

Esercizi di Algebra Lineare (Petronio 14/15)

18 ottobre 2014

Esercizio 1 Esibire una base dello spazio vettoriale assegnato:

(a) $\{x \in \mathbb{R}^3 : 7x_1 + 5x_2 - 4x_3 = 0\}$

(b) $\left\{x \in \mathbb{R}^3 : \begin{array}{l} 3x_1 + 2x_2 - 5x_3 = 0 \\ -4x_1 + x_2 + 7x_3 = 0 \end{array} \right\}$

(c) $\left\{x \in \mathbb{R}^3 : \begin{array}{l} -4x_1 + 2x_2 + 5x_3 = 0 \\ 3x_1 + 7x_2 - 2x_3 = 0 \\ x_1 + 27x_2 + 4x_3 = 0 \end{array} \right\}$

(d) $\{x \in \mathbb{R}^4 : 5x_1 - 8x_2 + 2x_3 - 3x_4 = 0\}$

(e) $\{x \in \mathbb{R}^4 : 9x_1 + 4x_3 - 5x_4\}$

(f) $\left\{x \in \mathbb{R}^4 : \begin{array}{l} 5x_1 + 3x_2 + 7x_3 - 2x_4 = 0 \\ 8x_1 - 7x_2 + 4x_3 - 5x_4 = 0 \end{array} \right\}$

(g) $\left\{x \in \mathbb{R}^4 : \begin{array}{l} 2x_1 + 4x_2 + 7x_3 - 9x_4 = 0 \\ -5x_1 + 6x_2 + 3x_3 + 2x_4 = 0 \\ 9x_1 + 2x_2 + 11x_3 - 20x_4 = 0 \end{array} \right\}$

(h) $\left\{x \in \mathbb{R}^n : \sum_{j=1}^n (-1)^{j+1} x_j = 0\right\}$

(j) $\left\{x \in \mathbb{R}^n : \sum_{j=1}^n (-1)^j x_j = \sum_{j=1}^n (1 + (-1)^{j+1}) x_j = 0\right\}$

(k) $\{A \in \mathcal{M}_{3 \times 2}(\mathbb{R}) : a_{11} + a_{32} - 2a_{21} = a_{12} - a_{31} + 5a_{21} = 0\}$

$$(l) \left\{ A \in \mathcal{M}_{2 \times 3}(\mathbb{R}) : \begin{array}{l} a_{11} - 2a_{12} + 3a_{23} = 0 \\ -a_{11} + a_{12} + 2a_{13} = 0 \\ a_{12} + a_{21} + a_{23} = 0 \\ a_{21} + 4a_{23} + 2a_{13} = 0 \end{array} \right\}$$

$$(m) \{A \in \mathcal{M}_{n \times n}(\mathbb{R}) : \text{tr}(A) = 0\}$$

$$(n) \mathcal{S}_n = \{A \in \mathcal{M}_{n \times n}(\mathbb{R}) : {}^t A = A\}$$

$$(o) \mathcal{A}_n = \{A \in \mathcal{M}_{n \times n}(\mathbb{R}) : {}^t A = -A\}$$

$$(p) \{p(t) \in \mathbb{R}_{\leq d}[t] : p(1) = 0\}$$

$$(q) \{p(t) \in \mathbb{R}_{\leq 4}[t] : 2p(1) + p''(-1) = p(2) - 3p'(1) = 0\}$$

$$(r) \{p(t) \in \mathbb{R}_{\leq d}[t] : p(-t) = p(t)\}$$

$$(s) \{p(t) \in \mathbb{R}_{\leq d}[t] : p(-t) = -p(t)\}$$

$$(t) \{p(t) \in \mathbb{R}_{\leq 3}[t] : 6p(t) - 3tp'(t) + 2p''(t) = 0\}$$

$$(u) \mathcal{F}(\{a, b, c\}, \mathbb{R})$$

$$(v) \{f \in \mathcal{F}(\{a, b, c\}, \mathbb{R}) : 3f(a) - 2f(b) + 5f(c) = 0\}$$

$$(w) \left\{ f \in \mathcal{F}(\{a, b, c\}, \mathbb{R}) : \begin{array}{l} 2f(a) + 5f(b) + 3f(c) = 0 \\ 7f(a) - 3f(b) + 2f(c) = 0 \end{array} \right\}$$

Esercizio 2. Completare i vettori $v_1, \dots, v_k$ assegnati a una base dello spazio vettoriale V dato.

$$(a) V = \mathbb{R}^4, v_1 = \begin{pmatrix} 4 \\ 5 \\ -6 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -3 \\ 4 \\ 1 \\ 7 \end{pmatrix}$$

$$(b) V = \mathbb{R}^4, v_1 = \begin{pmatrix} 3 \\ -1 \\ 2 \\ 5 \end{pmatrix}, v_2 = \begin{pmatrix} 4 \\ 2 \\ -1 \\ 3 \end{pmatrix}, v_3 = \begin{pmatrix} 2 \\ -4 \\ 5 \\ 7 \end{pmatrix}$$

$$(c) \ V = \{x \in \mathbb{R}^4 : 4x_1 + 3x_2 + 5x_3 - 2x_4 = 0\}, \ v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 6 \end{pmatrix}$$

$$(d) \ V = \{x \in \mathbb{R}^4 : 3x_1 - 5x_2 + x_3 - 2x_4 = 0\}, \ v_1 = \begin{pmatrix} 1 \\ 1 \\ 4 \\ 1 \end{pmatrix}, \ v_2 = \begin{pmatrix} 2 \\ -1 \\ -1 \\ 5 \end{pmatrix}$$

$$(e) \ V = \{x \in \mathbb{R}^4 : 7x_1 + 2x_2 - 4x_3 + 5x_4 = 0\},$$

$$v_1 = \begin{pmatrix} 1 \\ -4 \\ 1 \\ 1 \end{pmatrix}, \ v_2 = \begin{pmatrix} 2 \\ 1 \\ -3 \\ 4 \end{pmatrix}$$

$$(f) \ V = \left\{ x \in \mathbb{R}^4 : \begin{array}{l} x_1 + x_2 + x_3 - x_4 = 0 \\ 2x_1 + 5x_2 - x_3 + x_4 = 0 \end{array} \right\}, \ v_1 = \begin{pmatrix} 4 \\ -2 \\ 1 \\ 3 \end{pmatrix}$$

$$(g) \ V = \mathcal{S}_2, \ v_1 = \begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix}, \ v_2 = \begin{pmatrix} -1 & 1 \\ 1 & 4 \end{pmatrix}$$

$$(h) \ V = \mathbb{R}[t], \ v_1 = -4 + 3t + 5t^4, \ v_2 = 3 + t^2 - t^7, \ v_3 = 5 + t + t^2 - t^3$$

$$(i) \ V = \mathbb{R}_{\leq 3}[t], \ v_1 = -4 + 3t + 5t^3, \ v_2 = 3 + t^2 - t^3, \ v_3 = 5 + t + t^2 - t^3$$

$$(j) \ V = \{p(t) \in \mathbb{R}_{\leq 4}[t] : p'(2) = p(-1)\}, \ v_1 = 5 + \frac{1}{2}t + 4t^2 - 3t^3 + t^4$$

Esercizio 3. Estrarre dai vettori $v_1, \dots, v_k$ assegnati una base dello spazio vettoriale V dato.

$$(a) \ V = \mathbb{R}^3, \ v_1 = \begin{pmatrix} 4 \\ 6 \\ -10 \end{pmatrix}, \ v_2 = \begin{pmatrix} -6 \\ -9 \\ 15 \end{pmatrix}, \ v_3 = \begin{pmatrix} 5 \\ -7 \\ 2 \end{pmatrix}$$

$$v_4 = \begin{pmatrix} -1 \\ 13 \\ -12 \end{pmatrix}, \ v_5 = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}, \ v_6 = \begin{pmatrix} \pi \\ \sqrt{17} \\ e \end{pmatrix}$$

$$(b) \quad V = \{x \in \mathbb{R}^4 : x_1 + x_2 = x_3 + x_4\}, \quad v_1 = \begin{pmatrix} 3 \\ -7 \\ 1 \\ -5 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 6 \\ 9 \\ -1 \end{pmatrix},$$

$$v_3 = \begin{pmatrix} 0 \\ 32 \\ 25 \\ 7 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 6 \\ -1 \\ 1 \\ 4 \end{pmatrix}, \quad v_5 = \begin{pmatrix} 7 \\ 8 \\ -5 \\ 20 \end{pmatrix}$$

$$(c) \quad V = \mathbb{R}_{\leq 3}[t], \quad v_1 = -t, \quad v_2 = 1 - t + t^2, \quad v_3 = 5 + 2t - t^3, \quad v_4 = 6 + t^2 - t^3, \\ v_5 = -8t + 5t^2 + t^3, \quad v_6 = 2 - t + 3t^2 - 4t^3$$

$$(d) \quad V = \mathcal{A}_3, \quad v_1 = \begin{pmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ 3 & 2 & 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 & -7 & 4 \\ 7 & 0 & -1 \\ -4 & 1 & 0 \end{pmatrix},$$

$$v_3 = \begin{pmatrix} 0 & 9 & -10 \\ -9 & 0 & 3 \\ 10 & -3 & 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 & -1 & 3 \\ 1 & 0 & 2 \\ -3 & -2 & 0 \end{pmatrix},$$

$$v_5 = \begin{pmatrix} 0 & -7 & 11 \\ 7 & 0 & 3 \\ -11 & -3 & 0 \end{pmatrix}$$

$$(e) \quad V = \{x \in \mathbb{R}^3 : 2x_1 + 5x_2 + 7x_3 = 0\},$$

$$v_1 = \begin{pmatrix} -13 \\ 1 \\ 3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -1 \\ -8 \\ 6 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}$$

Esercizio 4. Nello spazio V dato considerare i sottospazi W e Z assegnati e calcolare le dimensioni di W , Z , $W \cap Z$, $W + Z$, verificando la formula di Grassmann.

$$(a) \quad V = \mathbb{R}^4, \quad W = \text{Span} \left(\begin{pmatrix} -1 \\ 4 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \\ 3 \end{pmatrix} \right),$$

$$Z = \text{Span} \left(\begin{pmatrix} 3 \\ 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -9 \\ 3 \\ 0 \\ 8 \end{pmatrix} \right)$$

$$(b) \quad V = \mathbb{R}^4, \quad W = \text{Span} \left(\begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 7 \\ 2 \\ -1 \\ -1 \end{pmatrix} \right),$$

$$Z = \left\{ x \in \mathbb{R}^4 : \begin{array}{l} 2x_1 + x_2 - 3x_3 + 5x_4 = 0 \\ 3x_1 - x_2 + 2x_3 + x_4 = 0 \end{array} \right\}$$

$$(c) \quad V = \mathbb{R}^4, \quad W = \left\{ x \in \mathbb{R}^4 : \begin{array}{l} 2x_1 - x_2 + 3x_3 + 4x_4 = 0 \\ 3x_1 + 2x_2 - 5x_3 - x_4 = 0 \\ -5x_1 - 8x_2 + 21x_3 + 11x_4 = 0 \end{array} \right\},$$

$$Z = \left\{ x \in \mathbb{R}^4 : \begin{array}{l} x_1 + 3x_2 - 2x_3 + 5x_4 = 0 \\ 3x_1 + 2x_2 - 2x_3 + 4x_4 = 0 \end{array} \right\}.$$

$$(d) \quad V = \mathbb{R}^5, \quad W = \text{Span}(2e_1 + e_2 - e_3 + 3e_4 + 4e_5, 5e_1 + 2e_2 - e_3 + e_4 - e_5),$$

$$Z = \left\{ x \in \mathbb{R}^5 : \begin{array}{l} x_2 + x_3 - x_4 + x_5 = 0 \\ x_1 - x_2 + 3x_3 - x_4 + 2x_5 = 0 \end{array} \right\}$$

$$(e) \quad V = \mathbb{R}_{\leq 4}[t], \quad W = \text{Span}(1 - t + t^2 + t^3, 2 + t - 2t^3 + 3t^4),$$

$$Z = \{p(t) \in V : p(-1) = p''\left(\frac{1}{6}\right) = 0\}$$