Geometria 8/4/14

Merc + Ven : Lisca (es. diago)

Giov : io 8:30 - 10:00

— o —

③⓪ $\mathbb{C}^2, \langle . | . \rangle_{\mathbb{C}^2}$: ortonormalizzare

$$\underset{w_1}{\begin{pmatrix} 1+i \\ -2 \end{pmatrix}} \quad \underset{w_2}{\begin{pmatrix} 3-i \\ 2i \end{pmatrix}}$$

$$u_1 = \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{1+1+4}} \begin{pmatrix} 1+i \\ -2 \end{pmatrix} = \dots$$

$$\tilde{u}_2 = w_2 - \langle w_2 | u_1 \rangle \cdot u_1 = \begin{pmatrix} 3-i \\ 2i \end{pmatrix} - \frac{(3-i)(1-i) + 2i\cdot(-2)}{6} \begin{pmatrix} 1+i \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 3-i \\ 2i \end{pmatrix} - \frac{2-4i-4i}{6} \begin{pmatrix} 1+i \\ -2 \end{pmatrix} = \dots = \frac{2}{3} \begin{pmatrix} 2 \\ 1-i \end{pmatrix}$$

$$u_2 = \frac{\tilde{u}_2}{\|\tilde{u}_2\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 1-i \end{pmatrix}.$$

30bis In $\mathbb{C}^2$ ortonormalizzare $\underset{w_1}{\begin{pmatrix} 2-i \\ 1+3i \end{pmatrix}} \underset{w_2}{\begin{pmatrix} 1+i \\ 3-2i \end{pmatrix}}$.

$$u_1 = \frac{1}{\sqrt{15}} \begin{pmatrix} 2-i \\ 1+3i \end{pmatrix}$$

So che $u_2 \perp u_1$, $\|u_2\|$ e

$$(u_1, u_2) = \begin{pmatrix} (>0) & * \\ 0 & (>0) \end{pmatrix} \cdot (w_1, w_2)$$

$$\Rightarrow \det(u_1, u_2) = \lambda \cdot \det(w_1, w_2) \qquad \lambda > 0. \quad \lambda \in \mathbb{R}$$

Altra strategia: trovo v_2 che sia $\perp$ a u_1;
trovo $\alpha \in \mathbb{C}$ t.c. $\det(u_1, \alpha v_2) = \lambda \cdot \det(w_1, w_2)$ $\lambda > 0, \lambda \in \mathbb{R}$
e infine normalizzo.

$$w_1 = \begin{pmatrix} 2-i \\ 1+3i \end{pmatrix} \qquad v_2 = \xcancel{\begin{pmatrix} 1+3i \\ -2+i \end{pmatrix}} = \begin{pmatrix} -1+3i \\ 2+i \end{pmatrix}$$

$$\det\left(\begin{pmatrix} 2-i \\ 1+3i \end{pmatrix} \alpha \begin{pmatrix} -1+3i \\ 2+i \end{pmatrix} \right) = \lambda \cdot \det \begin{pmatrix} 2-i & 1+i \\ 1+3i & 3-2i \end{pmatrix}$$

$$\alpha = \lambda \cdot (6-11i)$$

$$\alpha = 6-11i$$

$$\Rightarrow u_2 = (6-11i) \begin{pmatrix} -1+3i \\ 2+i \end{pmatrix} \Big/ \| \ldots \| -$$

(31) Calcolare proiez ortog di $\underset{v}{\begin{pmatrix} 2 \\ i \\ -1 \end{pmatrix}}$ su Span$\left(\underset{w_1}{\begin{pmatrix} i \\ 0 \\ 1+i \end{pmatrix}} \underset{w_2}{\begin{pmatrix} 0 \\ 1 \\ 1-i \end{pmatrix}}\right)$

$$(w_1, w_2) \overset{G-S}{\rightsquigarrow} (u_1, u_2)$$

$$p_W(v) = \langle v | u_1 \rangle \cdot u_1 + \langle v | u_2 \rangle \cdot u_2 \qquad (\ldots)$$

Oss: W ha equaz:

$$-(1+i)z_1 - (1+i)z_2 + i z_3 = 0$$

$$\Rightarrow W^{\perp} = \cancel{\begin{pmatrix} 1+i \\ 1+i \\ -i \end{pmatrix}^{\perp}} = \begin{pmatrix} 1-i \\ 1-i \\ i \end{pmatrix}^{\perp}$$

$$\Longrightarrow P_W(v) = v - P_{W^\perp}(v) = \begin{pmatrix} 2 \\ i \\ -1 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 2 \\ i \\ -1 \end{pmatrix} \middle| \begin{pmatrix} 1-i \\ 1-i \\ i \end{pmatrix} \right\rangle}{\left\| \begin{pmatrix} 1-i \\ 1-i \\ i \end{pmatrix} \right\|^2} \cdot \begin{pmatrix} 1-i \\ 1-i \\ i \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ i \\ -1 \end{pmatrix} - \frac{(1+i)2 + (1+i) \cdot i - i(-1)}{5} \begin{pmatrix} 1-i \\ 1-i \\ i \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 5-3i \\ 2i-5 \\ -1-i \end{pmatrix}.$$

(32) $A_k = \begin{pmatrix} 2 & i\sqrt{3} \\ -i\sqrt{3} & k \end{pmatrix} \qquad f_k = \langle \cdot | \cdot \rangle_{A_k}$

$\left(f_k(v,w) = {}^t\bar{w} \cdot A_k \cdot v \right)$

sempre sesquilineare

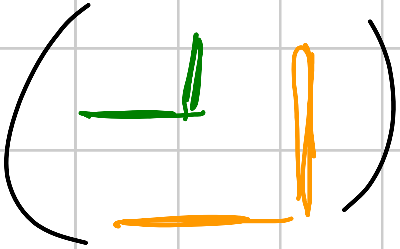
(a) Per quali k la f_k è hermitiana?

f_k hermitiana $\iff$ A_k hermitiana $\left(A_k^* = A_k\right)$

$\iff k \in \mathbb{R}$

(b) Per quali k la f_k è def. pos?

f_k def. pos. $\iff$ $\underline{d_1 > 0}$, $\underline{d_2 > 0}$



$\iff 2k-3>0 \iff k > \frac{3}{2}$.

(c) + (d) : $\langle \cdot | \cdot \rangle = f_3$ $(k=3)$

(c) Trovare $v \in \mathbb{C}^2$ con $\underline{v_1 \in i \cdot \mathbb{R}}$, $\|v\| = 1$, $v \perp \begin{pmatrix} 1-i \\ 1+i \end{pmatrix}$ (rispetto a $\langle \cdot | \cdot \rangle$).

Prima imponiamo $\perp \begin{pmatrix} 1-i \\ 1+i \end{pmatrix}$ e $\underline{\begin{pmatrix} i \\ \alpha \end{pmatrix}}$ poi ± normalizziamo;

$$\begin{pmatrix} 1+i, & 1-i \end{pmatrix} \cdot \begin{pmatrix} 2 & i\sqrt{3} \\ -i\sqrt{3} & 3 \end{pmatrix} \cdot \begin{pmatrix} i \\ \alpha \end{pmatrix} = 0$$

(conti): $\alpha = \frac{1}{2}(\sqrt{3}-1)$

$$\Rightarrow \pm \frac{1}{\sqrt{k}} \cdot \begin{pmatrix} 2i \\ \sqrt{3}-1 \end{pmatrix}$$

$$k = (-2i, \sqrt{3}-1) \begin{pmatrix} 2 & i\sqrt{3} \\ -i\sqrt{3} & 3 \end{pmatrix} \begin{pmatrix} 2i \\ \sqrt{3}-1 \end{pmatrix} = \dots$$

(d) Calcolare proiez. ortog di $\begin{pmatrix} i \\ 1 \end{pmatrix}$ su Span$\begin{pmatrix} 1 \\ i \end{pmatrix}$:

$$\frac{\left\langle \begin{pmatrix} i \\ 1 \end{pmatrix} \middle| \begin{pmatrix} 1 \\ i \end{pmatrix} \right\rangle}{\left\| \begin{pmatrix} 1 \\ i \end{pmatrix} \right\|^2} \cdot \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= \underbrace{\frac{(1,-i)\begin{pmatrix} 2 & i\sqrt{3} \\ -i\sqrt{3} & 3 \end{pmatrix}\begin{pmatrix} i \\ 1 \end{pmatrix}}{(1,-i)\begin{pmatrix} 2 & i\sqrt{3} \\ -i\sqrt{3} & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix}}}_{\frac{i}{2\sqrt{3}-5}} \cdot \begin{pmatrix} 1 \\ i \end{pmatrix}$$

㉝ Trovare v che soddisfa certe condizioni.

(c) $V = \mathbb{C}^2, \langle \cdot | \cdot \rangle_A \qquad A = \begin{pmatrix} 3 & 1+2i \\ 1-2i & 5 \end{pmatrix}$

(A hermitiana ; $d_1 = 3 > 0$

$d_2 = 15 - 5 > 0$

$\Rightarrow$ prod. scal.)

Trovare v unitario, $\perp \begin{pmatrix}3-i\\2+5i\end{pmatrix}$, $v_1 \in \mathbb{R}$ _

Impongo $\begin{pmatrix}1\\\alpha\end{pmatrix}$ sia $\perp \begin{pmatrix}3-i\\2+5i\end{pmatrix}$ poi $\pm$ normalizzo;

$$(3+i, 2-5i)\begin{pmatrix}3 & 1+2i\\1-2i & 5\end{pmatrix}\begin{pmatrix}1\\\alpha\end{pmatrix}=0$$

da cui $\alpha = \frac{1}{445}(-119+48i)$

$$\Rightarrow \pm \frac{1}{\sqrt{k}} \cdot \begin{pmatrix} 445 \\ -119+48i \end{pmatrix}$$

$$k = (445, -119-48i) \begin{pmatrix} 3 & 1+2i \\ 1-2i & 5 \end{pmatrix} \begin{pmatrix} 445 \\ -119+48i \end{pmatrix} =$$

(e) $V = \mathbb{C}^3, \langle \cdot | \cdot \rangle_{\mathbb{C}^3}$ v unitario,

somma $\underset{\text{II}}{\text{coord.}}$ reale , , $\overset{\text{I}}{\text{perp.}}$ a $\underset{\text{III}}{\begin{pmatrix} 2 \\ 1+i \\ 3-i \end{pmatrix}}$ e $\begin{pmatrix} 3+i \\ -2i \\ 1-i \end{pmatrix}$

Oss: la condiz. si conserve moltiplicando per $\lambda \in \mathbb{C}$.

Dunque: cerco v che soddisfa III;

divido per somma coordinate

± normalizzo.

$$\begin{pmatrix} 2 \\ 1+i \\ 3-i \end{pmatrix}, \begin{pmatrix} 3+i \\ -2i \\ 1-i \end{pmatrix}; \xcancel{\begin{pmatrix} (1+i)(1-i)-(3-i)(-2i) \\ -(2(1-i)-(3-i)(3+i)) \\ 2(-2i)-(1+i)(3+i) \end{pmatrix} = \begin{pmatrix} 4+6i \\ 8+2i \\ -2-8i \end{pmatrix}}$$

invece $\begin{pmatrix} 2-3i \\ 4-i \\ -1+4i \end{pmatrix}$.

Somma delle coord: già reale.

$$\Longrightarrow \pm \frac{1}{\sqrt{4+9+16+1+1+16}} \begin{pmatrix} 2-3i \\ 4-i \\ -1+4i \end{pmatrix}.$$

$\underbrace{\hphantom{xxxxxxxxxxxx}}_{4,7}$

(f) $V=\mathbb{C}^3$, $\langle\cdot|\cdot\rangle_{\mathbb{C}^3}$, v unitario, $v \perp \begin{pmatrix}1+i\\2\\3-i\end{pmatrix}, \begin{pmatrix}3+i\\1-i\\-2i\end{pmatrix}$

$v_3 \in i\cdot\underline{\mathbb{R}}$ _

Strategie

→ $w=\begin{pmatrix}\alpha\\\beta\\i\end{pmatrix}$; impongo $\perp \begin{pmatrix}\vdots\end{pmatrix}\begin{pmatrix}\vdots\end{pmatrix}$, trovo α,β e ± normalizzo

→ trovo $w \perp \begin{pmatrix}\vdots\end{pmatrix}\begin{pmatrix}\vdots\end{pmatrix}$, prendo $\frac{i}{w_3}\cdot w$, e ± normalizzo _

$$\begin{pmatrix} 1+i \\ 2 \\ 3-i \end{pmatrix}, \begin{pmatrix} 3+i \\ 1-i \\ -2i \end{pmatrix} \rightsquigarrow \xcancel{\begin{pmatrix} 2\cdot(-2i) - (3-i)(1-i) \\ -\left((1+i)(-2i) - (3-i)(3+i)\right) \\ (1+i)(1-i) - 2(3+i) \end{pmatrix} = \begin{pmatrix} -2 \\ 8+2i \\ -4-2i \end{pmatrix}}$$

invece $\begin{pmatrix} 1 \\ -4+i \\ 2-i \end{pmatrix}$; poi

$$i\cdot(2+i)\begin{pmatrix} 1 \\ -4+i \\ 2-i \end{pmatrix} = \begin{pmatrix} -1+2i \\ \dots \\ 5i \end{pmatrix}$$

$$\Rightarrow \pm \frac{1}{\sqrt{115}} \begin{pmatrix} 2i-1 \\ 2-8i \\ 5i \end{pmatrix} .$$

③④ Trovare la proiez. ortog. su W _

(c) $\mathbb{C}^3, \langle\cdot|\cdot\rangle_{\mathbb{C}^3}$, $W = \text{Span}\begin{pmatrix} 2-i \\ 1+i \\ 3-2i \end{pmatrix}$

$$P_W(z) = \frac{(2+i)z_1 + (1-i)z_2 + (3+2i)z_3}{4+1 \quad +1+1 \quad +9+4} \cdot \begin{pmatrix} 2-i \\ 1+i \\ 3-2i \end{pmatrix}$$

$$= \frac{1}{20}\begin{pmatrix} 5 & 1-3i & 8+i \\ 1+3i & 2 & 1+5i \\ 8-i & 1-5i & 13 \end{pmatrix} \cdot z$$

$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{A}$; falls $A^* = A$ ✓

$A \cdot A = A$.

(d) $V = \mathbb{C}^3$, $W = \{z \in \mathbb{C}^3 : iz_1 + (1-2i)z_2 + (3+i)z_3 = 0\}$

$$\Rightarrow W = \begin{pmatrix} -i \\ 1+2i \\ 3-i \end{pmatrix}^{\perp}$$

$$\Rightarrow P_W(z) = z - P_{W^\perp}(z) = z - \frac{iz_1 + (1-2i)z_2 + (3+i)z_3}{1 \quad +1+4 \quad +9+1} \cdot \begin{pmatrix} -i \\ 1+2i \\ 3-i \end{pmatrix}$$

$$= \underbrace{\frac{1}{16}\begin{pmatrix} 15 & 2+i & \cdot \\ 2-i & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}}_{A} \cdot z$$

$$A^* = A = A \cdot A .$$

(7) $\mathbb{C}^3, \langle \cdot | \cdot \rangle$

$$W : \begin{cases} z_1 + 2z_2 + iz_3 = 0 \\ (2-i)z_1 + 2iz_2 - z_3 = 0 \end{cases}$$

W = retta ortog a vett. $\begin{pmatrix}1\\2\\-i\end{pmatrix}$ e $\begin{pmatrix}2+i\\-2i\\-1\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}2\cdot(-1)-(-1)(2+i)\\-(1\cdot(-1)-(-i)(2+i))\\1\cdot(-2i)-2(2+1)\end{pmatrix}=\begin{pmatrix}a\\b\\c\end{pmatrix}$$

$$\Rightarrow p_W(z)=\frac{\bar{a}\cdot z_1+\bar{b}\cdot z_2+\bar{c}\cdot z_3}{\sqrt{|a|^2+|b|^2+|c|^2}}\cdot\begin{pmatrix}a\\b\\c\end{pmatrix} -$$

$$(\dots)$$

(h) $V = \mathbb{C}_{\leq 1}[U]$

$$\langle p(z) | q(z) \rangle = p(2) \cdot \overline{q(2)} + 2\, p(-i) \cdot \overline{q(-i)}$$

(Prod. scal : lin sx ✓

antilin dx ✓

hermitiano ✓

def. pos : $\langle p(z) | p(z) \rangle = |p(2)|^2 + 2\,|p(-i)|^2$

sempre ≥ 0 ; nullo solo $p(2) = p(-i) = 0$

dunque per $p(z) = 0$ (si usa deg ≤ 1) _)

Trovare proiez. ortog. su $\mathrm{Span}\left(1-2i+(3+i)\cdot z\right)$;

$$p_W(\alpha+\beta z)=\frac{(\alpha+2\beta)\cdot\overline{\left(1-2i+2(3+i)\right)}+2\cdot(\alpha-i\beta)\overline{\left(1+2i-i(3+i)\right)}}{|\,7\,|^2+2\cdot|\,2-i\,|^2}\cdot$$

$$\cdot\left(1-2i+(3+i)\cdot z\right)$$

$$=\frac{7\,(\alpha+2\beta)+2\,(2+i)\,(\alpha-i\beta)}{49+10}\left(1-2i+(3+i)\cdot z\right)=\dots$$