+2-2k)$$
$$= (t-1)(t^2-2t+3-2k)$$

$\lambda_1 = 1$ $\qquad \lambda_{2,3} = 1 \pm \sqrt{1-3+2k}$
$$= 1 \pm \sqrt{2(k-1)}$$

$k<1$: non diago su $\mathbb{R}$ (sì su $\mathbb{C}$)

$k=1$: c'è il solo autovalore 1 ma A non è diagonale $\Longrightarrow$ non è diagonalizzabile.

$$m.g.(1) = \dim(\operatorname{Ker}(1\cdot I_3 - A))$$

$$= 3 - \operatorname{rank}(1\cdot I_3 - A)$$

$$= 3 - \operatorname{rank}(A - 1\cdot I_3)$$

$$= 3 - \operatorname{rank}\left(\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -2 & 2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

$$= 3 - \text{rank} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -2 & 2 & 0 \end{pmatrix} = 3 - 2 = \underline{1}$$

$$\left(\text{mentre } m.a.(1) = 3 \right) .$$

Per $k > 1$ ho 3 autoval. distinti

$\Longrightarrow A$ è diagonalizzabile .

(9f) $A = \begin{pmatrix} 2 & 2(k-1) & 1-k \\ -k-4 & 2k-9 & 4-k \\ -2(k+4) & 2(k-7) & 6-k \end{pmatrix}$

$$p_A(t) = \det \begin{pmatrix} t-2 & 2(1-k) & -(1-k) \\ k+4 & t-2k+9 & k-4 \\ 2(k+4) & 14-2k & t+k-6 \end{pmatrix}$$

Sarrus: $2 \cdot 3 \cdot 3 + 2 \cdot 2 \cdot 2 + 2 \cdot 2 \cdot 2 + 2 \cdot 2 \cdot 2$

$+ 2 \cdot 3 \cdot 2 + 2 \cdot 3 \cdot 2 = 18 + 24 + 24 = 66$ termini

NON SI FA COSÌ : si usano le proprietà del determinante facendo operaz su righe e colonne.

ATTENZIONE

si fanno le operazioni su $t \cdot I - A$

(NON : prima solo su A, ottenendo A'
e poi facendo $\det(t \cdot I - A')$)

$$p_A(t) = \det \begin{pmatrix} t-2 & 2(1-k) & -(1-k) \\ k+4 & t-2k+9 & k-4 \\ 2(k+4) & 14-2k & t+k-6 \end{pmatrix} = \dots$$

$$r_3 \rightsquigarrow r_3 - 2 \cdot r_2$$

$$= \det \begin{pmatrix} t-2 & 2(1-k) & -(1-k) \\ k+4 & t-2k+9 & k-4 \\ 0 & -2t+2k-4 & t-k+2 \end{pmatrix}$$

$$= (t-k+2)\det\begin{pmatrix} t-2 & 2(1-k) & -(1-k) \\ k+4 & t-2k+8 & k-4 \\ 0 & -2 & 1 \end{pmatrix} = \dots$$

$C_2 \rightsquigarrow C_2 + 2C_3$

$$\dots = (t-k+2)\det\begin{pmatrix} t-2 & 0 & k-1 \\ k+4 & t+1 & k-4 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= (t-k+2)(t-2)(t+1)$$

$\Longrightarrow \quad \lambda_1 = k-2 \qquad \lambda_2 = 2 \qquad \lambda_3 = -1$

Certamente diagonalizzabile a meno che:

$k=4$:

$m.a.(2) = 2 \longrightarrow m.g.(2) = 3 - \mathrm{rank}(A - 2 \cdot I_3)$

$m.a.(-1) = 1$ ✓

$$= 3 - \mathrm{rank}\begin{pmatrix} 0 & 6 & -3 \\ -8 & -3 & 0 \\ -16 & -6 & 0 \end{pmatrix}$$

$= 3 - 2 = 1$ <u>NO</u>

$k=1$: $m.a.(2) = 1$ ✓

$$m.a.(-1) = 2 \longrightarrow m.g.(-1) = 3 - \operatorname{rank}(A + I_3)$$

$$= 3 - \operatorname{rank}\begin{pmatrix} 3 & 0 & 0 \\ -5 & -6 & 3 \\ -10 & -12 & 6 \end{pmatrix}$$

$$= 3 - 2 = 1$$

NO