Esercizî di Algebra Lineare (Petronio 14/15)

12 ottobre 2012

Nello spazio vettoriale V assegnato considerare l'insieme W descritto e stabilire se W sia un sottospazio vettoriale di V.

Esercizio 1 $V = \mathbb{R}^n$

(a) $W = \left\{ x : \sum_{j=1}^{n} \sqrt{j-1} \cdot x_j = 0 \right\}$

(b) $W = \left\{ x : \sum_{j=1}^{n} x_j^j = 0 \right\}$

(c) $W = \{ x : x_1^2 = x_2^2 \}$

(d) $W = \{ x : x_1^3 = x_2^3 \}$

(e) $W = \{ x : \cos(x_1 + \ldots + x_n) = 1 \}$

Esercizio 2. $V = \mathcal{M}_{m \times n}(\mathbb{R})$

(a) $W = \{ A : (A)_{1,1} \cdot (A)_{m,n} = 0 \}$

(b) $W = \left\{ A : \sum_{j=1}^{\min\{m,n\}} (A)_{j,n+1-j} = 0 \right\}$

(c) $W = \left\{ A : \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\sqrt{i}-\sqrt{j}}{\sqrt{i}+\sqrt{j}} \cdot (A)_{i,j} = 0 \right\}$

(d) $W = \left\{ A : \sum_{j=1}^{\min\{m,n\}} (A)_{j,j}^2 = 0 \right\}$

(e) $W = \{A : \ |(A)_{i,j}| \leqslant 7 \text{ per } 1 \leqslant i \leqslant m, \ 1 \leqslant j \leqslant n\}$

Esercizio 3. $V = \mathbb{R}[t]$

(a) $\{p(t) : \ p(-3) + p''(2) = 0\}$

(b) $\{p(t) : \ p(1) \cdot p'''(-1) = 0\}$

(c) $\left\{ p(t) : \ \sum_{n=0}^{+\infty} (-1)^n \cdot t^n \cdot p^{(n)}(t) = 0 \right\}$

(d) $\{p(t) : \ |p(t)| \leqslant 1 + e^t \ \forall t \in \mathbb{R}\}$

(e) $\{p(t) : \ \deg\left(2p'(t) - 3t^2 p'''(t)\right) \leqslant 5\}$

Esercizio 4. $V = \mathcal{F}(\{a, b, c\}, \mathbb{R})$

(a) $W = \{f : \ f(c) \geqslant 2f(a)\}$

(b) $W = \{f : \ 3f(a) - 7f(b) = 0\}$

(c) $W = \{f : \ f(a) \cdot f(b) + f(a) \cdot f(c) + f(b) \cdot f(c) = 0\}$

(d) $W = \{f : \ \ln\left(1 + |4f(a) - 3f(c)|\right) = 0\}$

(e) $W = \{f : \ |f(b) + f(c)| \leqslant |f(a)|\}$

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Nello spazio vettoriale V dato considerare i vettori $v, w_1, \ldots, w_k$ indicati, stabilire se v appartenga al sottospazio $W = \mathbf{Span}(w_1, \ldots, w_k)$ e in tal caso dire se l'espressione di v come combinazione lineare di $w_1, \ldots, w_k$ sia unica.

Esercizio 5. $V = \mathbb{R}^2$

(a) $v = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$, $w_1 = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$

(b) $v = \begin{pmatrix} \sqrt{6} \\ -2 \end{pmatrix}$, $w_1 = \begin{pmatrix} -3 \\ \sqrt{6} \end{pmatrix}$

(c) $v = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$, $w_1 = \begin{pmatrix} 4 \\ -11 \end{pmatrix}$, $w_2 = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$

(d) $v = \begin{pmatrix} -2 \\ 7 \end{pmatrix}$, $w_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $w_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $w_3 = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$

Esercizio 6. $V = \mathbb{R}^3$

(a) $v = \begin{pmatrix} 4 \\ 5 \\ -2 \end{pmatrix}$, $w_1 = \begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix}$

(b) $v = \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix}$, $w_1 = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$, $w_2 = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$

(c) $v = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$, $w_1 = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$, $w_2 = \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}$

(d) $v = \begin{pmatrix} -1 \\ 3 \\ -8 \end{pmatrix}$, $w_1 = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$, $w_2 = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$, $w_3 = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$

(e) $v = \begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix}$, $w_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $w_2 = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$, $w_3 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

(f) $v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $w_1 = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$, $w_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$, $w_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, $w_4 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

Esercizio 7. $V = \mathbb{R}[t]$

(a) $v = 1 - 2t + 3t^2$, $w_1 = 4 + t - t^2$, $w_2 = -3 + 2t + t^2$

(b) $v = 4 - 3t + t^2$, $w_1 = 2 + t - t^2$, $w_2 = 2 - 9t + 5t^2$

(c) $v = 7 + t - t^2$, $w_1 = 5 - t + 2t^2$, $w_2 = -2 - 2t + 3t^2$, $w_3 = 1 - 5t + 8t^2$

(d) $v = 2t^2$, $w_1 = 1 + t$, $w_2 = 1 - t + t^2$, $w_3 = 3t - t^2$

Esercizio 8. $V = \mathcal{M}_{2\times 2}(\mathbb{R})$

(a) $v = \begin{pmatrix} 2 & 3 \\ -1 & 7 \end{pmatrix}$, $w_1 = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$, $w_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $w_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,

(b) $v = \begin{pmatrix} 5 & -1 \\ 2 & 1 \end{pmatrix}$, $w_1 = \begin{pmatrix} 7 & 1 \\ -1 & 2 \end{pmatrix}$, $w_2 = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$, $w_3 = \begin{pmatrix} 1 & 0 \\ -6 & 2 \end{pmatrix}$,

(c) $v = \begin{pmatrix} 3 & 0 \\ 3 & 4 \end{pmatrix}$,
$w_1 = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$, $w_2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $w_3 = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}$, $w_4 = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$

(c) $v = \begin{pmatrix} 4 & 1 \\ -3 & 2 \end{pmatrix}$,
$w_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $w_2 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$, $w_3 = \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}$, $w_4 = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}$

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Esercizio 9. Nello spazio vettoriale V dato considerare il sottospazio vettoriale W e i vettori $w_1, \ldots, w_k$ indicati. Usando la definizione stabilire se $w_1, \ldots, w_k$ costituiscano un sistema di generatori per W, e in caso affermativo se ogni $w \in W$ abbia un'unica espressione come combinazione lineare di $w_1, \ldots, w_k$.

(a) $V = \mathbb{R}^3$, $W = \{x : \ x_1 + 3x_2 - 2x_3 = 0\}$,
$$w_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \ w_2 = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}, \ w_3 = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$

(b) $V = \mathbb{R}^3$, $W = \{x : \ 5x_1 + 2x_2 - x_3 = 0\}$, $w_1 = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$, $w_2 = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$

(c) $V = \mathbb{R}^3$, $W = \{x : \ 3x_1 - 2x_2 + x_3 = 0\}$, $w_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $w_2 = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$

(d) $V = \mathbb{R}^3$, $W = \{x : \ 4x_1 + 3x_2 - 5x_3 = 0\}$,
$$w_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \ w_2 = \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}, \ w_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

(e) $V = \mathbb{R}^4$, $W = \{x : \ x_1 + 3x_2 + x_3 - x_4 = 0\}$, $w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 5 \end{pmatrix}$, $w_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}$

(f) $V = \mathbb{R}^4$, $W = \{x : \ x_3 = 2x_1 + x_2, \ x_4 = x_1 - 3x_2\}$,
$$w_1 = \begin{pmatrix} 3 \\ -4 \\ 2 \\ 15 \end{pmatrix}, \ w_2 = \begin{pmatrix} -1 \\ 2 \\ 0 \\ -7 \end{pmatrix}$$

(g) $V = \mathbb{R}^4$, $W = \{x : \ x_3 = x_1 - 2x_2, \ x_4 = 3x_1 + 2x_2\}$,
$$w_1 = \begin{pmatrix} 1 \\ 2 \\ -3 \\ 7 \end{pmatrix}, \ w_2 = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 11 \end{pmatrix}, \ w_3 = \begin{pmatrix} 2 \\ -1 \\ 4 \\ 4 \end{pmatrix}$$