21/5/2015 Esercitazioni di Geometria

Foglio del 15/5/2015

Es. 1 (a) Dire se è regolare $\alpha : [-1,1] \to \mathbb{R}^2$

$$\alpha(t) = \begin{cases} (t^3, \sin(t)) & \text{per } t \in [-1,0) \\ (1-\cos(t), t^2) & \text{per } t \in [0,1] \end{cases}$$

$$\lim_{t \to 0^-} \alpha'(t) = (0,1)$$

$\Rightarrow \alpha'$ non è continua in 0.

$$\lim_{t \to 0^+} \alpha'(t) = (0,0)$$

$\Rightarrow \alpha$ <u>non</u> è regolare.

(b) $$\alpha(t) = \begin{cases} (\ln(1+t), t^3) & \text{per } t \in [-1,0) \\ (2t, 1-e^{t^2}) & \text{per } t \in [0,1] \end{cases}$$

$\alpha'(t)$ non è definita per $t = -1$

inoltre

$$\lim_{t \to 0^-} \alpha'(t) = (1,0) \qquad \lim_{t \to 0^+} \alpha'(t) = (2,0)$$

$\Rightarrow \alpha$ <u>non</u> è regolare

(c) $$\alpha(t) = \begin{cases} (\tan(t), \ln(\cos(t))) & t \in [-1,0) \\ (t, t^2) & t \in [0,1] \end{cases}$$

$$\lim_{t \to 0^-} \alpha(t) = \lim_{t \to 0^+} \alpha(t) = (0,0)$$

$$\lim_{t \to 0^-} \alpha'(t) = \lim_{t \to 0^+} \alpha(t) = (1,0)$$

$\Rightarrow \alpha$ è regolare

ES 2 Dire se le curve parametrizzate α, β definiscono la stessa curva orientata come sottoinsieme del piano:

$\alpha : [0,1] \to \mathbb{R}^2 \qquad \alpha(t) = (\cos(\pi \cdot t), \sin(\pi \cdot t))$

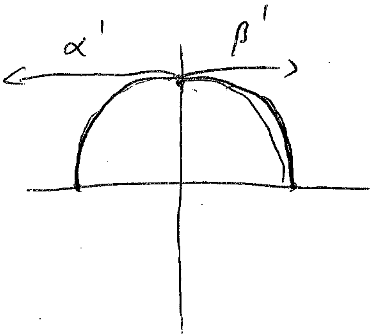
$\beta : [-1,1] \to \mathbb{R}^2 \qquad \beta(t) = (t, \sqrt{1-t^2})$

β' non è continua in quanto $\lim_{t \to -1} \beta'(t) = (1, \infty)$

Nell'intervallo aperto $(-1,1)$ β è regolare.

α, β definiscono lo stesso sottoinsieme di $\mathbb{R}^2$:

$\mathrm{Im}\,\alpha = \mathrm{Im}\,\beta = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1, y \geq 0\}$



Tuttavia non definiscono la stessa curva orientata. Infatti:

$\alpha(\frac{1}{2}) = \beta(0) = (0,1)$

$\alpha'(\frac{1}{2}) = (-\pi/2, 0) \qquad \beta'(0) = (1,0)$

quindi il cambio di parametro deve avere derivata negativa. Infatti se $\beta(s) = \alpha(\tau(s))$ |

$\beta'(0) = \alpha'(\tau(0))\, \tau'(0) =$

$= \alpha'(\frac{1}{2})\, \tau'(0) \quad \Rightarrow \tau'(0) = -\frac{\pi}{2}$

ES 3 Confrontare le lunghezze

$$\alpha : [0, 2\pi] \to \mathbb{R}^3 \qquad \alpha(t) = \begin{pmatrix} \sin t \\ \cos t \\ t^2 \end{pmatrix}$$

$$\beta : [0, \pi] \to \mathbb{R}^3 \qquad \beta(t) = \begin{pmatrix} \sin 2t \\ \cos 2t \\ 4t^2 \end{pmatrix}$$

Notiamo che $\beta(t) = \alpha(2t)$ ovvero, dato

$$\tau : [0, \pi] \to [0, 2\pi] \qquad \tau(t) = 2t$$

$$\beta(t) = \alpha(\tau(t)) \quad \Rightarrow \quad L(\alpha) = L(\beta)$$

perché β ottenuta da α per cambio di parametro.

$$\alpha'(t) = \begin{pmatrix} \cos t \\ -\sin t \\ 2t \end{pmatrix} \qquad L(\alpha) = \int_0^{2\pi} \|\alpha'(t)\|\, dt = \int_0^{2\pi} \sqrt{1 + 4t^2}\, dt$$

ES 4 Calcolare $\int_\alpha \sqrt{1 + x^2 + y^2}$ $\quad \alpha : [0,1] \to \mathbb{R}^2$ data da

$$\alpha(t) = (t \cos t, t \sin t)$$

$$\alpha'(t) = \begin{pmatrix} \cos t - t \sin t \\ \sin t + t \cos t \end{pmatrix} \qquad \|\alpha'(t)\| = \sqrt{1 + t^2}$$

$$\int_\alpha \sqrt{1 + x^2 + y^2} = \int_0^1 \sqrt{1 + t^2} \cdot \underbrace{\sqrt{1 + t^2}}_{= \|\alpha'(t)\|}\, dt = \int_0^1 (1 + t^2)\, dt = t + \frac{t^3}{3} \Big|_0^1 = \frac{4}{3}$$

ES 5 $\int_\alpha x y^2 \qquad \alpha : [0, \frac{\pi}{2}] \to \mathbb{R}^2 \qquad \alpha(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

$$\|\alpha'(t)\| = 1$$

$$\int_0^{\pi/2} \cos t \sin^2 t \cdot 1 \cdot dt = \int_0^{\pi/2} \frac{1}{3} \frac{d}{dt} (\sin^3 t)\, dt = \frac{\sin^3 t}{3} \Big|_0^{\pi/2} = \frac{1}{3}$$

ES. 6 $\int_\alpha \sqrt{12+x}$

$\alpha : [0,1] \to \mathbb{R}^2$

$\alpha(t) = \begin{pmatrix} 3t^2 \\ 1+t^3 \end{pmatrix}$

$\alpha'(t) = \begin{pmatrix} 6t \\ 3t^2 \end{pmatrix}$ $\qquad \|\alpha'(t)\| = 3t\sqrt{4+t^2}$

$$\int_\alpha \sqrt{12+x} = \int_0^1 \sqrt{12+3t^2} \cdot 3t\sqrt{4+t^2}\, dt =$$

$$= 3\sqrt{3} \int_0^1 t(4+t^2)\, dt = 3\sqrt{3}\left(2t^2 + \frac{t^4}{4}\right)\Big|_0^1 = 3\sqrt{3}\left(2+\frac{1}{4}\right) = \frac{27}{4}\sqrt{3}$$

ES 7 $\alpha : [0,1] \to \mathbb{R}^2$ $\qquad \alpha(t) = \begin{pmatrix} t \\ e^t \end{pmatrix}$

$$\int_\alpha \sqrt{1+y^2} = \int_0^1 \sqrt{1+e^{2t}} \cdot \sqrt{1+e^{2t}}\, dt =$$

$$= \int_0^1 (1+e^{2t})\, dt = \left(t + \frac{1}{2}e^{2t}\right)\Big|_0^1 = 1 + \frac{1}{2}e^2 - \frac{1}{2} =$$

$$= \frac{1}{2}(e^2 - 1)$$

ES. 8 $\alpha : [0,1] \to \mathbb{R}^2$ $\qquad \alpha(t) = (t, t^2)$

$$\int_\alpha x = \int_0^1 t\sqrt{1+4t^2}\, dt = \frac{2}{3} \cdot \frac{1}{8}\left(1+4t^2\right)^{3/2}\Big|_0^1 =$$

$$= \frac{1}{12}\left(5\sqrt{5} - 1\right)$$

ES. 9 $\quad \alpha : [0, \frac{\pi}{2}] \longrightarrow \mathbb{R}^2 \quad \alpha(t) = \begin{pmatrix} \sin t \\ 2\cos t \end{pmatrix}$

$$\int_\alpha xy = \int_0^{\pi/2} \sin t \, 2\cos t \sqrt{\cos^2 t + 4\sin^2 t} \, dt =$$

$$= \int_0^{\pi/2} 2 \sin t \cos t \sqrt{1 + 3\sin^2 t} \, dt = \frac{2}{3} \cdot \frac{1}{3} \cdot \left(1 + 3\sin^2 t\right)^{3/2} \Big|_0^{\pi/2} =$$

$$= \frac{2}{9}(8 - 1) = \frac{14}{9}$$

ES. 10 Sia $C = \{(x,y) \in \mathbb{R}^2 : \log(1 + x\cos(y)) + \cos x = 1\}$

Provare che $C \ni (0,0)$ e che C è una curva vicino a $(0,0)$. Trovare retta tangente in $(0,0)$

Dim: $\quad C = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : f(x,y) = 0 \right\}$

$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in C$, $\operatorname{grad}_{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}} f \neq 0 \quad \Rightarrow \quad C$ è curva vicino a $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ con tangente $\left(\operatorname{grad}_{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}} f\right)^\perp$

• $(0,0) \in C$?

$$\underbrace{\log(1+0)}_{0} + \underbrace{\cos 0}_{1} = 1 \quad \checkmark$$

$$f(x,y) = \log(1 + x\cos(y)) + \cos x - 1$$

$$\frac{\partial f}{\partial x} = \frac{\cos y}{1 + x\cos y} - \sin x \qquad \frac{\partial f}{\partial x}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 1$$

$$\frac{\partial f}{\partial y} = \frac{-x\sin y}{1 + x\cos y} \qquad \frac{\partial f}{\partial y}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$$

$\Rightarrow$ è curva, tangente parallela a $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

ES 11 $C = \{(x,y) \in \mathbb{R}^2 \mid 2xy^2 + 3x^2y = 1\}$

esistono punti di C in cui $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$ è tangente?

$$\begin{cases} 2xy^2 + 3x^2y = 1 \\ \left\langle \begin{pmatrix} -1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2y^2 + 6xy \\ 4xy + 3x^2 \end{pmatrix} \right\rangle = 0 \end{cases}$$

$$\begin{cases} 2xy^2 + 3x^2y = 1 \\ -\cancel{2}(y^2 + 3xy) + \cancel{4}^{\,2}(4xy + 3x^2) = 0 \end{cases}$$

$$\begin{cases} 2xy^2 + 3x^2y = 1 \\ 6x^2 + 5xy - y^2 = 0 \end{cases} \qquad \begin{cases} (1) \\ (6x - y)(x + y) = 0 \end{cases}$$

$$\begin{cases} x = -y \\ 2x^3 - 3x^3 = 1 \end{cases} \quad \vee \quad \begin{cases} \cancel{x} = 6x \\ 72x^3 + 18x^3 = 1 \end{cases}$$

$$\begin{cases} x = -1 \\ y = 1 \end{cases} \quad \vee \quad \begin{cases} x = 90^{-1/3} \\ y = 6 \cdot 90^{-1/3} \end{cases}$$

ES 12 $\alpha : [\frac{\pi}{2}, \pi) \to \mathbb{R}^2$

$$\alpha(t) = \begin{pmatrix} \sin t \\ \cos t + \log(\operatorname{tg}(t/2)) \end{pmatrix}$$

Mostrare che α ha lunghezza infinita. Trovare ~~$\tau : [0, +\infty) \to [\frac{\pi}{2}, \pi]$~~

$\tau : [0, +\infty) \to [\frac{\pi}{2}, \pi)$ t.c. $\alpha \circ \tau$ sia in parametro d'arco. Calcolare la curvatura di α in ogni pto.

Lunghezza fino a $t < \pi$ è

$$\sigma(t) = \int_{\pi/2}^{t} \|\alpha'(\theta)\| \, d\theta =$$

$$\alpha'(t) = \begin{pmatrix} \cos t \\ -\sin t + \dfrac{1}{tg(t/2)} \cdot \left(1 + tg^2(t/2)\right) \cdot \dfrac{1}{2} \end{pmatrix} =$$

$$= \begin{pmatrix} \cos t \\ \dfrac{1 + tg^2(t/2)}{2\, tg(t/2)} - \sin t \end{pmatrix} = \begin{pmatrix} \cos t \\ \dfrac{1}{\sin t} - \sin t \end{pmatrix}$$

$$\|\alpha'(t)\| = \sqrt{\cos^2 t + \frac{1}{\sin t^2} - 2 + \sin t^2} =$$

$$= \sqrt{\frac{1}{\sin t^2} - 1} = \sqrt{\frac{\cos^2 t}{\sin^2 t}} = -\frac{\cos t}{\sin t}$$

$t \geq \frac{\pi}{2}$

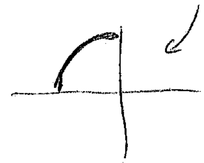
$$\sigma(t) = \int_{\pi/2}^{t} \|\alpha'(s)\|\, ds = \int_{\pi/2}^{t} -\frac{\cos(s)}{\sin(s)}\, ds =$$

$$= -\log \sin s \Big|_{\pi/2}^{t} = -\log \sin t$$

$\tau(s)$ è l'inverso di σ

$s = -\log(\sin t) \qquad \sin t = e^{-s} \qquad \left(\frac{\pi}{2} \leq t < \pi\right)$

$\tau(s) = \pi - \arcsin(e^{-s})$



Curvatura

$$K = \frac{\det(\alpha', \alpha'')}{\|\alpha'\|^3} \qquad \alpha'' = \begin{pmatrix} -\sin t \\ -\dfrac{\cos t}{\sin t^2} - \cos t \end{pmatrix}$$

$$K(t) = \ldots = \frac{-\cot g^2 t}{-\cot g^3 t} = tg(t)$$

ES 13 $r, \lambda > 0$ $\quad \alpha: \mathbb{R} \to \mathbb{R}^2$ $\quad \alpha(0) = \begin{pmatrix} r \\ 0 \end{pmatrix}$

e α in coord. polari soddisfa:

$\rho = r e^{\lambda \cdot \theta}$. Trovare $\tau : (s_0, \infty) \to \mathbb{R}$, $\tau(0) = 0$

che la param. d'arco e calcolare K in ogni punto.

$$\alpha(t) = 2 e^{\lambda t} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

$$\tau = \sigma^{-1} \qquad \sigma(t) = \int_0^t \|\alpha'(s)\| \, ds =$$

$$\alpha'(t) = 2 \begin{pmatrix} \lambda e^{\lambda t} \cos t - e^{\lambda t} \sin t \\ \lambda e^{\lambda t} \sin t + e^{\lambda t} \cos t \end{pmatrix}$$

$$\|\alpha'(t)\| = 2 e^{\lambda t} \sqrt{1+\lambda^2}$$

$$\sigma(t) = \int_0^t 2 e^{\lambda s} \sqrt{1+\lambda^2} \, ds = 2\sqrt{1+\lambda^2} \cdot \frac{e^{\lambda s}}{\lambda} \Big|_0^t =$$

$$= \frac{2}{\lambda} \sqrt{1+\lambda^2} \left(e^{\lambda t} - 1 \right)$$

$$\frac{2}{\lambda} \sqrt{1+\lambda^2} \left(e^{\lambda t} - 1 \right) = s$$

$$e^{\lambda t} = \frac{\lambda s}{2\sqrt{1+\lambda^2}} + 1 \qquad t = \underbrace{\frac{1}{\lambda} \log \left(\frac{\lambda s}{2\sqrt{1+\lambda^2}} + 1 \right)}_{= \tau(s)}$$

Curvatura:

$$\alpha''(t) = 2 e^{\lambda t} \begin{pmatrix} (\lambda^2 - 1) \cos t - 2\lambda \sin t \\ (\lambda^2 - 1) \sin t + 2\lambda \cos t \end{pmatrix}$$

$$K(t) = \frac{\det(\alpha', \alpha'')}{\|\alpha'\|^3} = \dots = \frac{1}{2\sqrt{1+\lambda^2}\, e^{\lambda t}}$$

ES. 14 $\rho : \mathbb{R} \to \mathbb{R}$ positiva, derivabile due volte. Trovare parametro d'arco e curvatura:

$$\alpha(t) = \rho(t) \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

$$\alpha'(t) = \begin{pmatrix} \rho'(t)\cos t - \rho(t)\sin t \\ \rho'(t)\sin t + \rho(t)\cos t \end{pmatrix}$$

$$\|\alpha'(t)\| = \sqrt{\rho'^2(t) + \rho^2(t)}$$

$$\sigma(t) = \int_0^t \sqrt{\rho'^2(s) + \rho^2(s)}\, ds$$

$$\tau(s) = \sigma^{-1}(s)$$

$$\alpha''(t) = \begin{pmatrix} \rho''(t)\cos t - 2\rho'(t)\sin t - \rho(t)\cos t \\ \rho''(t)\sin t + 2\rho'(t)\cos t - \rho(t)\sin t \end{pmatrix}$$

$$K(t) = \frac{-\rho(t)\rho''(t) + \rho'^2(t) + \rho^2(t)}{(\rho'^2(t) + \rho^2(t))^{3/2}}$$

ES. 15 Provare che per $v, w \in \mathbb{R}^3$

$$(v \wedge w) \wedge v = \|v\|^2 w - \langle w | v \rangle \cdot v$$

Se $v = 0$ o se $w = \lambda v$ è banalmente vero.

Se $w \neq \lambda v$ allora $(v, v\wedge w, (v\wedge w)\wedge v)$ è base ortogonale,

quindi $w = a v + b\, v\wedge w + c\,(v\wedge w)\wedge v \qquad a, b, c \in \mathbb{R}$

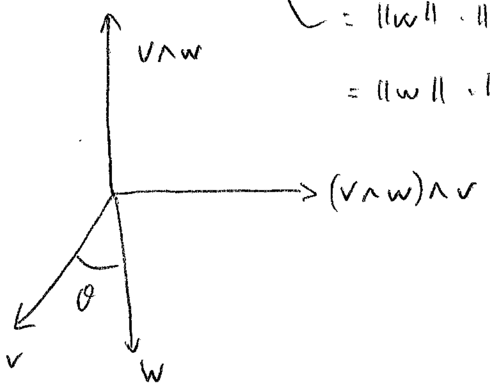
$\langle w | v \rangle = a \|v\|^2$

$0 = \langle w | v \wedge w \rangle = b \|v \wedge w\|^2$

$$\langle w \mid (v\wedge w)\wedge v\rangle = c\,\|(v\wedge w)\wedge v\|^2 = c\,\|v\|^2\cdot\|v\wedge w\|\sin\theta\cdot\|(v\wedge w)\wedge v\|$$

$$= \|w\|\cdot\|(v\wedge w)\wedge v\|\cdot\cos\left(\tfrac{\pi}{2}-\theta\right) =$$

$$= \|w\|\cdot\|(v\wedge w)\wedge v\|\cdot\sin\theta$$



$$\Rightarrow \left.\begin{aligned} a &= \langle w|v\rangle / \|v\|^2 \\ b &= 0 \\ c &= 1/\|w\|^2 \end{aligned}\right]$$

e otteniamo l'uguaglianza sostituendo w ∎

ES. 16 Calcolare curvatura e torsione di

$$\alpha(t) = \begin{pmatrix} 1+\cos t \\ 1-\sin t \\ \cos 2t \end{pmatrix}$$

$\sin 2t = 2\sin t\cos t$
$\cos 2t = \cos^2 t - \sin^2 t$

$$\alpha'\wedge\alpha'' = \begin{vmatrix} e_1 & e_2 & e_3 \\ -\sin t & -\cos t & -2\sin 2t \\ -\cos t & \sin t & -4\cos 2t \end{vmatrix} =$$

$$= \begin{pmatrix} 4\cos^3 t \\ 4\sin^3 t \\ -1 \end{pmatrix}$$

$$K(t) = \frac{\|\alpha'(t)\wedge\alpha''(t)\|}{\|\alpha'(t)\|^3} = \frac{\left(16\left(\cos^6 t+\sin^6 t\right)+1\right)^{1/2}}{\left(1+4\sin^2(2t)\right)^{3/2}}$$

$$\tau(s) = \frac{\langle\alpha'(t)\wedge\alpha''(t) \mid \alpha'''(t)\rangle}{\|\alpha'(t)\wedge\alpha''(t)\|^2} =$$

$$= \frac{\begin{vmatrix} \sin t & \cos t & 8 \sin 2t \\ -\sin t & -\cos t & -2 \sin 2t \\ -\cos t & \sin t & -4 \cos 2t \end{vmatrix}}{16(\cos^6 t + \sin^6 t) + 1}$$

ES. 17 (a) Trovare rif. di Frenet, curvatura e torsione in $\alpha(s_0)$

$$\alpha(s) = \begin{pmatrix} s \cos(s) \\ 1 + s^2 \\ \log(1+s) \end{pmatrix} \qquad s_0 = 0$$

$$t = \frac{\alpha'(s)}{\|\alpha'(s)\|} = \begin{pmatrix} \cos s - s \sin s \\ 2s \\ 1/1+s \end{pmatrix} / \|\alpha'(s)\|$$

$$t(s_0 = 0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\alpha'(0) \wedge \alpha''(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

$$\Rightarrow b(0) = \alpha'(0) \wedge \alpha''(0) / \|\alpha'(0) \wedge \alpha''(0)\| = \frac{1}{3} \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

$$n(0) = b(0) \wedge t(0) = \frac{1}{3\sqrt{2}} \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$$

$$k(0) = \frac{\|\alpha'(0) \wedge \alpha''(0)\|}{\|\alpha'(0)\|^3} = 3/2\sqrt{2}$$

$$\alpha'''(0) = \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$$

$$\tau(0) = \frac{\langle \alpha'(0) \wedge \alpha''(0) \mid \alpha'''(0) \rangle}{\|\alpha'(0) \wedge \alpha''(0)\|^2} = \frac{10}{9}$$