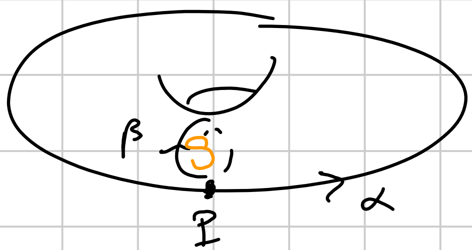
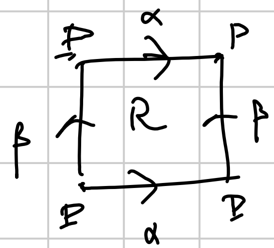
ETA 12/12/13

(1) $X =$ 



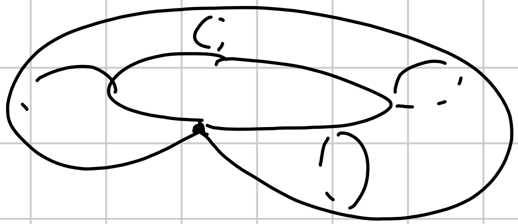
$\partial\alpha = \partial\beta = 0$

$\partial R = 0 \quad \partial S = \beta$

$$\begin{array}{ccccccccc} & & 2 & & 1 & & 0 & & \\ 0 & \to & \mathbb{Z}^2 & \xrightarrow[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}]{} & \mathbb{Z}^2 & \xrightarrow{0} & \mathbb{Z} & \to & 0 \end{array}$$

$H_0 = \mathbb{Z} \quad H_1 = \mathbb{Z} \quad H_2 = \mathbb{Z}$

$X \simeq$



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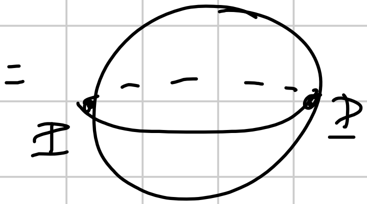

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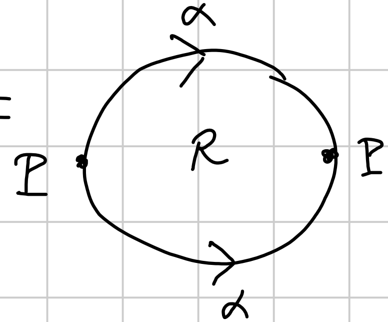
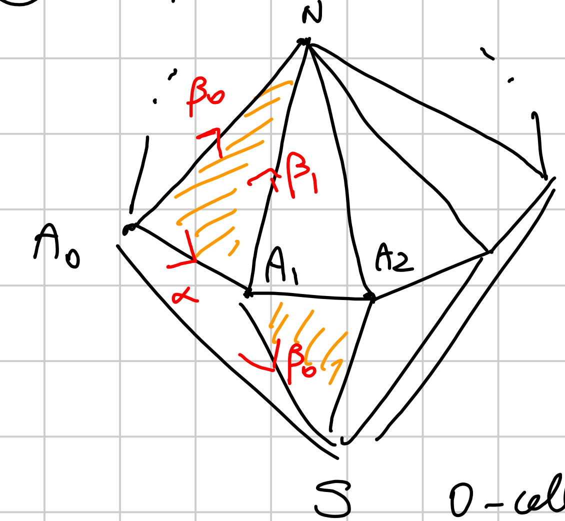


CW con
una 0-cella P
una 1-cella α
una 2-cella R
e $\partial_0 = \partial_1 = \partial_2 = 0$

$\Rightarrow C_* : \quad 0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$

$\Rightarrow H_* : \quad \mathbb{Z} \quad \mathbb{Z} \quad \mathbb{Z}$

4) Bipiramide B con base n-agono regolare



$$X = B \Big/ A_j A_{j+1} N = A_{j+1} A_{j+2} S$$

(indici mod n)

Provare che è una 3-varietà _ CW complesso con:

0-celle: $P = [A_j]$, $Q = [N] = [S]$

1-celle: $\alpha = [A_j, A_{j+1}]$ (tutti uguali)

$\beta_j = [A_j, N] = [A_{j+1}, S]$ $\quad j = 0 \dots n-1$

2-celle $T_j = [A_j\, A_{j+1}\, \alpha] = [A_{j+1}\, A_{j+2}\, S]$ $\quad j = 0, \dots, n-1$

3-celle B . $\quad (\chi = 2 - (n+1) + n - 1 = 0)$

(Fatto generale : $M^{(2k+1)}$ chiusa $\Rightarrow \chi(M) = 0$.)

$$\partial_1 \alpha = 0 \,, \; \partial_1 \beta_j = Q - P \qquad j = 0, \dots, n-1$$

$$\partial_2 T_j = \alpha + \beta_{j+1} - \beta_j \qquad \partial_3 B = 0$$

$$H_0 = \mathbb{Z} \qquad Z_1 = \langle \alpha\,,\ \underset{i<j}{\beta_i - \beta_j} \mid \_\_\_ \rangle$$

$$B_1 = \langle \underset{j=0 \dots n-1}{\alpha + \beta_{j+1} - \beta_j} \mid \_\_\_ \rangle$$

$$\alpha = \beta_{j+1} - \beta_j \quad \forall j$$

$$H_1 = Z_1 / B_1 \; : \langle \alpha, \beta_i - \beta_i \mid \qquad \beta_i - \beta_j = (\beta_i - \beta_{i+1}) + (\beta_{i+1} - \beta_{i+2}) +$$

$$\cdots + (\beta_{j-1} - \beta_j)$$

$$= (j-i)\alpha$$

$$0 = \beta_0 - \beta_0 = \beta_0 - \beta_n = n \cdot \alpha)$$

$$= \mathbb{Z}/n$$

$H_2 = 0$ $\qquad$ $\mathbb{Z}_2 = 0$ $\qquad$ $H_3 = \mathbb{Z}$ _

(X è 3-var orientabile)

(5) $X = \Sigma_g$ $\quad A = n$ punti in X. $H_*(X,A) = ?$

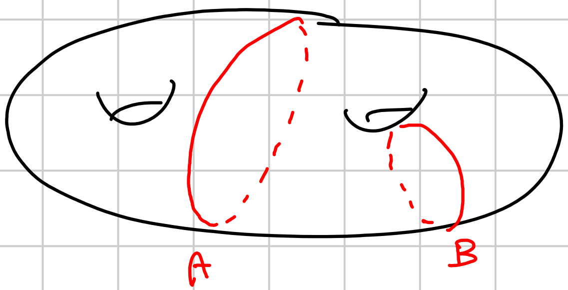
$$H_2(A) \to H_2(X) \to H_2(X,A) \to H_1(A) \to H_1(X) \to H_1(X,A) \to \tilde{H}_0(A)$$

$$\begin{array}{ccccccc} \| & \| & \| & \| & \| & & \| \\ 0 & \mathbb{Z} & \mathbb{Z} & 0 & \mathbb{Z}^{2g} & & \mathbb{Z}^{n-1} \end{array}$$

esatta corta
con quoziente libero
$\Rightarrow$ splitta
$\Rightarrow H_1(X,A)=\mathbb{Z}^{2g+n-1}$

$$\begin{array}{c} \downarrow \\ 0 = \tilde{H}_0(X) \\ \downarrow \\ 0 = \tilde{H}_0(X,A) \\ \downarrow \\ 0 \end{array}$$

(6) $X=\Sigma_2$



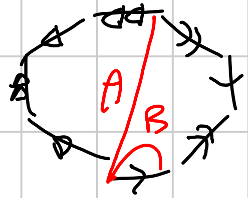
$H_*(X,A)$

$H_*(X,B)$

Sia $G\in\{A,B\}$

$$0 \to \underset{\substack{\| \\ 0}}{H_2(C)} \to \underset{\substack{\| \\ \mathbb{Z}}}{H_2(X)} \to H_2(X,C) \to \underset{\substack{\| \\ \mathbb{Z}}}{H_1(C)} \xrightarrow{j} \underset{\substack{\| \\ \mathbb{Z}^4}}{H_1(X)} \to H_1(X,C) \to \underset{\substack{\| \\ 0}}{\tilde{H}_0(C)}$$

$C=A \Rightarrow j=0$ protože $A=\partial D$ $\qquad D =$ 



$$C=B \Rightarrow j(1) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

$C=A \qquad 0 \to \underset{\substack{\| \\ \mathbb{Z}}}{H_2(X)} \to \underset{\substack{\| \\ \mathbb{Z}^2}}{H_2(X,A)} \to \underset{\substack{\| \\ \mathbb{Z}}}{H_1(A)} \to 0$

$$\begin{tikzcd}
0 \arrow[r] & H_1(X) \arrow[r] \arrow[d, equal] & H_1(X,A) \arrow[r] \arrow[d, equal] & 0 \\
 & \mathbb{Z}^4 & \mathbb{Z}^4 &
\end{tikzcd}$$

j iniettiva

$C = B$

$$\begin{tikzcd}
0 \arrow[r] & H_2(X) \arrow[r] \arrow[d, equal] & H_2(X,B) \arrow[r] \arrow[d, equal] & 0 \\
 & \mathbb{Z} & \mathbb{Z} &
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & H_1(B) \arrow[r, "j"] \arrow[d, equal] & H_1(X) \arrow[r] \arrow[d, equal] & H_1(X,B) \arrow[r] \arrow[d, equal] & 0 \\
 & \mathbb{Z} & \mathbb{Z}^4 & \mathbb{Z}^3 &
\end{tikzcd}$$

(perché $j(1)$ è un generatore di $\mathbb{Z}^4$)

⑦ $H_1(\mathbb{R},\mathbb{Q}) = H_1(\mathbb{R},\mathbb{Q};\mathbb{Z})$

$$H_1(\mathbb{Q}) \to H_1(\mathbb{R}) \to H_1(\mathbb{R},\mathbb{Q}) \xrightarrow{\cong} \tilde{H}_0(\mathbb{Q}) \to \tilde{H}_0(\mathbb{R})$$

$H_1(\mathbb{Q}) = 0$, $H_1(\mathbb{R}) = 0$, $\tilde{H}_0(\mathbb{R}) = 0$

$$\tilde{H}_0(\mathbb{Q}) = \left\{ \alpha : \mathbb{Q} \to \mathbb{Z} \ : \ \#\{q : \alpha(q) \neq 0\} < +\infty, \ \sum_{q\in\mathbb{Q}} \alpha(q) = 0 \right\}$$

$\tilde{H}_0(\mathbb{Q})$ = gruppo abeliano libero con base $\{\hat{q}\}_{\hat{q}\in\mathbb{Q}^+}$

$$\hat{q}(x) = \begin{cases} 1 & x = q \\ -1 & x = 0 \\ 0 & \text{altrimenti} \end{cases}$$

→ $H_1(\mathbb{R},\mathbb{Q})$ = gruppo abeliano libero con base $\{[0,q]\}_{q\in\mathbb{Q}^*}$

$[0,q]$ =

⑧ Sia $f: S^{2n} \circlearrowleft$ continua; provare che esiste $x \in S^{2n}$ t.c. o $f(x)=x$ o $f(x)=-x$.

Altrimenti: $f \simeq -id_{S^{2n}}$, $f \simeq id_{S^{2n}}$

ma $2n$ è pari $\deg(-id_{S^{2n}}) = -1$. Assurdo.

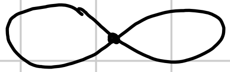
8bis $f : \mathbb{P}^{2n}(\mathbb{R}) \circlearrowleft$ continua ha punti fissi;
falso per $f : \mathbb{P}^{2n+1}(\mathbb{R}) \circlearrowleft$

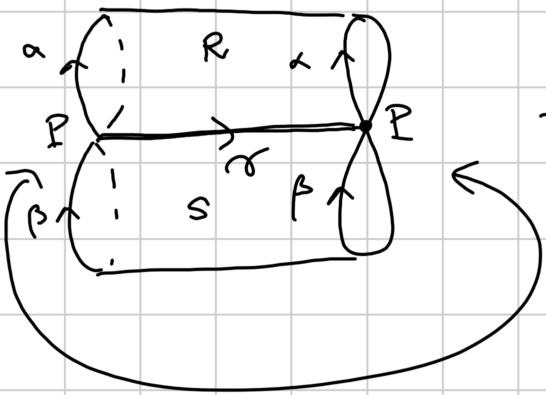
$$\begin{tikzcd}
S^{2n} \arrow[r, "\tilde{f}"] \arrow[d] \arrow[dr, dashed] & S^{2n} \arrow[d] \\
\mathbb{P}^{2n} \arrow[r, "f"] & \mathbb{P}^{2n}
\end{tikzcd}$$

Applico 8 a $\tilde{f}$.

Per $2n+1$ prendere f indotta da

$$\left.\underbrace{\begin{pmatrix} \begin{smallmatrix}0&1\\1&0\end{smallmatrix} & & \\ & \begin{smallmatrix}0&1\\1&0\end{smallmatrix} & \\ & & \ddots \end{pmatrix}}_{2n+2}\right\}2n+2$$

9 $S^1 \vee S^1 =$  ; $H_*(\underbrace{S^1 \times (S^1 \vee S^1)}_{X})$ –



$= X$

$\partial\alpha = \partial\beta = 0$

$\partial R = \partial S = 0$

$H_0 = \mathbb{Z},\ H_1 = \mathbb{Z}^3,\ H_2 = \mathbb{Z}^2$

Künneth:

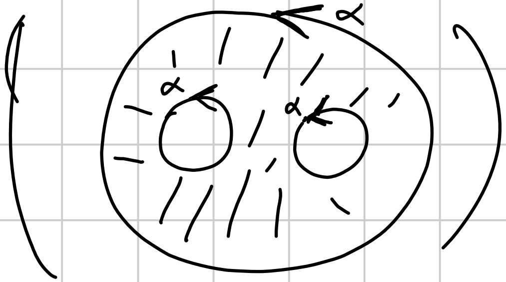
	S^1	$S^1 \vee S^1$
H_0	$\mathbb{Z}$	$\mathbb{Z}$
H_1	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$

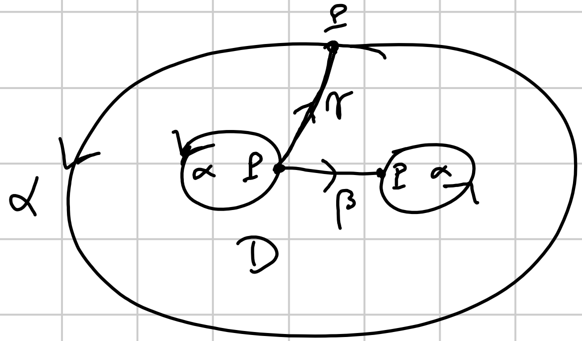
$H_0 = \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}$

$H_1 = \mathbb{Z} \otimes (\mathbb{Z} \oplus \mathbb{Z}) \oplus \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}^3$

$H_2 = \mathbb{Z} \otimes (\mathbb{Z} \oplus \mathbb{Z}) = \mathbb{Z}^2$

10 H_α





$$\overset{0}{P},\ \overbrace{\alpha,\ \beta,\ \gamma}^{1},\ \overset{2}{D}$$

$$\partial\alpha = \partial\beta = \partial\gamma = 0$$

$$\partial D = \alpha$$

$$\Rightarrow H_0 = \mathbb{Z}$$

$$H_1 = \mathbb{Z}^2 \qquad H_2 = 0$$

(11) $A \in GL(n,\mathbb{R})$; $A_* : H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \circlearrowleft$

$$\underbrace{H_n(\mathbb{R}^n)}_{=0} \longrightarrow H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \longrightarrow \underbrace{\underbrace{\tilde{H}_{n-1}(\mathbb{R}^n \setminus \{0\})}_{\cong S^{n-1}}}_{\mathbb{Z}} \longrightarrow \underbrace{\tilde{H}_{n-1}(\mathbb{R}^n)}_{=0}$$

isomorfismo dato dal bordo

$\Rightarrow H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ $\underbrace{\text{è}}_{\mathbb{Z} \text{ ed è}}$ canonicamente generato da qualsiasi n-simplesso σ con $0 \in \mathrm{int}(\sigma)$ e σ orientato come $\mathbb{R}^n$

$\Rightarrow A_* = \mathrm{sgn}(\det(A)) \cdot \mathrm{id}_{\mathbb{Z}}$.

<u>Conseguenza</u>: la seguente definizione di orientaz. per una varietà topologica $M^{(n)}$ coincide con quella differenziabile se M è differenziabile:

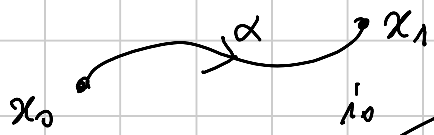
orientazione top. per M:

- scelta di un generatore per ogni $H_n(M, M \setminus \{x\}) \cong \mathbb{Z}$

$$H_*(M, M \setminus \{x\}) \underset{\text{escissione}}{\cong} H_*(U(x), U(x) \setminus \{x\}) \underset{\text{omotopia}}{\cong} H_*(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$$

(sulle componenti connesse per archi)

• scelta (localmente) coerente :

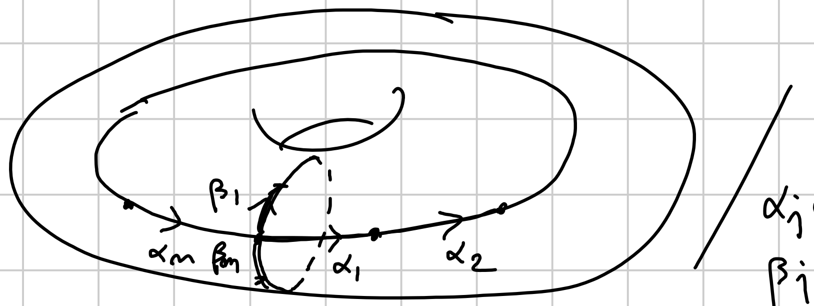


$$(M, M\setminus x_0) \xleftarrow{i_0} (M, M\setminus \alpha) \xrightarrow{i_1} (M, M\setminus x_1)$$

i_{0_*}, i_{1_*} sono isomorfismi per omotopia

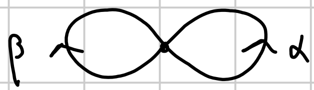
$\Longrightarrow$ $i_{0_*} \circ (i_{1_*})^{-1}$ manda il generatore scelto per $H_n(M, M\setminus x_1)$ in quello scelto per $H_n(M, M\setminus x_0)$

12

$X =$ 

$\alpha_j \sim \alpha_{j+1}$

$\beta_j \sim \beta_{j+1}$

X è ottenuto da 

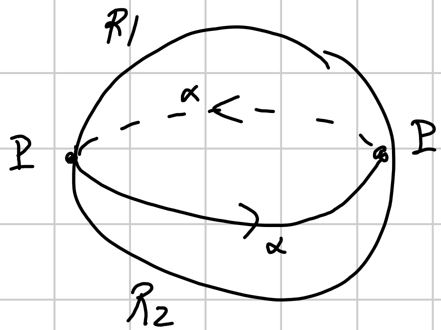
incollando una 2-cella lungo: $n \cdot \alpha + m \cdot \beta - n\alpha - m \cdot \beta = 0$

$\Rightarrow H_0 = \mathbb{Z} \qquad H_1 = \mathbb{Z}^2 \qquad H_2 = \mathbb{Z}$

(13) $S^2 /_{x \sim -x \text{ se } x \in S^1 \times \{0\}}$

$S^3 /_{x \sim -x \text{ se } x \in S^2 \times \{0\}}$

calcolare H_*



$\partial R_1 = \partial R_2 = 2\alpha$

$H_0 = \mathbb{Z}$

$H_1 = \mathbb{Z}/2$

$H_2 = \mathbb{Z} = \langle R_1 - R_2 \rangle$

P, α, R, B_1, B_2

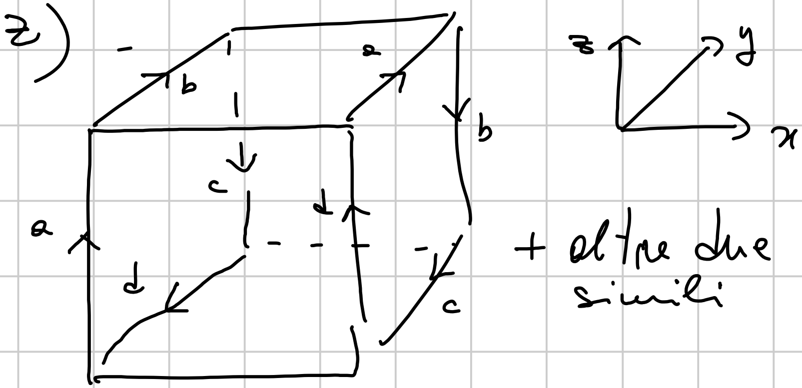
$0 \quad 1 \quad 2 \quad 3 \quad 3$

$\partial \alpha = 0 \quad \partial R = 2\alpha \quad \partial B_1 = \partial B_2 = 0$

$\mathbb{Z}, \mathbb{Z}/2, 0, \mathbb{Z}^2$

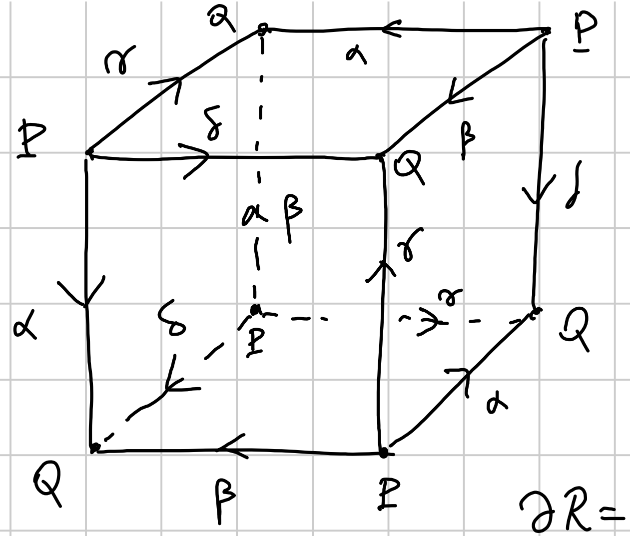
(14) $Q = [-1,1]^3 \qquad X = Q/\sim$

identifico la faccia con $x=-1$ (risp. $y=-1$, risp $z=-1$) con la faccia con $x=1$ (risp. $x=1$, risp. $z=1$) dando $1/4$ di giro positivo in direzione x (risp. y, risp. z)



+ altre due simili

$H_*(X)$



P, Q 0-celle

$\alpha, \beta, \gamma, \delta$ 1-celle

R, S, T 2-celle

B 3-celle

$$\partial\alpha = \partial\beta = \partial\gamma = \partial\delta = Q - P$$

$$\partial R = \alpha - \delta + \beta - \gamma$$

$$\partial S = \alpha - \beta + \gamma - \delta \qquad \partial B = 0$$

$$\partial T = \alpha - \gamma + \delta - \beta$$

$$H_0 = \mathbb{Z} \quad H_3 = \mathbb{Z} \quad H_2 = 0 \quad (Z_2 = 0)$$

$H_1 : \langle \underset{c_1}{\underset{\shortparallel}{\alpha-\beta}}, \underset{c_2}{\underset{\shortparallel}{\alpha-\gamma}}, \underset{c_3}{\underset{\shortparallel}{\alpha-\delta}} \rangle$

$\partial R: \; c_2 + c_3 = c_1$

$\partial S: \; c_2 + c_3 + c_3 - c_2 = 0 \Rightarrow 2c_3 = 0$

$\partial T: \; c_2 + c_3 - c_3 + c_2 = 0 \quad 2c_2 = 0$

$$\Rightarrow H_1 = \mathbb{Z}/2 \oplus \mathbb{Z}/2$$