Esercizi di Geometria

$\boxed{4}$ 1.

(A) $B = A + A^* = \begin{pmatrix} -2y & -5\sqrt{3}\,i & -5\sqrt{6}\,i \\ 5\sqrt{3}\,i & 1 & \sqrt{2} \\ 5\sqrt{6}\,i & \sqrt{2} & 2x \end{pmatrix}$

$C = A - A^* = \begin{pmatrix} 2xi & 5\sqrt{3}\,i & 5\sqrt{6}\,i \\ 5\sqrt{3}\,i & 0 & \sqrt{2} \\ 5\sqrt{6}\,i & -\sqrt{2} & 2yi \end{pmatrix}$

(B) $d_1(B) = -2y$, $d_2(B) = -2y - 75$,

$d_3(B) = \det(B) =$ (sviluppo rispetto alla I^{a} col.)

$= -2y(2x-2) + 5\sqrt{3}i(10x\sqrt{3}i - 10\sqrt{3}i) + 5\sqrt{6}i(0)$

$= (-4y + 150)\cdot(x-1)$

B positiva $\Rightarrow d_2(B) > 0 \Rightarrow y < -75/2$

$\Rightarrow (-4y+150) > 0$. B pos. $\Rightarrow d_3(B) > 0$

$\Rightarrow x > 1$

Viceversa: $x > 1$ & $y < -75/2 \Rightarrow$

$d_1(B), d_2(B), d_3(B) > 0$, quindi

B positiva $\Leftrightarrow x > 1$ & $y < -75/2$

(C) ②

$$B\Big|_{z=1-\frac{3}{2}i} = \begin{pmatrix} 3 & -5\sqrt{3}\,i & -5\sqrt{6}\,i \\ 5\sqrt{3}\,i & 1 & \sqrt{2} \\ 5\sqrt{6}\,i & \sqrt{2} & 2 \end{pmatrix}$$

$$p_B(t) = \begin{vmatrix} t-3 & 5\sqrt{3}\,i & 5\sqrt{6}\,i \\ -5\sqrt{3}\,i & t-1 & -\sqrt{2} \\ -5\sqrt{6}\,i & -\sqrt{2} & t-2 \end{vmatrix} =$$

$$= (t-3)\left[t^2-3t+\cancel{2}-\cancel{2}\right] +$$

$$5\sqrt{3}\,i\left[5\sqrt{3}\,i\cdot t - 10\cancel{\sqrt{3}}i + 10\cancel{\sqrt{3}}\,i\right] +$$

$$-\,5\sqrt{6}\,i\left[-5\cancel{\sqrt{6}}\,i - 5\sqrt{6}\,i\,t + 5\cancel{\sqrt{6}}\,i\right] =$$

$$= t\,(t-3)^2 - 75t - 150t =$$

$$= t\,(t^2-6t-216) = t\,(t-18)(t+12)$$

$\lambda_1 = 0$: risolviamo con Cramer rispetto a x, y il sistema dato da $(B - 0\cdot I)\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$, ponendo $z=1$

$$x = \frac{\begin{vmatrix} 5\sqrt{6}\,i & -5\sqrt{3}\,i \\ -\sqrt{2} & 1 \end{vmatrix}}{\begin{vmatrix} 3 & -5\sqrt{3}\,i \\ 5\sqrt{3}\,i & 1 \end{vmatrix}} = \frac{5\sqrt{6}\,i - 5\sqrt{6}\,i}{3-75} = \frac{0}{-72} = 0$$

$$y = \frac{\begin{vmatrix} 3 & 5\sqrt{6}\,i \\ 5\sqrt{3}\,i & -\sqrt{2} \end{vmatrix}}{-72} = \frac{-3\sqrt{2} + 75\sqrt{2}}{-72} = -\sqrt{2}$$

$$\Rightarrow v_1 = \frac{1}{\sqrt{3}} \cdot \begin{pmatrix} 0 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

$\lambda_2 = 18$: risolviamo rispetto a x, z il sistema delle prime due righe di $(B - 18I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$, ponendo $y = 1$

$$x = \frac{\begin{vmatrix} 5\sqrt{3}\,i & -5\sqrt{6}\,i \\ 17 & \sqrt{2} \end{vmatrix}}{\begin{vmatrix} -15 & -5\sqrt{6}\,i \\ 5\sqrt{3}\,i & \sqrt{2} \end{vmatrix}} = \frac{90\sqrt{6}\,i}{-90\sqrt{2}} = -\sqrt{3}\,i$$

$$z = \frac{\begin{vmatrix} -15 & 5\sqrt{3}\,i \\ 5\sqrt{3}\,i & 17 \end{vmatrix}}{-90\sqrt{2}} = \frac{-255 + 75}{-90\sqrt{2}} = \sqrt{2}$$

$$\Rightarrow v_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} -i\sqrt{3} \\ 1 \\ \sqrt{2} \end{pmatrix}$$

$\lambda_3 = -12$: risolviamo rispetto a x, z il sistema dato dalle prime due righe di $(B + 12I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$ ponendo $y = 1$

$$x = \frac{\begin{vmatrix} 5\sqrt{3}\,i & -5\sqrt{6}\,i \\ -13 & \sqrt{2} \end{vmatrix}}{\begin{vmatrix} 15 & -5\sqrt{6}\,i \\ 5\sqrt{3}\,i & \sqrt{2} \end{vmatrix}} = \frac{5\sqrt{6}\,i - 65\sqrt{6}\,i}{-60\sqrt{2}} = i\sqrt{3}$$

$$z = \frac{\begin{vmatrix} 15 & 5\sqrt{3}\,i \\ 5\sqrt{3}\,i & -13 \end{vmatrix}}{-60\sqrt{2}} = \sqrt{2}$$

$$\Rightarrow \quad v_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} i\sqrt{3} \\ 1 \\ \sqrt{2} \end{pmatrix}$$

(D) $$C\Big|_{z=0} = \begin{pmatrix} 0 & 5\sqrt{3}\,i & 5\sqrt{6}\,i \\ 5\sqrt{3}\,i & 0 & \sqrt{2} \\ 5\sqrt{6}\,i & -\sqrt{2} & 0 \end{pmatrix}$$

$$\begin{vmatrix} t & -5\sqrt{3}\,i & -5\sqrt{6}\,i \\ -5\sqrt{3}\,i & t & -\sqrt{2} \\ -5\sqrt{6}\,i & \sqrt{2} & t \end{vmatrix} = t\,(t^2+2) +$$

$$+\ 5\sqrt{3}\,i\,(-5\sqrt{3}\,i\,t + 10\sqrt{3}\,i)$$

$$-\ 5\sqrt{6}\,i\,(5\sqrt{6}\,i + t\cdot 5\sqrt{6}\,i) =$$

$$= t\,(t^2+2+225) = t\,(t+\sqrt{227}\,i)(t-\sqrt{227}\,i)$$

Risolviamo rispetto a x, y il sistema dato dalle prime due righe di $B\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$, ponendo $z = 1$.

$$x = \frac{\begin{vmatrix} -5\sqrt{6}\,i & 5\sqrt{3}\,i \\ -\sqrt{2} & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 5\sqrt{3}\,i \\ 5\sqrt{3}\,i & 0 \end{vmatrix}} = \frac{5\sqrt{6}\,i}{75} = \frac{\sqrt{6}}{15}\,i$$

$$y = \frac{\begin{vmatrix} 0 & -5\sqrt{6}\,i \\ 5\sqrt{3}\,i & -\sqrt{2} \end{vmatrix}}{75} = \frac{-75\sqrt{2}}{75} = -\sqrt{2}$$

$$\Rightarrow \quad v_1 = \begin{pmatrix} \sqrt{6}/15 \; i \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

2.(A)

$$\begin{vmatrix} t-2k^2+k+19 & 3k^2-4k-44 & k^2-k-10 \\ 8 & t-k^2-19 & -4 \\ -2k^2+2k-12 & 6k^2-8k+32 & t+k^2-2k+5 \end{vmatrix} =$$

$$\overset{\mathrm{I}c \to \mathrm{I}c+2\cdot\mathrm{III}c}{=} \begin{vmatrix} t-k-1 & 3k^2-4k-44 & k^2-k-10 \\ 0 & t-k^2-19 & -4 \\ 2t-2k-2 & 6k^2-8k+32 & t+k^2-2k+5 \end{vmatrix}$$

$$\overset{\mathrm{III}R \to \mathrm{III}R-2\mathrm{I}R}{=} \begin{vmatrix} t-k-1 & 3k^2-4k-44 & k^2-k-10 \\ 0 & t-k^2-19 & -4 \\ 0 & 120 & t-k^2+25 \end{vmatrix} \overset{B(t,k)}{=}$$

$$= (t-k-1)\left(t^2-(2k^2-6)t+k^4-5\right)$$

$$= (t-(k+1))(t-(k^2-5))(t-(k^2-1))$$

(B) $k+1 = k^2-5 \iff$

$k^2-k-6=0 \iff k = \frac{1\pm 5}{2} \begin{cases} 3 \\ -2 \end{cases}$

$k+1 = k^2-1 \iff k^2-k-2=0 \iff$

$k = \frac{1\pm 3}{2} \begin{cases} 2 \\ -1 \end{cases}$

$\cancel{k^2}-1 = \cancel{k^2}-5$ mai

$\Rightarrow$ A ha 3 autovalori distinti per $k \notin \{-1,-2,2,3\}$

$$B(0,-1) = \begin{pmatrix} 0 & -37 & -8 \\ 0 & -20 & -4 \\ 0 & 120 & 24 \end{pmatrix}$$ ha rango 2

$\Rightarrow$ per $k=-1$ l'autovalore $k+1=0$ ha molteplicità geometrica $=1$ $\Rightarrow$ A non è diag. le.

$$B(3,2) = \begin{pmatrix} 0 & -40 & -8 \\ 0 & -20 & -4 \\ 0 & 120 & 24 \end{pmatrix}$$ ha rango 1

$\Rightarrow$ per $k=2$ l'autovalore $k+1=3$ ha molt. tà geometrica $=2$. L'altro autovalore $k^2-5=-1$ ha molt. tà geom. $=1$

$\Rightarrow$ A è diag. le per $k=2$

$$B(-1,-2) = \begin{pmatrix} 0 & -24 & -4 \\ 0 & -24 & -4 \\ 0 & 120 & 20 \end{pmatrix}$$ ha rango = 1

$\Rightarrow$ come nel caso precedente A diag.le per $k=-2$

$$B(4,3) = \begin{pmatrix} 0 & -29 & -4 \\ 0 & -24 & -4 \\ 0 & 120 & 20 \end{pmatrix}$$ ha rango 2

$\Rightarrow$ per $k=3$ A non è diag.le.

$\boxed{6}$ 2. $\alpha : (-1,+\infty) \to \mathbb{R}^3$

$$\alpha(s) = \begin{pmatrix} s \cdot e^s \\ \ln(1+s) \\ s+2s^2+s^3 \end{pmatrix}$$

(A) $\ln(1+s_1) = \ln(1+s_2) \Rightarrow s_1 = s_2$

$\Rightarrow \alpha$ è semplice

le componenti di α sono C^1. Inoltre,

$$\alpha'(s) = \begin{pmatrix} e^s(s+1) \\ 1/s+1 \\ 1+4s+3s^2 \end{pmatrix}$$ $\Rightarrow$ la prima (e la seconda) componente di $\alpha'(s)$ non è mai nulla

$\Rightarrow \alpha$ è regolare.

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(B)

$$\int_\beta z\,dy = \int_0^1 \left(\frac{s+2s^2+s^3}{1+s}\right) ds = \int_0^1 (s+s^2)\, ds$$

$$= \left[\tfrac{1}{2}s^2 + \tfrac{1}{3}s^3\right]_0^1 = 5/6$$

(C)

$$\int_\gamma \frac{x\,dx + y\,dy}{x^2+y^2} = \frac{1}{2}\int_\gamma d\left(\ln(x^2+y^2)\right)$$

$$= \frac{1}{2}\ln(x^2+y^2)\Big|_1^2 = \frac{1}{2}\ln\left(\frac{4\cdot e^4 + \ln(3)^2}{e^2 + \ln(2)^2}\right).$$

(D)

$$t(0) = \alpha'(0) / \|\alpha'(0)\| = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\alpha''(s) = \begin{pmatrix} e^s\cdot(s+2) \\ -1/(s+1)^2 \\ 4+6s \end{pmatrix} \quad \alpha'''(s) = \begin{pmatrix} e^s\cdot(s+3) \\ 2/(s+1)^3 \\ 6 \end{pmatrix}$$

$$\alpha'(0)\wedge\alpha''(0) = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 1 & 1 \\ 2 & -1 & 4 \end{vmatrix} = \begin{pmatrix}5\\-2\\-3\end{pmatrix}$$

$$\Rightarrow b(0) = \frac{1}{\sqrt{38}}\begin{pmatrix}5\\-2\\-3\end{pmatrix}$$

$$n(0) = b(0)\wedge t(0) = \frac{1}{\sqrt{114}}\begin{vmatrix} e_1 & e_2 & e_3 \\ 5 & -2 & -3 \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{\sqrt{114}}\begin{pmatrix}1\\-8\\7\end{pmatrix}$$

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$$k(0) = \frac{\| \alpha'(0) \wedge \alpha''(0) \|}{\| \alpha'(0) \|^3} = \frac{\sqrt{38}}{3\sqrt{3}}$$

$$\tau(0) = \frac{\langle \alpha'(0) \wedge \alpha''(0) \mid \alpha'''(0) \rangle}{\| \alpha'(0) \wedge \alpha''(0) \|^2} =$$

$$= \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -2 \\ -3 \end{pmatrix} \Big/ 38 = \frac{1}{38} (15 - 4 - 18)$$

$$= -7/38$$