ETA 7/11/13

$A \subset X$ sottocomplesso

$$0 \to C(A) \xrightarrow{i} C(X) \xrightarrow{p} C(X,A) \to 0$$

Induce

$$\dots \to H_n(A) \xrightarrow{i_{n*}} H_n(X) \xrightarrow{p_{n*}} H_n(X,A) \xrightarrow{\partial_n} H_{n-1}(A) \to \dots$$

$d_n([z]) = [u]$

$$
\begin{tikzcd}
 & w \quad C_n(X) \arrow[r, "p_n"] \arrow[d, "\partial^X"] & z \quad C_n(X,A) \\
u \quad C_{n-1}(A) \arrow[r, "i_{n-1}"] & C_{n-1}(X) &
\end{tikzcd}
$$

——— 0 ———

$$z \in C_n(X,A) \Rightarrow z = \sum_{\sigma \in X^{[n]} \setminus A^{[n]}} n_\sigma \cdot \sigma$$

$$w = \sum_{\sigma \in X^{[n]}} n_\sigma \cdot \sigma \quad \text{dove } n_\sigma = 0 \text{ per } \sigma \in A^{[n]}$$

$$z \in Z_n(X,A) \Rightarrow \partial^{(X,A)} z = 0$$

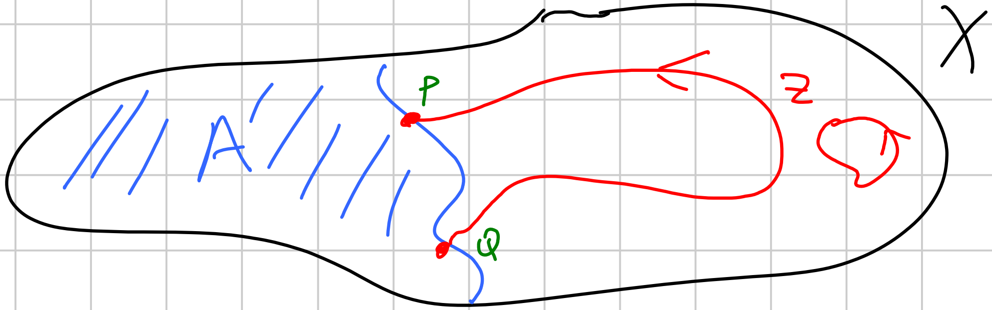
$$\Rightarrow \partial^X w = \sum_{\substack{\sigma \in X^{[n]},\ \tau \subset \sigma \\ \tau \in A^{[n-1]}}} m_\sigma \cdot \varepsilon(\sigma,\tau) \cdot \tau +$$

$$+ \sum_{\substack{\sigma \in X^{[n]},\ \tau \subset \sigma \\ \tau \in X^{[n-1]} \setminus A^{[n-1]}}} m_\sigma \cdot \varepsilon(\sigma,\tau) \cdot \tau$$

questo deve essere nullo

$$\Rightarrow d_n\left(\sum n_\sigma \cdot \sigma\right) = \sum_{\substack{\sigma, \tau \in A^{(n-1)} \\ \tau < \sigma}} n_\sigma \cdot \varepsilon(\sigma,\tau) \cdot \tau.$$

Cioè: $d_n : H_n(X,A) \to H_{n-1}(A)$
si ottiene applicando l'operatore ∂
(per costruzione si trovano solo $n-1$
simplessi di A).

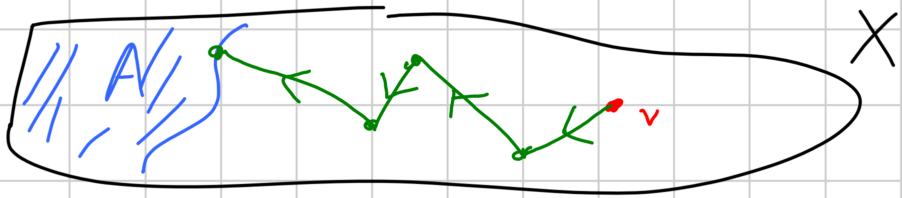


$$\partial z = [P - Q] \in H_0(A)$$

<u>Omologia ridotta</u>: Se X è connesso

$H_0(X) \simeq \mathbb{Z}$ canonicamente gen. da $[v]$, $v \in X^{[0]}$

$H_0(X,A) = 0$ se $A \neq \emptyset$



<u>Def</u>: Se $X = X_1 \sqcup \dots \sqcup X_k$ (componenti connesse)

Sia e $v_i \in X_i^{(0)}$

$$\tilde{H}_n(X,A) = \begin{cases} H_n(X,A) & n > 0 \\ \left\{ \sum_{i=1}^{k} m_i [v_i] \in H_0(X,A) : \sum_i m_i = 0 \right\} \end{cases}$$

Oss: Se $A \neq \emptyset$ ho qualche $[v_j] = 0$

quindi $\tilde{H}_0(X,A) = H_0(X,A)$.

Invece se $A = \emptyset$ ho $H_0(X) \cong \mathbb{Z}^k$

e $\tilde{H}_0(X) \cong \mathbb{Z}^{k-1} = \{(n_1, \dots, n_k) \in \mathbb{Z}^k : \sum n_j = 0\}$.

$(\, H_n(X) = H_n(X, \emptyset).)$

Prop: la successione esatta di omologia

$$\dots \longrightarrow H_1(X,A) \xrightarrow{d_1} H_0(A) \xrightarrow{i_{0*}} H_0(X) \xrightarrow{p_{0*}} H_0(X,A) \longrightarrow 0$$

induce

$$\dots \longrightarrow \widetilde{H}_1(X,A) \xrightarrow{\widetilde{d}_1} \widetilde{H}_0(A) \xrightarrow{\widetilde{i}_{0*}} \widetilde{H}_0(X) \xrightarrow{\widetilde{p}_{0*}} \widetilde{H}_0(X,A) \longrightarrow 0 .$$

<u>Dim</u>: $\widetilde{H}_0(A) = \left\{ \sum_i n_i [a_i] : \sum_i n_i = 0 \right\}$

Provo che $Im(d_1) \subset \widetilde{H}_0(A)$.

Sia $[z] \in \widetilde{H}_1(X,A) = H_1(X,A)$

$z = \sum_i m_i \cdot e_i \; ; \quad \partial^{(X,A)} z = 0$

$$\Longrightarrow \partial^X z = \sum_i m_i \cdot \left(e_i(1) - e_i(0) \right) \qquad (1)$$

$$= \sum_{v \in A^{[0]}} \left(\sum_{j,\, e_j(1) = v} m_j - \sum_{j,\, e_j(0) = v} m_j \right) \quad (2)$$

$$+ \sum_{v \notin A^{[0]}} \left(\sum_{j,\, e_j(1) = v} m_j - \sum_{j,\, e_j(0) = v} m_j \right) \quad (3)$$

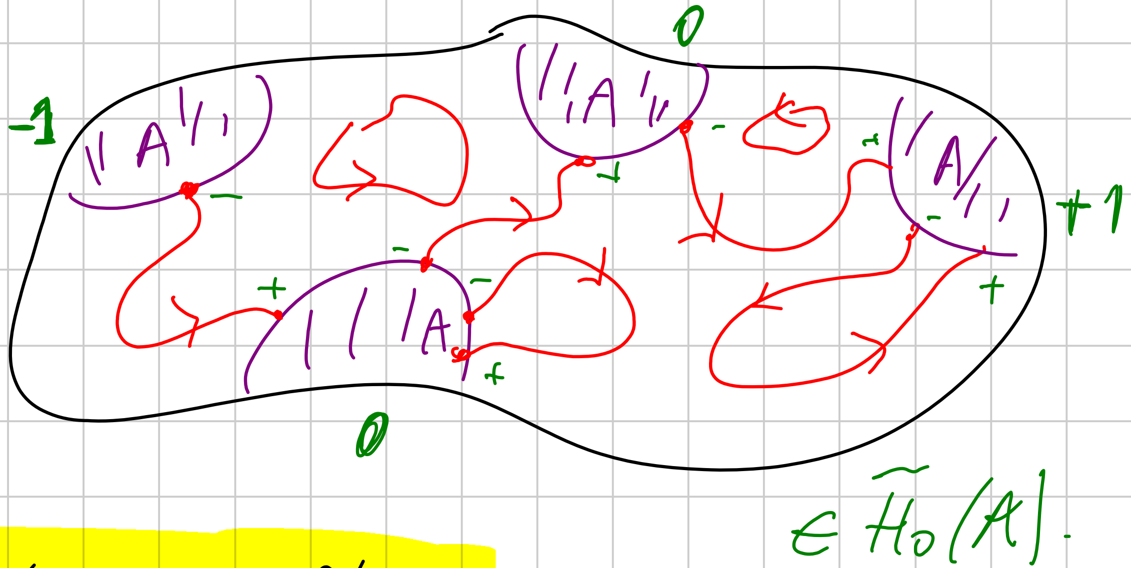
In (1) la somma di tutti i coeff è 0;

$\partial^X_A z = 0 \implies$ In (3) tutti i coeff sono 0

$\implies$ In (2) la somma di tutti i coeff è 0

$\implies [\partial^X z] \in \tilde{H}_0(A)$

$\quad \overset{\parallel}{d_1([z])}$



$$i_{0*}(\tilde{H}_0(A)) \subset \tilde{H}_0(X)$$

$$i_{0*}\left(\sum n_j [a_j]^A\right) = \sum n_j [a_j]^X$$

la condizione $\sum a_j = 0$ persiste

$$\boxed{\mathrm{Im}(\tilde{d}_1) \subset \mathrm{Ker}(\tilde{i}_{0*})}$$

ovvio $\tilde{i}_{0*} \circ \tilde{d}_1 = i_0\Big|_{\tilde{H}_0(A)}^{\tilde{H}_0(X)} \circ d_1\Big|^{\tilde{H}_0(A)}$

$$= \underbrace{i_0 \circ d_1}_{=0}\Big|^{\tilde{H}_0(X)} = 0$$

$\mathrm{Ker}(\tilde{i}_{0*}) \subset \mathrm{Im}(\tilde{d}_1)$.

$[z] \in \mathrm{Ker}(\tilde{i}_{0*}) \Rightarrow [z] \in \mathrm{Ker}(i_{0*})$

$\Rightarrow [z] \in \mathrm{Im}(d_1) = \mathrm{Im}(\tilde{d}_1)$.

$p_{0*}(\tilde{H}_0(X)) \subset \tilde{H}_0(X, A)$

$p_{0*}\left(\sum_i n_i [v_i]^X\right) = \sum_i n_i [v_i]^{(X,A)}$

condizione $\sum n_i = 0$ preservata.

$\operatorname{Im} \tilde{i}_{0*} \subset \operatorname{Ker} \tilde{p}_{0*}$ ovvio.

$\operatorname{Ker} \tilde{p}_{0*} \subset \operatorname{Im} \tilde{i}_{0*}$

$[z] \in \operatorname{Ker}(\tilde{p}_{0*}) \Rightarrow [z] \in \operatorname{Ker}(p_{0*})$

$\Rightarrow \exists [w] \in H_0(A)$ t.c.

$[z] = i_{0*}([w])$.

Bisogna vedere che $[w] \in \tilde{H}_0(A)$.
(Esercizio).

$\tilde{p}_{0*}$ suriettivo

$$\sum_i n_i [v_i]^{(X,A)} \in \tilde{H}_0(X,A)$$

$$\Rightarrow \text{è } \tilde{p}_{0*}\left(\sum_i n_i [v_i]^X \right).$$

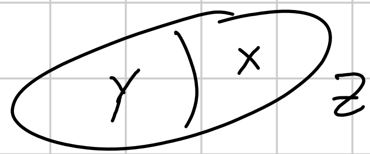
∎

Fatto: per $\tilde{H}_n$ valgono anche le:

- omotopia $f, g : (X,A) \to (Y,B)$

  $$f \simeq_A g \Rightarrow f_* = g_* : \tilde{H}_n(X,A) \to \tilde{H}_n(Y,B)$$

- escissione



$$\tilde{H}_n(Z,Y) \cong \tilde{H}_n(X, X \cap Y).$$

Teorema: $$\tilde{H}_n(S^m) = \begin{cases} \mathbb{Z} & \text{se } n = m \\ 0 & \text{altrimenti} \end{cases}$$

Cioè:
$$H_n(S^m) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = m = 0 \\ \mathbb{Z} & \begin{cases} n = 0 < m \\ n = m > 0 \end{cases} \\ 0 & \text{altrimenti} \end{cases}$$

Dim: $S^{m-1} = \partial D^m$; succ. esatta (D^m, S^{m-1}):

$$\underset{\substack{\parallel \\ 0}}{\tilde{H}_n(D^m)} \longrightarrow \tilde{H}_n(D^m, S^{m-1}) \longrightarrow \tilde{H}_{n-1}(S^{m-1}) \longrightarrow \underset{\substack{\parallel \\ 0}}{\tilde{H}_{n-1}(D^m)}$$

$(D^n \simeq \{pt\}$ e $\tilde{H}_n(pt) = 0 \ \forall n)$

$$\Longrightarrow \tilde{H}_n(D^n, S^{n-1}) \cong \tilde{H}_{n-1}(S^{n-1}) \quad (1)$$

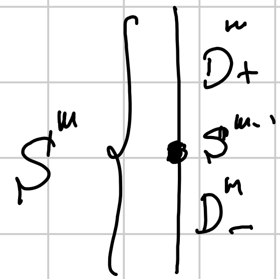
Ora $S^n = \{x \in \mathbb{R}^{n+1} : x_0^2 + \dots + x_n^2 = 1\}$

$$= \{x \in S^n : x_0 \geq 0\} \cup \{x \in S^n : x_0 \leq 0\} =: D^n_+ \cup D^n_-$$

inoltre $D^n_+ \cap D^n_- = S^{n-1}$; per escissione

$$\tilde{H}_n(S^n, D^n_+) \cong \tilde{H}_n(D^n_-, S^{n-1}) \qquad (2)$$

Per la successione esatta di (S^n, D^n_+) ho



$$\underbrace{\tilde{H}_n(D^n)}_{=0} \to \tilde{H}_n(S^n) \to \tilde{H}_n(S^n, D^n) \to \underbrace{H_{n-1}(D^n)}_{=0}$$

$$\Longrightarrow \quad \tilde{H}_n(S^n) \cong \tilde{H}_n(S^n, D^n) \qquad (3)$$

$$\tilde{H}_n(D^n, S^{n-1}) \cong \tilde{H}_{n-1}(S^{n-1}) \qquad (1)$$

$$\tilde{H}_n(S^n, D^n_+) \cong H_n(D^n_-, S^{n-1}) \qquad (2)$$

$$\tilde{H}_n(S^n) \cong \tilde{H}_n(S^n, D^n) \qquad (3)$$

$$\tilde{H}_n(S^n) = \tilde{H}_{n-1}(S^{n-1}) \qquad (1+2+3)$$

Scendo fino a

$$\tilde{H}_{n-m}(S^0) = \begin{cases} \mathbb{Z} & n-m=0 \\ 0 & n-m>0 \end{cases}$$

$$\tilde{H}_0(S^{n-m}) = \begin{cases} \mathbb{Z} & n-m=0 \\ 0 & n-m>0. \end{cases}$$

Morale $\begin{cases} \mathbb{Z} & n=m \\ 0 & \text{altrimenti.} \end{cases}$ $\blacksquare$

——— o ———

<u>Fatto</u>: la successione esatta di omotopia è funtoriale

Categoria delle coppie (K, L) con K complesso simpliciale e L sottocomplesso;

mappe = mappe simpliciali di coppie

$\rightsquigarrow$

Categoria delle successioni esatte di omomorfismi di gruppi abeliani;

mappe naturali

Segue dal fatto generale:

Prop: se ho successioni esatte corte

$0 \to \mathcal{C}' \xrightarrow{i} \mathcal{C} \xrightarrow{p} \mathcal{C}'' \to 0$ (con bordi $\partial, \partial', \partial''$)

$0 \to \mathcal{D}' \xrightarrow{j} \mathcal{D} \xrightarrow{q} \mathcal{D}'' \to 0$ (con bordi d, d', d'')

di complessi di cocatene e mappe di complessi di cocatene $\varphi' : \mathcal{C}' \to \mathcal{D}'$, $\varphi : \mathcal{C} \to \mathcal{D}$, $\varphi'' : \mathcal{C}'' \to \mathcal{D}''$ t.c.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}' \arrow[r, "i"] \arrow[d, "\varphi'"] & \mathcal{C} \arrow[r, "p"] \arrow[d, "\varphi"] & \mathcal{C}'' \arrow[r] \arrow[d, "\varphi''"] & 0 \\
0 \arrow[r] & \mathcal{D}' \arrow[r, "j"] & \mathcal{D} \arrow[r, "q"] & \mathcal{D}'' \arrow[r] & 0
\end{tikzcd}$$

commuta, allora commuta anche

$$
\begin{tikzcd}
\cdots \arrow[r] & H_n(\mathcal{C}') \arrow[r, "i_{n*}"] \arrow[d, "\varphi'_{n*}"] \arrow[dr, phantom, "\circlearrowleft"] & H_n(\mathcal{C}) \arrow[r, "p_{n*}"] \arrow[d, "\varphi_{n*}"] \arrow[dr, phantom, "\circlearrowleft"] & H_n(\mathcal{C}'') \arrow[r, "\partial_n"] \arrow[d, "\varphi''_{n*}"] \arrow[dr, phantom, "\circlearrowleft"] & H_{n-1}(\mathcal{C}') \arrow[r] \arrow[d, "\varphi'_{n-1*}"] & \cdots \\
\cdots \arrow[r] & H_n(\mathcal{D}') \arrow[r, "j_{n*}"] & H_n(\mathcal{D}) \arrow[r, "q_{n*}"] & H_n(\mathcal{D}'') \arrow[r, "\eta_n"] & H_{n-1}(\mathcal{D}') \arrow[r] & \cdots
\end{tikzcd}
$$

<u>Dim</u>: $\circlearrowleft \quad \circlearrowright$ ovvi: vale a livello di catene:

$$\varphi_n \circ i_n = j_n \circ \varphi'_n \Rightarrow \varphi_{n*} \circ i_{n*} = j_{n*} \circ \varphi'_{n*}$$

....

$\circlearrowleft$: $[z] \in H_n(\mathcal{C}'')$

$\delta_n([z]) = [u]$

$$
\begin{tikzcd}
 & {}_{w}C_n \arrow[r, "p_n"] \arrow[d, "\partial_n"] & {}_{z}C''_n \\
{}_{u}C'_{n-1} \arrow[r, "i_{n-1}"] & C_{n-1} &
\end{tikzcd}
$$

Devo provare che $\eta_n(\varphi''_{n*}([z])) = [\varphi'_{n-1}(u)]$.

$$\varphi''_{n*}([z]) = [\varphi''_n(z)]$$

$$\eta_n(\varphi''_{n*}([z])) = \eta_n([\varphi''_n(z)])$$

$$\begin{tikzcd}
 & D_n \arrow[r, "q_n"] \arrow[d, "d_n"] & D''_n \\
D'_{n-1} \arrow[r, "i_{n-1}"] & D_{n-1} &
\end{tikzcd}$$

$\varphi''_n(z)$

Ho:

$$\begin{tikzcd}
w \; C_n \arrow[r, "p_n"] \arrow[d, "\varphi_n"'] & z \; C''_n \arrow[d, "\varphi''_n"] \\
D_n \arrow[r, "q_n"] & D''_n \; \varphi''_n(z)
\end{tikzcd}$$

$\Longrightarrow$ pu sollevare $\varphi''_n(z)$ a D_n
va bene $\varphi_n(w)$

$$
\begin{tikzcd}
\varphi_n(w) & D_n \arrow[r, "g_n"] \arrow[d, "d_n"] & D''_n \\
D'_{n-1} \arrow[r, "j_{n-1}"] & D_{n-1} & \varphi''_n(z)
\end{tikzcd}
$$

Se provo che $j_{n-1}(\varphi'_{n-1}(u)) = d_n(\varphi_n(w))$
ho la condizione;

$$
\begin{tikzcd}
u \quad C'_{n-1} \arrow[r, "i_{n-1}"] \arrow[d, "\varphi'_{n-1}"] & C_{n-1} \arrow[d, "\varphi_{n-1}"] \\
D'_{n-1} \arrow[r, "i_{n-1}"] & D_{n-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
w \quad C_n \arrow[r, "\partial_n"] \arrow[d, "\varphi_n"] & C_{n-1} \arrow[d, "\varphi_{n-1}"] \\
D_n \arrow[r, "d_n"] & D_{n-1}
\end{tikzcd}
$$

$$d_n(\varphi_n(w)) = \varphi_{n-1}(\partial_n w) = \varphi_{n-1}(i_{n-1}(u)) . \qquad \boxtimes$$

Successione esatta di Mayer-Vietoris (versione omologica di Van Kampen) —

Sia K c.s. A_1, A_2 sottocomplessi, $K = A_1 \cup A_2$, $L = A_1 \cap A_2$. Ho

$$\begin{tikzcd} C(L) \arrow[r, "i^{(p)}"] \arrow[rr, bend right, "\ell"'] & C(A_p) \arrow[r, "j^{(p)}"] & C(K) \end{tikzcd} \qquad p = 1,2$$

Teorema: la successione

$$0 \longrightarrow C(L) \xrightarrow{(i^{(1)}, i^{(2)})} C(A_1) \oplus C(A_2) \xrightarrow{j^{(1)} - j^{(2)}} C(K) \longrightarrow 0$$

è esatta.

(Dimo rimandato.) Per il teo generale
(esatte corta di complessi di catene
$\rightsquigarrow$ esatte lunga in omologia)
abbiamo:

$$\ldots \to H_n(L) \to H_n(A_1) \oplus H_n(A_2) \to H_n(K) \to H_{n-1}(L) \to \ldots$$

Fatto: vale anche per $\widetilde{H}_n$; per $n=1$
e L connesso dice

$$H_1(L) \to H_1(A_1) \oplus H_1(A_2) \to H_1(K) \to \tilde{H}_0(L)$$

$$\tilde{H}_0(L) = 0$$

$$\Longrightarrow H_1(K) \cong \frac{H_1(A_1) \oplus H_1(A_2)}{H_1(L)}$$

È l'abelianizzata di Van Kampen _