Geometria 11/3/15

Ortonormalizzazione (G-S):

base $v_1, \dots, v_n \rightsquigarrow$ base ortonormale $w_1, \dots, w_n$:

$$w_1 = \frac{v_1}{\|v_1\|} \quad (\text{ok perché } v_1 \neq 0)$$

$$\tilde{w}_{k+1} = v_{k+1} - \sum_{i=1}^{k} \langle v_{k+1} | w_i \rangle \cdot w_i \quad , \quad w_{k+1} = \frac{\tilde{w}_{k+1}}{\|\tilde{w}_{k+1}\|}.$$

Verifica: provo per induzione su $k \geq 1$

(1) $w_1, \dots, w_k$ è ortonormale

(2) $\mathrm{Span}(w_1, \dots, w_k) = \mathrm{Span}(v_1, \dots, v_k)$.

Basta perché da (2) segue che $\tilde{w}_{k+1} \neq 0$ poiché

$v_{k+1} \notin \mathrm{Span}(v_1, \dots, v_k) = \mathrm{Span}(w_1, \dots, w_k)$.

Facile per $k = 1$. Passo induttivo:

1) $w_1, \dots, w_{k+1}$ ortonormale sapendolo per $w_1, \dots, w_k$; basta vedere che $\langle \tilde{w}_{k+1} | w_j \rangle = 0$ per $j = 1 \dots k$:

$$\langle \tilde{w}_{k+1} | w_j \rangle = \left\langle v_{k+1} - \sum_{i=1}^{k} \langle v_{k+1} | w_i \rangle \cdot w_i \,\middle|\, w_j \right\rangle$$

$$= \langle v_{k+1} | w_j \rangle - \underbrace{\sum_{i=1}^{k} \langle v_{k+1} | w_i \rangle \cdot \underbrace{\langle w_i | w_j \rangle}_{\delta_{ij}}}_{\langle v_{k+1} | w_j \rangle} = 0$$

✓

2) Provo che $\mathrm{Span}(w_1, \dots, w_{k+1}) = \mathrm{Span}(v_1, \dots, v_{k+1})$ sapendo $\mathrm{Span}(w_1, \dots, w_k) = \mathrm{Span}(v_1, \dots, v_k)$:

$$\mathrm{Span}(w_1, \dots, w_{k+1}) = \mathrm{Span}\Big(v_1, \dots, v_k, v_{k+1} - \underbrace{\sum_{i=1}^{k} (\dots) \cdot w_i}_{\in \mathrm{Span}(v_1, \dots, v_k)}\Big)$$

$$= \mathrm{Span}(v_1, \dots, v_k, v_{k+1}) \qquad \blacksquare$$

__Oss:__ $$\tilde{w}_{k+1} = v_{k+1} - \underbrace{\sum_{i=1}^{k} (v_{k+1} | w_i) \cdot w_i}_{\in \mathrm{Span}(v_1, \dots, v_k)}$$

$$w_{k+1} = \frac{\tilde{w}_{k+1}}{\| w_{k+1} \|}$$

$\Rightarrow$ la matrice di cambio di base è del tipo .

$$\begin{pmatrix} (>0) & * & * & \cdots & * \\ 0 & (>0) & * & & \vdots \\ 0 & 0 & (>0) & & \vdots \\ \vdots & \vdots & 0 & & \vdots \\ \vdots & \vdots & \vdots & & * \\ 0 & 0 & 0 & & (>0) \end{pmatrix} \qquad (\Rightarrow \det > 0).$$

<u>Cor</u>: se $\dim V < +\infty$ un sistema $v_1, \dots, v_k$ di vettori ortogonali non nulli si può completare a una base ortogonale.

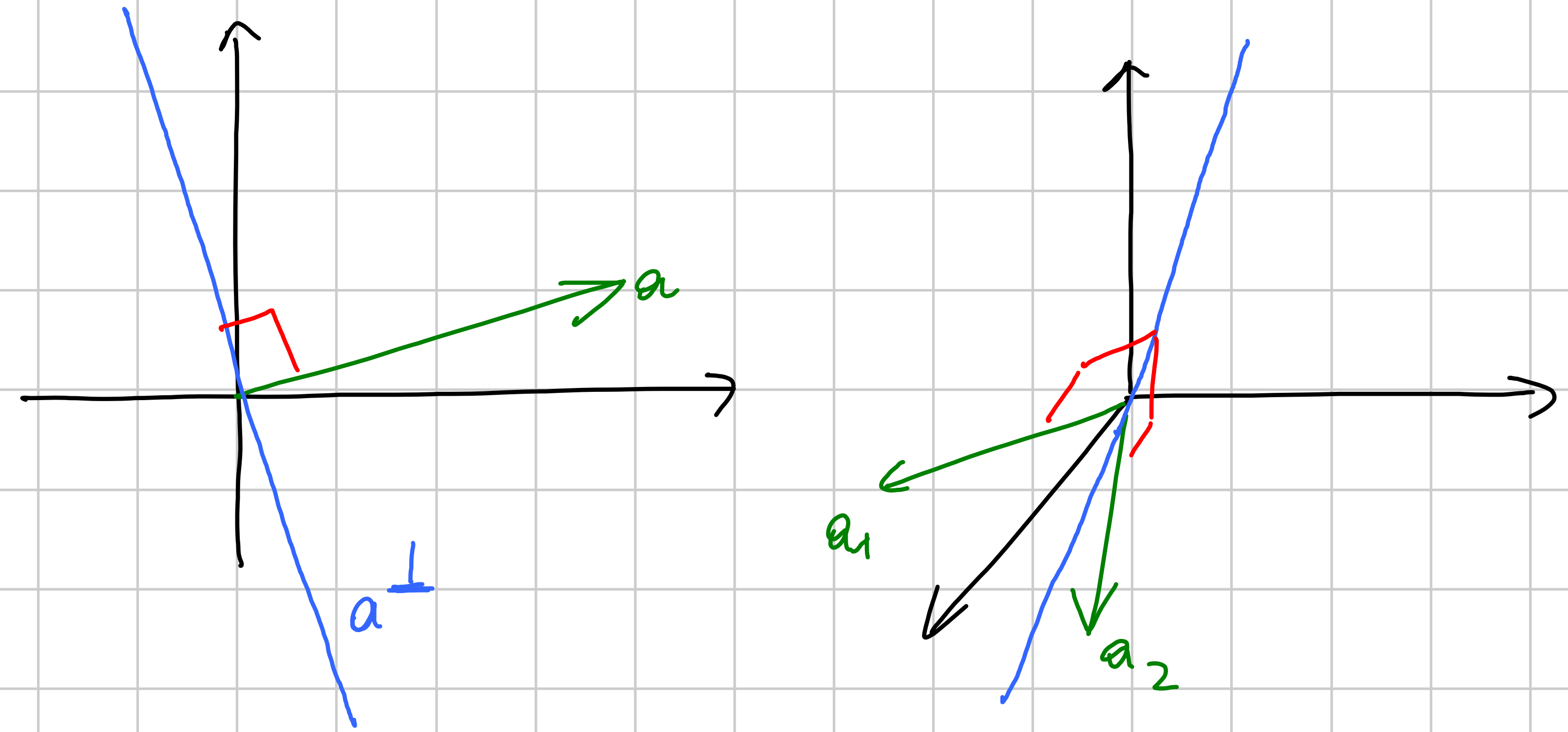
$\underline{\text{Dim}}$: $v_1, \dots, v_k \xrightarrow{\text{completo a base}} v_1, \dots, v_k, \dots, v_n$

$\xrightarrow{G-S} w_1 = \frac{v_1}{\|v_1\|},\ w_2 = \frac{v_2}{\|v_2\|}, \dots, w_k = \frac{v_k}{\|v_k\|}, w_{k+1}, \dots, v_n$

$\Rightarrow$ uso $v_1, \dots, v_k, w_{k+1}, \dots, w_n$. $\blacksquare$

$\underline{\text{Def}}$: Se $A \subset V$ definisco il suo ortogonale

$$A^\perp = \{ v \in V : \langle v | a \rangle = 0 \quad \forall a \in A \}$$

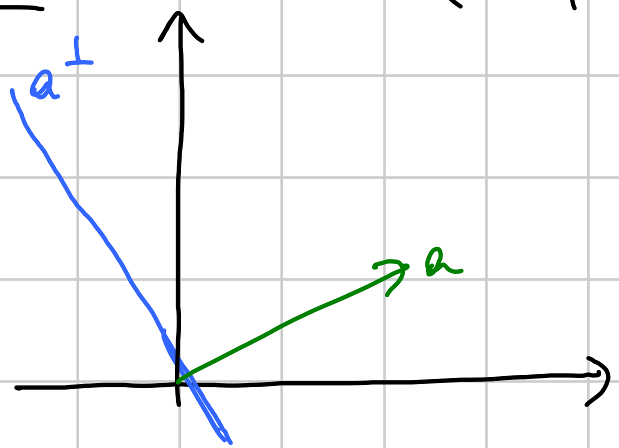

a
a⊥
a1
a2

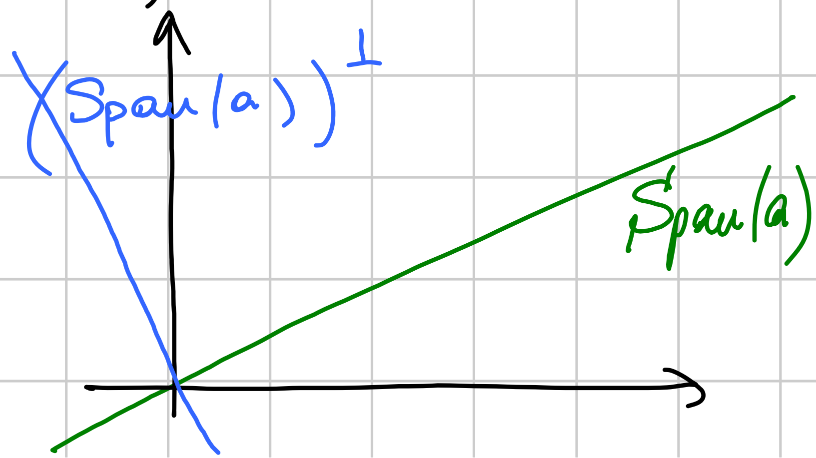
Oss: $A^\perp$ è sempre un sottospazio:

$v_1, v_2 \in A^\perp$ cioè $\langle v_1 | a \rangle = \langle v_2 | a \rangle = 0 \quad \forall a \in A$

$\Rightarrow \langle \lambda_1 v_1 + \lambda_2 v_2 | a \rangle = \lambda_1 \langle v_1 | a \rangle + \lambda_2 \langle v_2 | a \rangle = 0 \quad \forall a \in A.$

Oss: $A^\perp = \left(\mathrm{Span}(A) \right)^\perp$





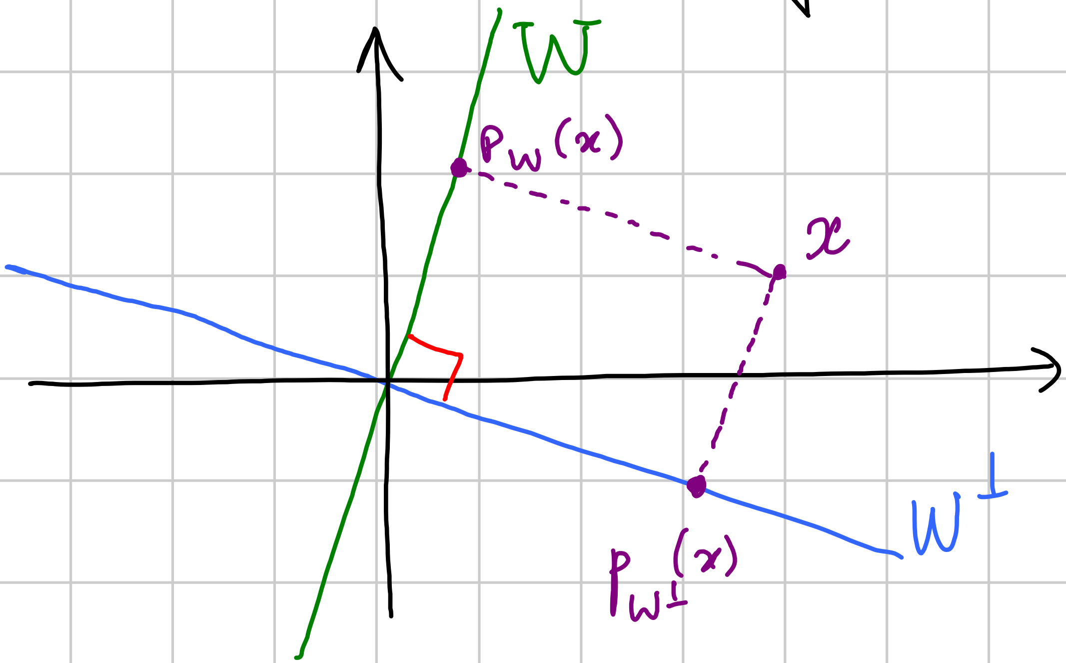
Infatti se $v \in A^\perp$ allora $\forall \lambda_1, \dots, \lambda_k \in \mathbb{R}$
$a_1, \dots, a_k \in A$ ho

$$\langle v | \lambda_1 a_1 + \dots + \lambda_k a_k \rangle = \lambda_1 \underbrace{\langle v / a_1 \rangle}_{0} + \dots + \lambda_k \underbrace{\langle v / a_k \rangle}_{0} = 0$$

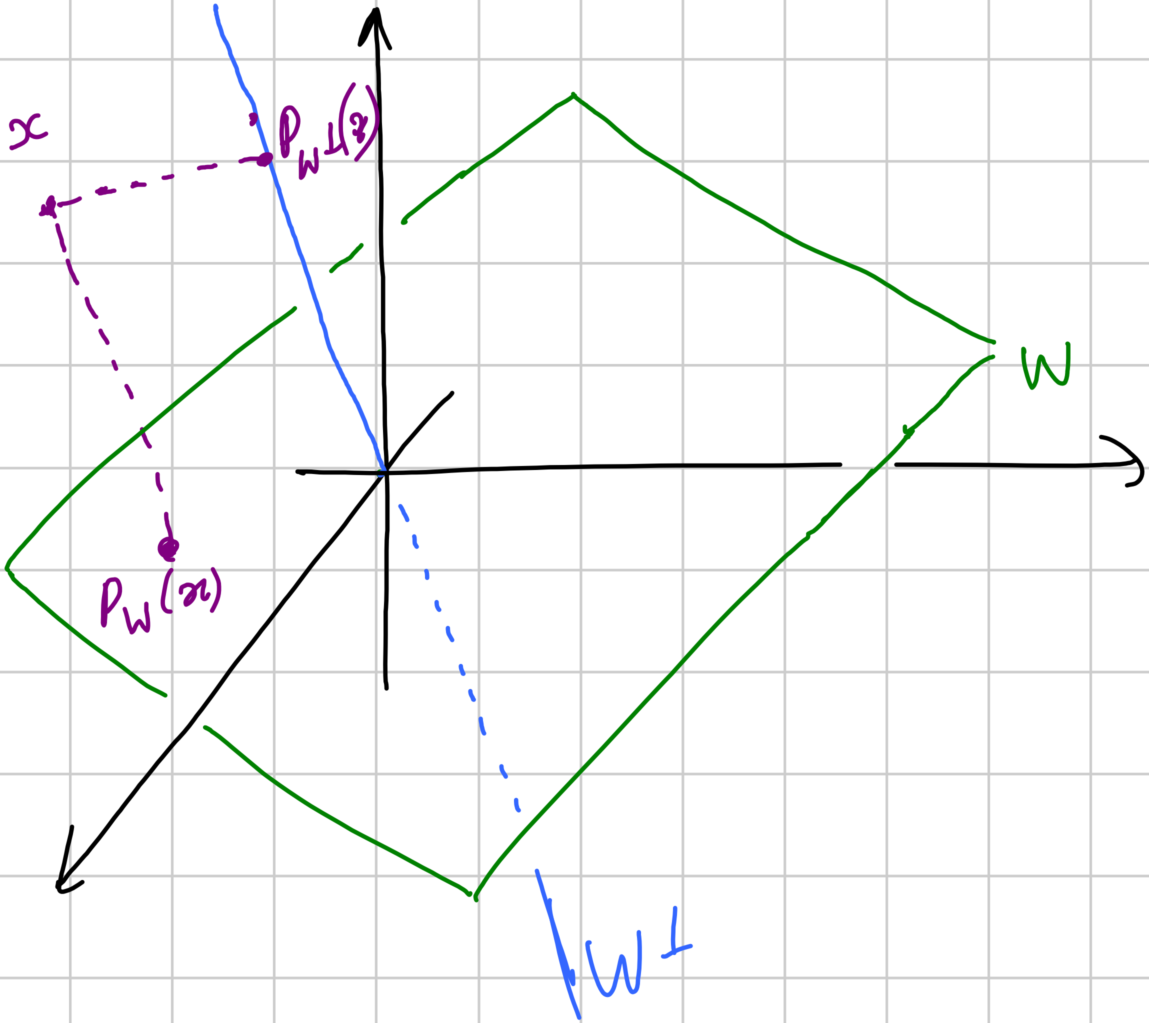
$\Longrightarrow v \in \left(\operatorname{Span}(A)\right)^\perp$. Viceversa ovvio:
in generale $A \subseteq B \Longrightarrow B^\perp \subseteq A^\perp$.

Proposizione: se $\dim(V) < +\infty$ e $W \subset V$ è un sottospazio ho $V = W \oplus W^\perp$.

(Somma diretta ortogonale.)



$P_W, P_{W^\perp}$ le proiezioni associate sono dette proiezioni ortogonali.


x
P_{W⊥}(x)
P_W(x)
W
W⊥

Dim: provo che $W \cap W^\perp = \{0\}$
(vero anche se $\dim V = +\infty$).

$$w \in W \cap W^\perp \Rightarrow \langle \underset{\substack{\cap \\ W^\perp}}{w} \mid \underset{\substack{\cap \\ W}}{w} \rangle = 0 \Rightarrow \|w\| = 0 \Rightarrow w = 0.$$

Provo che $V = W + W^\perp$ (usa $\dim(V) < +\infty$).
Scelgo base ortonormale $w_1, \dots, w_k$ di W
(prima a caso, poi G-S), e completo a base
ortonormale di V, sarà $w_1, \dots, w_k, w_{k+1}, \dots, w_n$.
So che

$$V = \underbrace{\mathrm{Span}(w_1, \dots, w_k)}_{\substack{\shortparallel \\ W}} \oplus \underbrace{\mathrm{Span}(w_{k+1}, \dots, w_n)}$$

per concludere basta provare che lui è $W^\perp$.

Affermo: $W^\perp = \mathrm{Span}(w_{k+1}, \dots, w_n)$

$\underline{\supseteq}$ Fissato $i > k$ ho

$\langle w_i | w_j \rangle = 0$ per $j = 1, \dots, k$

$\Rightarrow w_i \in (\{w_1, \dots, w_k\})^\perp$

$= (\mathrm{Span}(w_1, \dots, w_k))^\perp = W^\perp$

Ho provato che $w_i \in W^\perp \ \forall i > k$

$\Rightarrow \mathrm{Span}(w_{k+1}, \dots, w_n) \subset W^\perp$.

$\underline{\subset}$ Prendo $w \in W^\perp$. So che $w_1, \dots, w_n$ è base ortonormale di V

$$\Rightarrow w = \sum_{i=1}^{n} \underbrace{\langle w / w_i \rangle}_{\text{nulli per } i \leq k} \cdot w_i$$

$$= \sum_{i=k+1}^{n} \langle w | w_i \rangle \cdot w_i \in \operatorname{Span}(w_{k+1}, \dots, w_n).$$

$\blacksquare$

Prop: Se $W \subset V$ è sottosp ($\dim V < +\infty$) e $w_1, \dots, w_k$ è base ortogonale di W allora

$$P_W(v) = \sum_{i=1}^{k} \frac{\langle v | w_i \rangle}{\|w_i\|^2} \cdot w_i$$

Dim: Se $w_1, \dots, w_n$ è un completamento a base ortog. ho

$$v = \sum_{i=1}^{n} \frac{\langle v | w_i \rangle}{\|w_i\|^2} \cdot w_i$$

$$= \underbrace{\sum_{i=1}^{k} \frac{\langle v | w_i \rangle}{\|w_i\|^2} w_i}_{\substack{\cap \\ W}} + \underbrace{\sum_{i=k+1}^{n} \frac{\langle v | w_i \rangle}{\|w_i\|^2} \cdot w_i}_{\substack{\cap \\ W^\perp}}.$$

$\blacksquare$

Cor (disuguaglianza di Bessel):

Se $w_1, \dots, w_k$ è base ortogonale di W ho

$$\sum_{i=1}^{k} \frac{|\langle v | w_i \rangle|^2}{\|w_i\|^2} \le \|v\|^2 \qquad \forall v \in V.$$

Dim: $v = P_W(v) + P_{W^\perp}(v)$ e sono $\perp$ fra loro

$$\Rightarrow \|v\|^2 = \|P_W(v)\|^2 + 2 \cdot 0 + \|P_{W^\perp}(v)\|^2$$

$$\geq \|P_W(v)\|^2$$

$$= \left\| \sum_{i=1}^{k} \frac{\langle v | w_i \rangle}{\|w_i\|^2} \cdot w_i \right\|^2$$

e sono tutti ortog. fra loro

$$= \sum_{i=1}^{k} \left| \frac{\langle v | w_i \rangle}{\|w_i\|^2} \right|^2 \cdot \|w_i\|^2$$

$$= \sum_{i=1}^{k} \frac{|\langle v | w_i \rangle|^2}{\|w_i\|^2} .$$

$\boxed{///}$

Esempio: $W = \mathrm{Span}\left(\underset{v_1}{\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}} \right) \subset \mathbb{R}^3.$

Li trasformo in base ortogonale:

$$w_1 = v_1$$

$$w_2 = v_2 - \left\langle v_2 \,\middle|\, \frac{w_1}{\|w_1\|} \right\rangle \cdot \frac{w_1}{\|w_1\|}$$

$$= \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix} - \cancel{\frac{14+3-3}{14}} \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix}$$

(va bene perché: $5 \cdot 2 + (-2) \cdot 3 + 4 \cdot (-1) = 0$.)

$$P_W\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{2+3-1}{14}\begin{pmatrix}2\\3\\-1\end{pmatrix} + \frac{5-2+4}{45}\begin{pmatrix}5\\-2\\4\end{pmatrix}$$

$$= \frac{2}{7}\begin{pmatrix}2\\3\\-1\end{pmatrix} + \frac{7}{45}\begin{pmatrix}5\\-2\\4\end{pmatrix} = \dots$$

Esercizi foglio 9/3/15.

(1) Calcolare angolo e distanze:

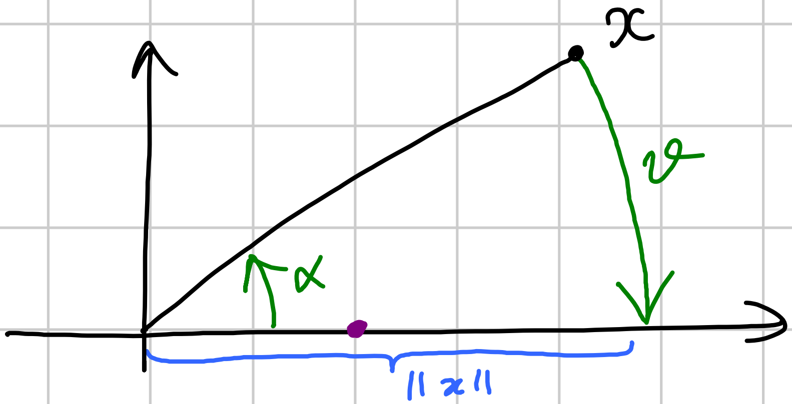
(b) $\begin{pmatrix}7\\2\\-3\end{pmatrix}, \begin{pmatrix}-1\\3\\4\end{pmatrix}$

$$d = \sqrt{8^2 + (-1)^2 + (-7)^2} = \sqrt{114}$$

$$\vartheta = \arccos \frac{7 \cdot (-1) + 2 \cdot 3 + (-3) \cdot 4}{\sqrt{49+4+9} \cdot \sqrt{1+9+16}} = \dots$$

(c) $\begin{pmatrix} \cdot \\ \cdot \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$ Non ha senso -

(2)



$$\frac{1}{\|x\|} \cdot R_\vartheta(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\vartheta = -\alpha$ $\qquad$ $\lambda = \frac{1}{\|x\|} = \frac{1}{\sqrt{x_1^2 + x_2^2}}$

$$\cos\alpha = \frac{x_1}{\sqrt{\ldots}} \qquad \sin\alpha = \frac{x_2}{\sqrt{\ldots}}$$

$$\Rightarrow R_\vartheta = \begin{pmatrix} x_1/\sqrt{\ldots} & x_2/\sqrt{\ldots} \\ -x_2/\sqrt{\ldots} & x_1/\sqrt{\ldots} \end{pmatrix}.$$

$$y \perp x \Leftrightarrow R_\vartheta(y) \perp \underbrace{R_\vartheta(x)}_{\in \mathbb{R}\cdot e_1}$$

$$\Longleftrightarrow R_v(y) \in \mathbb{R} \cdot e_2$$

$$\Longleftrightarrow R_v(y) = \begin{pmatrix} 0 \\ * \end{pmatrix}$$

$$\overset{\shortparallel}{\frac{1}{\sqrt{\dots}}\begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix}}$$

$$\Longleftrightarrow x_1 y_1 + x_2 y_2 = 0$$

(cioè $\langle x | y \rangle_{\mathbb{R}^2} = 0$, come già trovato via Carnot).

③ Provare che si ha un prod. scal. su V:

ⓐ $V = \mathbb{R}_{\leq 1}[t]$ $\quad (\dim = 2)$

$$f(p(t), q(t)) = 5\big(p(1) - 2\cdot p(-2)\big)\big(q(1) - 2\cdot q(-2)\big) + 3\big(p(2) - 7\cdot p(-1)\big)\cdot\big(q(2) - 7\cdot q(-1)\big)$$

$f: V \times V \longrightarrow \mathbb{R}$

- bilineare: fissato q è del tipo

$$p(t) \longmapsto \alpha \cdot p(1) + \beta \cdot p(-2) + \gamma \cdot p(2) + \delta \cdot p(-1)$$

$\Longrightarrow$ è lineare.

- simmetrica : ovvio ($\Rightarrow$ lin. a dx)

- def. pos :

$$f(p(t), p(t)) = 5 \cdot \left(p(1) - 2 \cdot p(-2)\right)^2 + 3 \cdot \left(p(2) - 7 p(-1)\right)^2$$

è sempre ≥ 0 ; nullo solo se :

$$\begin{cases} p(1) = 2\,p(-2) \\ p(2) = 7\,p(-1) \end{cases} \qquad \begin{cases} a_0 + a_1 = 2(a_0 - 2a_1) \\ a_0 + 2a_1 = 7(a_0 - a_1) \end{cases}$$

$$\begin{cases} a_0 = 5a_1 \\ \cancel{6}^{\,2} a_0 = \cancel{9}^{\,3} a_1 \end{cases} \qquad \begin{cases} a_0 = 0 \\ a_1 = 0 \end{cases}$$

dunque solo se $p(t) = 0$.

4. $V = C^0([0,1], \mathbb{R})$

$$\langle f | g \rangle = \int_0^1 f(t) \cdot g(t)\, dt$$

Considero $S, C \in V$

$$S(t) = \sin(2\pi t) \qquad C(t) = \cos(2\pi t)$$

Verifico che $\langle S | C \rangle = 0$;

$$\langle S, C \rangle = \int_0^1 S(t) \cdot C(t)\, dt = \int_0^1 \sin(2\pi t) \cdot \cos(2\pi t)\, dt$$

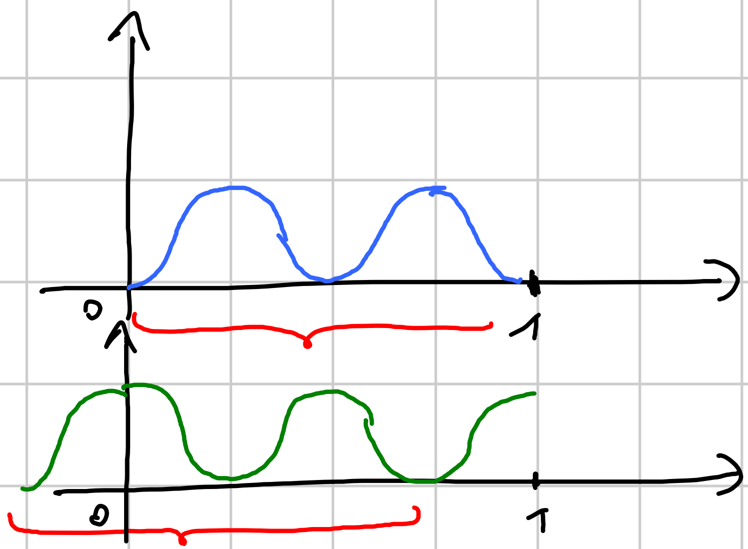
$$= \frac{1}{2}\int_0^1 \sin(4\pi t)\,dt = -\frac{1}{8\pi}\cos(4\pi t)\Big|_0^1$$

$$= -\frac{1}{8\pi}(1-1) = 0 .$$

Calcolo $\|S\|$ e $\|C\|$.

$$\|S\|^2 = \int_0^1 \underline{\sin^2}(2\pi t)\,dt$$

$$\|C\|^2 = \int_0^1 \underline{\cos^2}(2\pi t)\,dt$$



$\Longrightarrow \|S\| = \|C\|.$

Inoltre $\|S\|^2 + \|C\|^2 = \int_0^1 \left(\sin^2(2\pi t) + \cos^2(2\pi t)\right) dt = 1$

$\Longrightarrow \|S\| = \|C\| = 1/\sqrt{2}$.

⑥ Dire se f sia bil/simm/def pos :

(a) $\mathbb{R}^2$ $\quad f(x,y) = 7x_1y_2 + 8y_1y_2 - 5x_2y_1$

No bil $\quad f\left(0, \begin{pmatrix}1\\1\end{pmatrix}\right) = 8 \neq 0$

(b) $\mathbb{R}^2$ $\quad f(x,y) = -3x_1y_1 + 2x_1y_2 + 8x_2y_1 + 7x_2y_2$

Bil : Si

Simm : No

$$f\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right) = 2$$
$$\neq$$
$$f\left(\begin{pmatrix}0\\1\end{pmatrix} \begin{pmatrix}1\\0\end{pmatrix}\right) = 8$$

(c) $f(x,y) = 7x_1y_1 - 3x_1y_2 - 3x_2y_1 - 4x_2y_2$

Bil Si

Simm Si

Def. pos : $7x_1^2 - 6x_1x_2 - 4x_2^2 > 0 \quad \forall x \neq 0$?

$f\begin{pmatrix}0\\1\end{pmatrix} = -4$ No

(d) $f(x,y) = 3x_1y_1 + 4x_1y_2 + 4x_2y_1 + 5x_2y_2$

<u>Bil</u> Sì <u>Simm</u> Sì

Def pos: $3x_1^2 + 8x_1x_2 + 5x_2^2 > 0 \quad \forall x \neq 0$?

<u>No</u>: $f\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$

(Anzi: $f\begin{pmatrix} 3 \\ -2 \end{pmatrix} = 27 - 48 + 20 < 0$)