Es. Alg Lin. 25/11/2014

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Es. 2

$f : \mathbb{R}^3 \to \mathbb{R}^2$

f definita su $\underset{v_1}{\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}}$, $\underset{v_2}{\begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}}$, $\underset{v_3}{\begin{pmatrix} 1 \\ 0 \\ 2-t^2 \end{pmatrix}}$

<u>se</u> $\{v_1, v_2, v_3\}$ sono base $\Rightarrow$ f è definita in modo unico

Verifico se v_1, v_2, v_3 sono indip.

- v_1, v_2 sono indip. (facile)
- cerco di capire se v_3 è indip. da v_1, v_2

$$v_3 = \begin{pmatrix} 1 \\ 6 \\ 2+t^2 \end{pmatrix} = \boxed{2} \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \boxed{-1} \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 9-t^2 \end{pmatrix}$$

trovat. guardando le 1ª 2 coord.

v_3 è dip. da v_1, v_2 sse $9-t^2=0$ $(t=\pm 3)$

Se $t \neq \pm 3 \Rightarrow v_1, v_2, v_3$ sono base $\Rightarrow \exists!\, f$

Se $t = \pm 3$ $\quad v_3 = 2v_1 - v_2$

$$f(v_1) = \begin{pmatrix} 4 \\ -7 \end{pmatrix} \qquad f(v_2) = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$$f(v_3) = \begin{pmatrix} 3 \\ -15 \end{pmatrix} = \begin{pmatrix} t \\ -15 \end{pmatrix} \Rightarrow t = 3$$

Se $t = -3 \Rightarrow \nexists\, f$ che soddisfa

Se $t = 3 \Rightarrow \exists\, \infty\, f$ che soddisfano

Es. 3 $\quad v_1 = \begin{pmatrix} 3 \\ 8 \end{pmatrix} \quad v_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad w_1 = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} \quad w_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad w_3 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

$f : \mathbb{R}^3 \to \mathbb{R}^2 \qquad f(x) = \begin{pmatrix} 3x_1 - 5x_2 \\ -x_1 + 2x_2 \\ 2x_1 - 3x_2 \end{pmatrix}$

$v = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \qquad f(v) = \begin{pmatrix} -19 \\ 7 \\ -12 \end{pmatrix}$

Verificare $\quad [f(v)]_{\mathcal{C}} = [f]_{B}^{\mathcal{C}} \cdot [v]_B$

$[v]_B \qquad v = \lambda_1 v_1 + \lambda_2 v_2 \qquad \lambda_1 = 19 \quad \lambda_2 = -30$

$\quad \shortparallel$ $\qquad\qquad\qquad\qquad\qquad$ risolvo il sistema

$\begin{pmatrix} 19 \\ -30 \end{pmatrix}$

$[f(v)]_{\mathcal{C}}$ $\qquad \begin{pmatrix}-19\\7\\-12\end{pmatrix} = a_1 w_1 + a_2 w_2 + a_3 w_3$

$\downarrow$ (risolvo il sistema)

$= \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} = \begin{pmatrix}78\\-71\\130\end{pmatrix}$

$[f]^{\mathcal{C}}_{\mathcal{B}}$ $\qquad \left(fB = \mathcal{C}\,[f]^{\mathcal{C}}_{\mathcal{B}} \right)$

$f(v_1) = f\begin{pmatrix}3\\8\end{pmatrix} = \begin{pmatrix}-31\\13\\-18\end{pmatrix} = 132\begin{pmatrix}4\\1\\0\end{pmatrix} - 119\begin{pmatrix}1\\1\\2\end{pmatrix} + 220\begin{pmatrix}-2\\0\\1\end{pmatrix}$

$$f(v_2) = f\begin{pmatrix}2\\5\end{pmatrix} = \begin{pmatrix}-19\\9\\-11\end{pmatrix} = 81\begin{pmatrix}4\\1\\0\end{pmatrix} - 73\begin{pmatrix}1\\1\\2\end{pmatrix} + 135\begin{pmatrix}-2\\0\\1\end{pmatrix}$$

$$[f]^{\mathcal{E}}_{\mathcal{B}} = \begin{pmatrix}132 & 81\\-119 & -73\\220 & 135\end{pmatrix}$$

$$\begin{pmatrix}78\\-71\\130\end{pmatrix} = \begin{pmatrix}132 & 81\\-119 & -73\\220 & 135\end{pmatrix}\begin{pmatrix}19\\-30\end{pmatrix}$$

questo calcolo è la verifica richiesta

ES. (4) $f: V \to W$ $\quad \dim V = \dim W = n$

- Suppongo $\exists\ \mathcal{B}, \mathcal{C}$ basi di V, W t.c.

$[f]^{\mathcal{C}}_{\mathcal{B}}$ è invertibile. Considero $g = \left([f]^{\mathcal{C}}_{\mathcal{B}}\right)^{-1}$

g definisce un'app. lineare da $W \xrightarrow{\gamma} V$

(risp. a basi $\mathcal{C}, \mathcal{B}$)

$g = (m_{ij})$

Se $\mathcal{C} = \{w_1, \dots, w_n\}$ $\quad \mathcal{B} = \{v_1, \dots, v_n\}$

$$\gamma : w_1 \longmapsto \sum m_{i1} v_i$$

$$\vdots$$

$$w_n \longmapsto \sum m_{in} v_i$$

$$[\gamma]^{\mathcal{B}}_{\mathcal{C}} = g \qquad \text{per costruzione}$$

$$[f \circ \gamma]^{\mathcal{C}}_{\mathcal{C}} = [f]^{\mathcal{C}}_{\mathcal{B}} \cdot [\gamma]^{\mathcal{B}}_{\mathcal{C}} =$$

$$= [f]^{\mathcal{C}}_{\mathcal{B}} \cdot g = Id_n$$

$\Rightarrow$ γ è l'inversa di f

• Viceversa se f è invertibile

sia $g = f^{-1} : W \to V$

$$\exists \quad g \circ f = \mathrm{Id}_V$$

$$\mathrm{Id}_n = [g \circ f]_{\mathcal{B}} = [g]^{\mathcal{B}}_{\mathcal{C}} \cdot [f]^{\mathcal{C}}_{\mathcal{B}}$$

$\Rightarrow \ [g]^{\mathcal{B}}_{\mathcal{C}}$ è l'inversa di $[f]^{\mathcal{C}}_{\mathcal{B}}$

Es. 5 a) $V = \mathbb{R}^2$

$$B = \left\{ \overset{v_1}{\begin{pmatrix} 2 \\ -5 \end{pmatrix}}, \overset{v_2}{\begin{pmatrix} 3 \\ 7 \end{pmatrix}} \right\}$$

$$B' = \left\{ \underset{w_1}{\begin{pmatrix} 4 \\ 1 \end{pmatrix}}, \underset{w_2}{\begin{pmatrix} 2 \\ -3 \end{pmatrix}} \right\}$$

$$v_1 = a_{11} w_1 + a_{21} w_2$$

$$\begin{pmatrix} 2 \\ -5 \end{pmatrix} = -\frac{2}{7} \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \frac{11}{7} \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$v_2 = a_{21} w_1 + a_{22} w_2$$

$$\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \frac{23}{14} \begin{pmatrix} 4 \\ 1 \end{pmatrix} - \frac{25}{14} \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$[\mathrm{Id}]_B^{B'} = \begin{pmatrix} -2/7 & 23/14 \\ 11/7 & -25/14 \end{pmatrix}$$

$$w_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \frac{25}{29} \begin{pmatrix} 2 \\ -5 \end{pmatrix} + \frac{22}{29} \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

$$w_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \frac{23}{29} \begin{pmatrix} 2 \\ -5 \end{pmatrix} + \frac{4}{29} \cdot \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

$$[Id]^{B}_{B'} = \begin{pmatrix} 23/29 & 23/29 \\ 22/29 & 4/29 \end{pmatrix} \qquad \boxed{B' = B\,[Id]^{B}_{B'}}$$

si verifica che $\begin{pmatrix} 23/29 & 23/29 \\ 22/29 & 4/29 \end{pmatrix} \begin{pmatrix} -2/7 & 23/14 \\ 4/7 & -25/14 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(b)

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = -\frac{10}{3} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + 5/3 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = 5/3 \,(\,''\,) + 1 \,(\,''\,) - 1/3 \,(\,''\,)$$

$$v_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = -2/3 \,(\,''\,) - 1 \,(\,''\,) + 4/3 \,(\,''\,)$$

Matrice di cambio base da B' a B è

$$M = \begin{pmatrix} -10/3 & 5/3 & -2/3 \\ -2 & 1 & -1 \\ 5/3 & -1/3 & 4/3 \end{pmatrix}$$

Analogamente trovo la matrice da B a B':

$$N = \begin{pmatrix} -1 & 2 & 1 \\ -1 & 10/3 & 2 \\ 1 & -5/3 & 0 \end{pmatrix}$$

$$M \cdot N = \begin{pmatrix} 1 & \overset{0}{\cancel{-\frac{60}{9}+\frac{50}{9}+\frac{10}{9}}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$