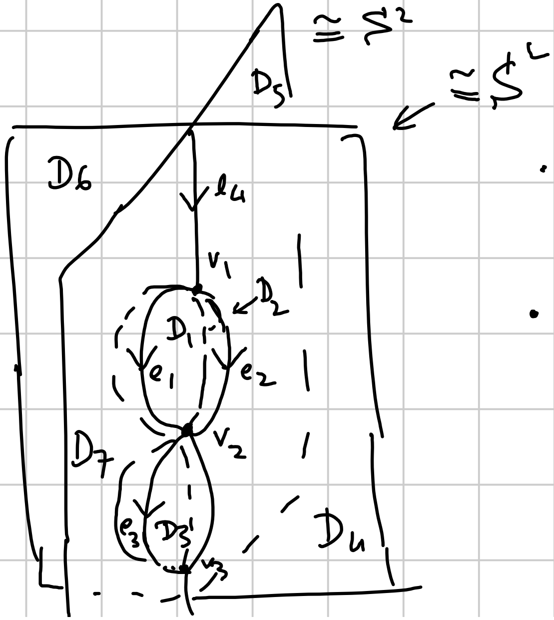
ETA 17/12/13

(2)

$X =$



B_1
B_2

azione:

- rotaz di π sul bordo delle palle sup.
- rotaz. di $\pi/2$ sul bordo delle palle inf

$Y/\sim$

$\partial_1 e_1 = v_2 - v_1 \quad \partial_1 e_3 = v_3 - v_2 \quad \partial_2 D_1 = e_1 - e_2 \quad \partial_2 D_2 = e_2 - e_1$

$\partial_1 e_2 = v_2 - v_1 \quad \partial_1 e_4 = v_1 - v_3 \quad \partial_2 D_3 = 0 \quad \partial_2 D_4 = e_2 + e_3 + e_4 = \partial D_6$

$\partial D_5 = e_1 + e_3 + e_4 = \partial D_7$

$\partial_3 B_1 = 2D_1 + 2D_2$

$\partial_3 B_2 = 4D_3$

$$H_0 = \mathbb{Z} \qquad H_1 = \frac{\mathbb{Z}\langle e_1 - e_2, e_1 + e_3 + e_4 \rangle}{\langle e_1 - e_2, e_1 + e_3 + e_4 \rangle} = 0$$

$$H_2 = \frac{\langle D_1 + D_2, D_3, D_6 - D_4, D_7 - D_5, D_1 + D_4 - D_5 \rangle}{\langle 2(D_1 + D_2), 4D_3 \rangle}$$

$$= \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}^3$$

$H_3 = 0$

③ $X = S^3/_{Q_8}$ $\quad \pi_1(X) \cong Q_8$

$H_1(X) = Ab(Q_8)$

$Q_8 = \mathbb{Z}/4 \ltimes \mathbb{Z}/2 \qquad \mathbb{Z}/4 = \{1, i, -1, -i\} = \langle i \rangle$

$j \cdot i \, j^{-1} = j \cdot i \cdot (-j) = -j \cdot k = -i$ (finite)

$Ab(Q_8)$: $[i,j] = i \cdot j \cdot (-i) \cdot (-j) = k \cdot k = -1$

$\{\underset{\mathbb{1}}{\pm 1}, \underset{I}{\pm i}, \underset{J}{\pm j}, \underset{K}{\pm k}\} = Q_8/_{\{1,-1\}}$

$= \langle I, J \mid I^2 = J^2 = 1 \quad (I \cdot J = k) \rangle = \mathbb{Z}/2 \oplus \mathbb{Z}/2$

$S^3 = \{ a + ib + jc + k d \in \mathbb{H} : a^2 + b^2 + c^2 + d^2 = 1 \}$

Trovo dom. fond. $\subset S^3_+ = \{ a \geq 0 \}$

$S^3_+ \cong D^3$

$p : S^3_+ \to D^3 \qquad a + ib + jc + kd \mapsto (b, c, d)$

$h : D^3 \to S^3_+ \qquad (x, y, z) \mapsto \sqrt{1 - (x^2 + y^2 + z^2)} + ix + jy + kz$

Azione residua su D^3:

$I : (x, y, z) \xrightarrow{h} (\ldots) \xrightarrow{\pm i} (\ldots) \xrightarrow{p} D^3$

$\pm (-x + i\sqrt{\ldots} - jz + ky) \longrightarrow \begin{cases} \sqrt{\ldots} & x \leq 0 \\ -(\ ) & x \geq 0 \end{cases}$

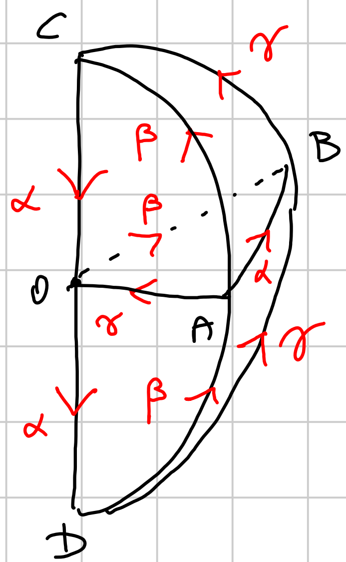
$$I : (x,y,z) \longmapsto \begin{cases} (-\sqrt{\ }, z, -y) & x \geq 0 \\ (\sqrt{\ }, -z, y) & x \leq 0 \end{cases}$$

$$J : (x,y,z) \longmapsto \begin{cases} (-z, -\sqrt{\ }, x) & y \geq 0 \\ (z, \sqrt{\ }, -x) & y \leq 0 \end{cases}$$

$$K : (x,y,z) \longmapsto \begin{cases} (y, -x, -\sqrt{\ }) & z \geq 0 \\ (-y, x, \sqrt{\ }) & z \leq 0 \end{cases}$$

Usando I ottengo $x \geq 0$. Se $y \leq 0$ ottengo $y \geq 0$ usando J se $z \geq 0$ (non tocco $x \geq 0$), e usando K se $z \leq 0$ (non tocco $x \geq 0$).

Dom. fond.:



$ABC \cong ODB$ R

$ABD \cong COA$ S

$OBC \cong DAO$ T

$\mathcal{B}$

$A = O = B = C = D =: P$

$U =$ tutto

$\partial_0 P = 0$; $\partial_1 \alpha = \partial_1 \beta = \partial_1 \gamma = 0$; $\partial_2 R = \alpha + \gamma - \beta$ $\partial_3 U = 0$

$\partial_2 S = \gamma - \alpha - \beta$

$\partial_2 T = \alpha + \beta + \gamma$

$H_0 = \mathbb{Z}$ $\qquad H_1 = \dfrac{\langle \alpha, \beta, \gamma \rangle}{\begin{array}{c}\beta = \alpha + \gamma \\ 2\alpha = 0 \\ 2(\alpha+\gamma) = 0\end{array}} = \mathbb{Z}/2 \oplus \mathbb{Z}/2$

$H_2 = 0 \qquad H_3 = \mathbb{Z}$

—— o ——

$0 \longrightarrow K \xrightarrow{i} H \xrightarrow{j} A \rightarrow$ ris. lib.

$$\mathrm{Ext}(A,G) = \frac{\mathrm{Hom}(K,G)}{\mathrm{Im}(i^*)}$$

$$0 \leftarrow \mathrm{Hom}(K,G) \xleftarrow{i^*} \mathrm{Hom}(H,G) \leftarrow \mathrm{Hom}(A,G) \leftarrow 0$$

può non essere esatta qui

<u>Lem</u>: se $0 \to K \xrightarrow{i} H \to A \to 0$ splitta allora

$\mathrm{Ext}(A,G) = 0$ (cioè l'altra è esatta).

Vero in partic. se A è libero.

<u>Dim</u>: $\exists\, q : H \to K$ t.c. $q \circ i = id_K$

Se $\alpha \in \mathrm{Hom}(K,G)$ pongo $\beta = \alpha \circ q \in \mathrm{Hom}(H,G)$

e ho $\alpha = i^*(\beta)$. $\boxtimes$

<u>Thm</u>: $\exists$ split esatta

$0 \to \mathrm{Ext}(H_{n-1},G) \to H^n(C;G) \to \mathrm{Hom}(H_n,G) \to 0$

<u>Pf</u>: $h_n : H^n(C;G) \to \mathrm{Hom}(H_n,G)$

$h_n([\varphi])([u]) = \varphi$ ben def, surj.

Splitta:

$0 \to Z_n \to C_n \to B_{n-1} \to 0$ splitta

$\Rightarrow \exists\, q_n : C_n \to Z_n$ identità su Z_n. Sia

$j_n : Hom(H_n; G) \to C^n(G)$

$j_n(\eta)(w) = \eta([q_n(w)])$

Affermo che $j_n(\eta) \in Z^n(G)$

$(\delta_n(j_n(\eta)))(u) = j_n(\eta)(\partial_{n+1} u)$

$= \eta([q_n(\partial_{n+1} u)])$

$$\underbrace{\underbrace{\underbrace{\qquad\qquad}_{\partial_{n+1} u}}_{0}}_{0}$$

$\Rightarrow$ Possiamo definire $J_n : \mathrm{Hom}(H_n, G) \to H^n(G)$

$$J_n(\eta) = [j_n(\eta)]$$

Dà uno splitting:

$$h_n \circ J_n = id_{H^n(G)}$$

$$h_n(J_n(\eta))([u]) = j_n(\eta)(u) = \eta([q_n u]) = \eta([u])$$

Resta da vedere: $\operatorname{Ker}(h_n) \cong \operatorname{Ext}(H_{n-1}, G)$.

Consideriamo:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "\partial_{n+1}=0"] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "\partial_{n+1}"] & B_n \arrow[r] \arrow[d, "\partial_n=0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r] \arrow[d] & C_n \arrow[r, "\partial_n"] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Dualizzando ottengo con righe esatte per il lemma.

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 & \mathrm{Hom}(Z_{n+1},G) \arrow[l] \arrow[u] & C^{n+1}(G) \arrow[l] \arrow[u] & \mathrm{Hom}(B_n,G) \arrow[l] \arrow[u] & 0 \arrow[l] \\
0 & \mathrm{Hom}(Z_n,G) \arrow[l] \arrow[u, "0"'] & C^n(G) \arrow[l] \arrow[u, "\delta_n"'] & \mathrm{Hom}(B_{n-1},G) \arrow[l] \arrow[u, "0"'] & 0 \arrow[l] \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}
$$

Ho succ. esatta corta di complessi di cocatene da cui l'esatta lunga; però $(\mathrm{Hom}(Z_*,G))$ $(\mathrm{Hom}(B_*,G))$ hanno i bordi nulli $\Longrightarrow$

ognuno coincide con la propria omologia : dunque

$$\dots \leftarrow Hom(B_n,G) \leftarrow Hom(Z_n,G) \leftarrow H^n(G) \leftarrow Hom(B_{n-1},G) \leftarrow \dots \quad (*)$$

definita da:

$\varphi \longmapsto \eta$

$$\begin{tikzcd}
 & C^{n+1}(G) & Hom(B_n,G) \arrow[l, "\partial^*_{n+1}"'] & 0 \arrow[l] \\
0 & Hom(Z_n,G) \arrow[l] & C^n(G) \arrow[l] \arrow[u, "\delta_n"'] &
\end{tikzcd}$$

$\eta \in Hom(B_n,G)$, $\varphi \in Hom(Z_n,G)$, $\psi \in C^n(G)$

Devo avere $\partial^*_{n+1}\eta = \delta_n \psi$, $\psi = \varphi \circ q_n$

$\Rightarrow \eta = \psi|_{B_n} = \varphi \circ q_n|_{B_n} = \varphi|_{B_n}$

$\Rightarrow$ la mappa $\mathrm{Hom}(B_n, G) \longleftarrow \mathrm{Hom}(Z_n, G)$

è i_n^* da

$$0 \to B_n \xrightarrow{i_n} Z_n \to H_n \to 0$$

dunque da ⊛ trovo esatta:

$$0 \leftarrow \mathrm{Ker}(i_n^*) \longleftarrow H^n(G) \longleftarrow \underset{\underset{\displaystyle \mathrm{Ext}(H_{n-1}, G)}{\shortparallel}}{\frac{\mathrm{Hom}(B_{n-1}, G)}{\mathrm{Im}(i_{n-1}^*)}} \longleftarrow 0$$

$\Rightarrow$ per concludere basta vedere che

$$\mathrm{Ker}(i_n^*) \longleftarrow H^n(G) \quad \text{è } h_n$$

Infatti: $\mathrm{Ker}(i_n^*)$ = gli omomorfismi $Z_n \to G$ che si annullano su B_n

$i_n : B_n \to Z_n$ = gli omomorfismi $Z_n/B_n = H_n \to G$

$$\Rightarrow \mathrm{Ker}(i_n^*) = \mathrm{Hom}(H_n, G)$$

(inoltre la mappa è la stessa) — $\blacksquare$

Oss: (1) $\mathrm{Ext}(A \oplus B, G) = \mathrm{Ext}(A, G) \oplus \mathrm{Ext}(B, G)$

(2) $\mathrm{Ext}(\mathbb{Z}, G) = 0$

$$0 \to 0 \to \mathbb{Z} \to \mathbb{Z} \to 0$$
$$0 \leftarrow 0 \leftarrow G \leftarrow G \leftarrow 0$$

(3) $\mathrm{Ext}(\mathbb{Z}/m, G) = G/mG$

$$0 \to \mathbb{Z} \xrightarrow{m\cdot} \mathbb{Z} \to \mathbb{Z}/m \to 0$$
$$0 \leftarrow G \xleftarrow{m\cdot} G \leftarrow$$

(4) $\mathrm{Ext}(\mathbb{Z}/m, \mathbb{Z}) = \mathbb{Z}/m \neq 0 = \mathrm{Ext}(\mathbb{Z}, \mathbb{Z}/m)$

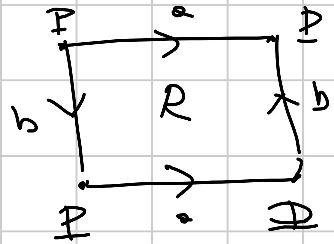
(non symm) .

(5) $\mathrm{Ext}(A, \mathbb{Z}) = \mathrm{Tor}(A)$

(6) Già visto usando $D(n,k)$, $E(k)$

$$H^n(X;\mathbb{Z}) = \frac{H_n(X;\mathbb{Z})}{\mathrm{Tor}(H_n(X;\mathbb{Z}))} \oplus \mathrm{Tor}(H_{n-1}(X;\mathbb{Z}))$$

<u>Esempio</u>: $K =$



$\partial_0 P = 0$, $\partial_1 a = \partial_1 b = 0$, $\partial_2 R = 2b$

$\Rightarrow H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z} \oplus \mathbb{Z}/2$, $H_2 = 0$

$\delta \hat{P} = 0 \quad \delta \hat{a} = 0 \quad \delta \hat{b} = 2\hat{R} \quad \delta \hat{R} = 0$

$\Rightarrow H^0 = \mathbb{Z} \quad H^1 = \mathbb{Z} \quad H^2 = \mathbb{Z}/2$

Calcolo diretto delle coomologie.

Teoria simpliciale: $X = |K|$

$C^n(X) \cong \mathbb{Z}^{K^{[n]}}$ generato da $\hat{\sigma}$: $\sigma \in K^{[n]}$

$$(\delta_n \hat{\sigma})(\tau) = \hat{\sigma}(\partial_{n+1} \tau) = \begin{cases} \pm 1 & \text{se } \partial \tau \supset \pm \sigma \\ 0 & \text{altrimenti.} \end{cases}$$

$\sigma \in K^{[n]}$
$\tau \in K^{[n+1]}$

Teoria singolare : ok

Propietà di H^* (usate per la teoria cellulare):

• coomologia ridotta : $\tilde{H}^n(G)$ ottenuta dualizzando il complesso aumentato

$$\to C_2 \to C_1 \to C_0 \to \underset{\substack{\parallel \\ C_{-1}}}{\mathbb{Z}} \to 0 \to \cdots$$

$\sum n_i p_i \mapsto \sum_i n_i$

$\tilde{H}^n \cong H^n \quad \forall n > 0$

H^0 : funzioni $X \to G$ costanti sulle comp. connesse

$\tilde{H}^0$: $H^0 /$ funzioni costanti

- LES : $0 \to C_n(A) \to C_n(X) \to C_n(X,A) \to 0$

dà $0 \leftarrow C^n(A;G) \leftarrow C^n(X;G) \leftarrow C^n(X,A;G) \leftarrow 0$

esatta, da cui

$$\cdots \leftarrow H_n(A) \longleftarrow H_n(X) \longleftarrow H_n(X,A) \xleftarrow{\Delta_n} H_{n+1}(A) \leftarrow \cdots$$

indotte da

$A \hookrightarrow X$

$(X,\emptyset) \hookrightarrow (X,A)$

$$\left(\Delta_n[\varphi]\right) = \left[u \longmapsto \varphi(\partial_{n+1} u)\right]$$

$\varphi : C_n \to G$

$u \in C_{n+1}(X)$

$\partial_{n+1} u \in C_n(A)$

- funtorialità: $f : (X,A) \to (Y,B)$

$\Rightarrow f^* : H^*(Y,B) \to H^*(X,A)$

$$f^*(\varphi)(\sigma) = \varphi(f(\sigma))$$

$(f \circ g)^* = g^* \circ f^*$, $id^* = id$

$\Longrightarrow$ H^* funtore controvariante

$$\mathrm{TOP}_2 \longrightarrow \text{succ. gruppi ab.}$$

- omotopia : $f \simeq g \Longrightarrow f^* = g^*$
- escissione (stesso enunciato)
- dimensione $H^n(\{pt\}; G) = \begin{cases} G & n = 0 \\ 0 & \text{altrimenti} \end{cases}$
- le proprietà elencate caratterizzano H^*

Coomologia cellulare: X CW complesso

Omologia: $C_n(X) = \mathbb{Z}^{X^{[n]}} \cong H_n(X^{(n)}, X^{(n-1)})$

$$H_n(X^{(n)}, X^{(n-1)}) \longrightarrow H_{n-1}(X^{(n-1)}, X^{(n-2)})$$

def. che usa il grado $S^{n-1} \longrightarrow S^{n-1}$

Coomologia: $C^n(X) = \mathbb{Z}^{X^{[n]}} \cong H^n(X^{(n)}, X^{(n-1)})$

(uso che per UCT $\tilde{H}^k(S^n) \cong \tilde{H}^k(D^{n+1}, S^n) = \begin{cases} G & k=n \\ 0 & \text{altrimenti} \end{cases}$)

Il cobordo:

$$\cdots \to \tilde{H}^n(X^{(n)}, X^{(n-1)}; G) \to \tilde{H}^{n+1}(X^{(n+1)}, X^{(n)}; G) \to \cdots$$

$$\tilde{H}^n(X^{(n)}; G)$$

è 1

indotta da $(X^n, \emptyset) \hookrightarrow (X^{(n)}, X^{(n-1)})$

parte di LES per $(X^{(n+1)}, X^{(n)})$ (l'omomorf. di bordo)

• Mayer-Vietoris $X = A \cup B$
(sottocomplessi oppure $X = \mathrm{int}(A) \cup \mathrm{int}(B)$)

$$\Rightarrow \dots \to H^n(X;G) \xrightarrow{\Phi} H^n(A;G) \oplus H^n(B;G)$$
$$\xrightarrow{\Psi} H^n(A \cap B;G) \xrightarrow{\textcircled{d}} H^{n+1}(X;G) \to \dots$$

$\Phi = (i_A^*, i_B^*)$ $\quad A \xrightarrow{i_A} X \quad B \xrightarrow{i_B} X$

$\Psi = j_A^* - j_B^*$ $\quad A\cap B \xrightarrow{j_A} A \quad A \cap B \xrightarrow{j_B} B$

$\textcircled{d}([\omega]) = [\lambda]$ $\quad \omega : C_n(A \cap B) \to G$, $\delta\omega = 0$

$\lambda : C_{n+1}(X) \to G$

per $u \in C_{n+1}$ scrivo $\underset{\substack{\cap \\ C_{n+1}(A)}}{a} + \underset{\substack{\cap \\ C_{n+1}(B)}}{b}$

$\lambda u = \omega(\partial a)$ indipendente dall'espressione di u come $a+b$.

(Avevo sbagliato: controllare)