Algebra Lineare - 23/10/13

Teo (Grassmann): $\dim(V) < +\infty$, $Z, W \subset V$ sttosp

$$\Rightarrow \underset{n}{\dim(Z)} + \underset{m}{\dim(W)} = \underset{p}{\dim(Z \cap W)} + \dim(Z + W).$$

Dim: Sia $p = \dim(Z \cap W)$, $n = \dim(Z)$, $m = \dim(W)$

Tesi: $\dim(Z + W) = n + m - p$.

Prendo base $u_1, \dots, u_p$ di $Z \cap W$. Ora $Z \cap W \subset Z$
e $Z \cap W \subset W$

Completo a basi:

$u_1, \dots, u_p, z_1, \dots, z_{n-p}$ base di Z

$u_1, \dots, u_p, w_1, \dots, w_{m-p}$ base di W.

Affermo che

$\underbrace{u_1, \dots, u_p}_{p}, \underbrace{z_1, \dots, z_{n-p}}_{n-p}, \underbrace{w_1, \dots, w_{m-p}}_{m-p}$ sono base di $Z+W$

$\Longrightarrow \dim(Z+W) = p + n - p + m - p = n + m - p$

$\Longrightarrow$ OK.

• lin. indip: Sia

$$\alpha_1 u_1 + \dots + \alpha_p u_p + \beta_1 z_1 + \dots + \beta_{n-p} z_{n-p} + \gamma_1 w_1 + \dots + \gamma_{m-p} w_{m-p} = 0$$

Allora:

$$(*) \quad \underbrace{\alpha_1 u_1 + \dots + \alpha_p u_p + \beta_1 z_1 + \dots + \beta_{n-p} z_{n-p}}_{\in Z} = -\underbrace{(\gamma_1 w_1 + \dots + \gamma_{m-p} w_{m-p})}_{\in W}$$

$\Rightarrow$ appartiene a $Z \cap W$

$$\Rightarrow -(\gamma_1 w_1 + \dots + \gamma_{m-p} w_{m-p}) = \delta_1 u_1 + \dots + \delta_p u_p$$

$$\Rightarrow \delta_1 u_1 + \dots + \delta_p u_p + \gamma_1 w_1 + \dots + \gamma_{m-p} w_{m-p} = 0$$

$$\Rightarrow \delta_1 = \dots = \delta_p = \gamma_1 = \dots = \gamma_{m-p} = 0$$

(i vett sono base di W)

Sostituendo in $\otimes$ ho

$$\alpha_1 u_1 + \dots + \alpha_p u_p + \beta_1 z_1 + \dots + \beta_{m-p} z_{m-p} = 0$$

$$\Rightarrow \alpha_1 = \dots = \alpha_p = \beta_1 = \dots = \beta_{m-p} = 0 \quad . \quad \underline{\text{OK}}$$

(base di Z)

• generano: se $t \in Z + W$, ho $t = \underset{\substack{\cap \\ Z}}{z} + \underset{\substack{\cap \\ W}}{w}$

$$\Longrightarrow \quad \begin{aligned} z &= \alpha_1 u_1 + \dots + \alpha_p u_p + \beta_1 z_1 + \dots + \beta_{n-p} z_{n-p} \\ w &= \gamma_1 u_1 + \dots + \gamma_p u_p + \delta_1 w_1 + \dots + \delta_{m-p} w_{m-p} \end{aligned}$$

$$t = z + w = (\alpha_1 + \gamma_1) u_1 + \dots + (\alpha_p + \gamma_p) u_p + \\ + \beta_1 z_1 + \dots \quad + \delta_1 w_1 + \dots$$

<u>OK</u> $\boxed{\checkmark}$

Esempio: $V=\mathbb{R}^4$ $\quad Z=\mathrm{Span}\left(\begin{pmatrix}1\\2\\-1\\3\end{pmatrix},\begin{pmatrix}-1\\2\\1\\0\end{pmatrix}\right)$

$$W=\mathrm{Span}\left(\begin{pmatrix}0\\1\\-1\\0\end{pmatrix},\begin{pmatrix}1\\2\\1\\-1\end{pmatrix}\right)$$

$\dim Z=\dim W=2$

$Z\cap W$: $\quad \alpha\begin{pmatrix}1\\2\\-1\\3\end{pmatrix}+\beta\begin{pmatrix}-1\\2\\1\\0\end{pmatrix}=\gamma\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}+\delta\begin{pmatrix}1\\2\\1\\-1\end{pmatrix}$

$\underline{\mathrm{IV}}$: $\delta=-3\alpha$ $\qquad$ I : $\beta=4\alpha$

$$\mathrm{II}: \quad 12\alpha - \gamma = 0 \qquad \Rightarrow \alpha = \beta = \gamma = 0$$

$$\mathrm{III}: \quad 6\alpha + \gamma = 0 \qquad \Rightarrow Z \cap W = \{0\}$$

$$G: \quad \underset{\dim Z}{2} + \underset{\dim W}{2} = \underset{\dim Z\cap W}{0} + \underset{\dim(Z+W)}{4}$$

(Oss: Se $Z = \mathrm{Span}(z_1, \dots, z_n)$, $W = \mathrm{Span}(w_1, \dots, w_m)$ $\Rightarrow Z + W = \mathrm{Span}(z_1, \dots w_1, \dots)$)

Infatti: $Z + W = \mathrm{Span}\left(\begin{pmatrix}1\\2\\-1\\3\end{pmatrix}\begin{pmatrix}-1\\2\\1\\0\end{pmatrix}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\begin{pmatrix}1\\2\\1\\-1\end{pmatrix}\right) = \mathbb{R}^4.$

$\checkmark \quad \checkmark \quad \checkmark \quad \checkmark$

Esempio: $Z = \mathrm{span}\left(\begin{pmatrix}1\\2\\-1\\3\end{pmatrix}, \begin{pmatrix}-2\\1\\3\\1\end{pmatrix}\right)$ $W = \mathrm{span}\left(\begin{pmatrix}1\\4\\-1\\5\end{pmatrix}, \begin{pmatrix}4\\5\\-5\\7\end{pmatrix}\right)$

$\dim Z = \dim W = 2$

$Z \cap W$: $\alpha \begin{pmatrix}1\\2\\-1\\3\end{pmatrix} + \beta \begin{pmatrix}-2\\1\\3\\1\end{pmatrix} = \gamma \cdot \begin{pmatrix}1\\4\\-1\\5\end{pmatrix} + \delta \cdot \begin{pmatrix}4\\5\\-5\\7\end{pmatrix}$

Le soluzioni sono tutte e sole quelle del tipo

$\alpha = t \quad \beta = -t \quad \gamma = -t \quad \delta = t \qquad t \in \mathbb{R}$

$$\Rightarrow Z \cap W = \left\{ t \begin{pmatrix} 3 \\ 1 \\ -4 \\ 2 \end{pmatrix} : t \in \mathbb{R} \right\} = \operatorname{Span}\begin{pmatrix} 3 \\ 1 \\ -4 \\ 2 \end{pmatrix}$$

$$\Rightarrow \dim Z \cap W = 1.$$

$$G: \quad \begin{array}{ccccccc} 2 & + & 2 & = & 1 & + & 3 \\ Z & & W & & Z \cap W & & Z+W \end{array}$$

Infatti $Z+W = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ -1 \\ 5 \end{pmatrix}, \cancel{\begin{pmatrix} 4 \\ 5 \\ -5 \\ 7 \end{pmatrix}} \right) \Rightarrow \dim = 3$

— o —

<u>Def</u>: Se V, W sono sp. vett. $f: V \to W$ è detta lineare se ("rispetta le operaz. di sp. vett."):

- $f(0) = 0$
- $f(v_1 + v_2) = f(v_1) + f(v_2)$
- $f(\lambda v) = \lambda \cdot f(v)$

<u>Oss</u>: La proprietà $f(0) = 0$ segue da $f(\lambda \cdot v) = \lambda \cdot f(v)$.

<u>Oss</u>: Posso unificare:

$$f(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 f(v_1) + \lambda_2 f(v_2).$$

Esempi: $f: \mathbb{R}^2 \to \mathbb{R}^3$

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5x_1 - 3x_2 \\ 2x_1 + 4x_2 \\ \pi x_1 + 7x_2 \end{pmatrix}$$

ciascuna componente di $f(x)$ è pol. I grado omogeneo nelle compo di x

È lineare:

$$f(\lambda x + \mu y) \overset{?}{=} \lambda \cdot f(x) + \mu f(y)$$

$$\begin{array}{ccc} \| & & \| \end{array}$$

$$f\left(\begin{pmatrix} \lambda x_1 + \mu y_1 \\ \lambda x_2 + \mu y_2 \end{pmatrix}\right) \qquad \lambda \cdot \begin{pmatrix} 5x_1 - 3x_2 \\ 2x_1 + 4x_2 \\ \pi x_1 + 7x_2 \end{pmatrix} + \mu \begin{pmatrix} 5y_1 - 3y_2 \\ 2y_1 + 4y_2 \\ \pi y_1 + 7y_2 \end{pmatrix}$$

$$\|$$

$$\begin{pmatrix} 5(\lambda x_1+\mu y_1)-3(\lambda x_2+\mu y_2) \\ 2(\lambda x_1+\mu y_1)+4(\lambda x_2+\mu y_2) \\ \pi(\lambda x_1+\mu y_1)+7(\lambda x_2+\mu y_2) \end{pmatrix} = \begin{pmatrix} 5\lambda x_1+5\mu y_1-3\lambda x_2-3\mu y_2 \\ \vdots \\ \end{pmatrix} \checkmark$$

Es: $f: \mathbb{R}^4 \to M_{2\times 2}(\mathbb{R})$

$$f(x) = \begin{pmatrix} 5x_1-7x_2+x_3 & e\,x_1-\sqrt{3}\,x_4 \\ -9x_2+x_3 & x_1+x_2+x_3+x_4 \end{pmatrix}$$

lineare

Es: $f: \mathbb{R}_{\leq 2}[t] \to \mathbb{R}^2$, $f(p(x)) = \begin{pmatrix} 2p''(1) - p(0) \\ p'(-1) \end{pmatrix}$

$$f(a_0 + a_1 t + a_2 t^2) = \begin{pmatrix} 4a_2 - a_0 \\ a_1 - 2a_2 \end{pmatrix} \quad \text{lineare}$$

Es: $X = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$

$$f: X \to \mathbb{R}^2, \quad f(x) = \begin{pmatrix} \log(e^{5x_1 - \pi x_2}) \\ 2x_1 + (x_1 + x_2)^2 - x_3^2 \end{pmatrix} = \begin{pmatrix} 5x_1 - \pi x_2 \\ 2x_1 \end{pmatrix}$$

$\mathbb{R}$ espressione lineare mascherata.

Def: Se $f: V \to W$ è lineare pongo

$$\left(\begin{array}{l} \mathrm{Ker}(f) = \{ v \in V : f(v) = 0 \} \subset V \quad \text{nucleo} \\ \mathrm{Im}(f) = \{ f(v) : v \in V \} = \{ w \in W : \exists v \in V \text{ con } f(v) = w \} \subset W \\ \text{immagine} \end{array} \right)$$

Prop: $\operatorname{Ker} f \subset V$ e $\operatorname{Im} f \subset W$ sono sottosp.

Dim: •) $v_1, v_2 \in \operatorname{Ker} f$, cioè $f(v_1) = f(v_2) = 0$

$$\Longrightarrow f(\underbrace{\lambda_1 v_1 + \lambda_2 v_2}) = \lambda_1 \underset{0}{\underset{\shortparallel}{f(v_1)}} + \lambda_2 \underset{0}{\underset{\shortparallel}{f(v_2)}} = 0$$

$$\Longrightarrow \lambda_1 v_1 + \lambda_2 v_2 \in \operatorname{Ker} f.$$

•) $w_1, w_2 \in \operatorname{Im}(f)$, cioè $w_1 = f(v_1)$ $w_2 = f(v_2)$

$$\Rightarrow \underbrace{\lambda_1 w_1 + \lambda_2 w_2} = \lambda_1 f(v_1) + \lambda_2 f(v_2) = f(\lambda_1 v_1 + \lambda_2 v_2)$$

$$\Rightarrow \lambda_1 w_1 + \lambda_2 w_2 \in \operatorname{Im}(f) \qquad \blacksquare$$

Esercitazioni

Esercizi 4/10/13

14) $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$F = \{ x \in \mathbb{R}^2 \mid f(x) = 0 \}$ $G = \{ x \in \mathbb{R}^2 \mid g(x) = 0 \}$

$u : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $u(x) = f(x) \cdot g(x)$

$u(x) = f(x) \cdot g(x) = 0 \iff f(x) = 0$ o $g(x) = 0$

$$\{ x \in \mathbb{R}^2 \mid u(x) = 0 \} \subseteq F \cup G$$

$$\Uparrow$$

$$\cup \Leftarrow \Longleftarrow \qquad \Rightarrow$$

$$F \cup G$$

$$\Rightarrow \{ x \in \mathbb{R}^2 \mid u(x) = 0 \} = F \cup G$$

"tale che" come :

$$i : \mathbb{R}^2 \to \mathbb{R} \qquad i(x) = f(x)^2 + g(x)^2$$

$$\Rightarrow i(x) = 0 \iff f(x) = g(x) = 0$$

$\Rightarrow \{x \in \mathbb{R} \mid i(x) = 0\} = F \cap G$

Esercizi dell' 11/10/13

1) (a) W è un s.s. perché definito da un'eq.ne lineare omogenea nelle componenti.

(b) $n \geq 2$ $\quad (1, 1, 0, \dots, 0) \in W$

$\quad (1, -1, 0, \dots, 0) \in W$

$(1, 1, 0, \dots, 0) + (1, -1, 0, \dots, 0) = (2, 0, 0, \dots, 0)$

$2^2 = 4 \neq 0^2 = 0 \implies \notin W$

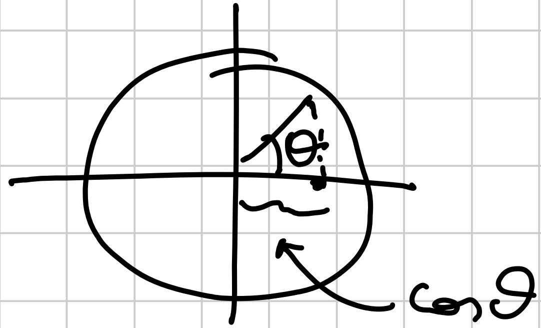
$\implies$ W non è s.s. vettoriale!

(c) $n \geq 2$: $\quad x_1^3 = x_2^3 \iff x_1 = x_2$

$\implies W = \{x \in \mathbb{R}^n : x_1 = x_2\} = \{x \in \mathbb{R}^n : x_1 - x_2 = 0\}$

è s.s. perché definito da un'eq.ne lineare omogenea.

(d) $\cos(x_1 + \cdots + x_n) = 1$



$$\Updownarrow$$

$$x_1 + \cdots + x_n = k \cdot 2\pi, \quad k \in \mathbb{Z}$$

$$\frac{1}{2}x_1 + \cdots + \frac{1}{2}x_n = \frac{1}{2} \cdot k \cdot 2\pi = k \cdot \pi, \quad k \in \mathbb{Z}$$

$$\Rightarrow \left(\frac{1}{2}x_1, \dots, \frac{1}{2}x_n\right) = \frac{1}{2} \cdot (x_1, \dots, x_n) \notin W$$

sebbene $(x_1, \dots, x_n) \in W \Rightarrow W$ non è s.s.

Esercizio 2 $\qquad \mathbb{R}[t]$

(a) $p(-3) + p''(2) = 0$

eq. lineare omogenea nei coefficienti $\Rightarrow W$ è s.s.

$$p_1, p_2 \in W \quad \overset{?}{\Longrightarrow} \quad \lambda_1 p_1 + \lambda_2 p_2 \in W$$

$$\lambda_1, \lambda_2 \in \mathbb{R}$$

$$(\lambda_1 p_1 + \lambda_2 p_2)(-3) = \lambda_1 p_1(-3) + \lambda_2 p_2(-3)$$

$$[\lambda_1 p_1(t) + \lambda_2 p_2(t)]' = \lambda_1 p_1'(t) + \lambda_2 p_2'(t)$$

$$p_1(t) = \sum_{k=0}^{m} a_k t^k \, , \quad p_2(t) = \sum_{h=0}^{m} b_h t^h$$

$$\lambda_1 p_1(t) + \lambda_2 p_2(t) =$$

$$= \sum_{k=0}^{m} (\lambda_1 a_k) t^k + \sum_{h=0}^{m} (\lambda_2 b_h) t^h =$$

$$= \sum_{k=0}^{m} (\lambda_1 a_k + \lambda_2 b_k) t^k$$

$$\left(\lambda_1 p_1(t) + \lambda_2 p_2(t)\right)' =$$

$$= \sum_{k=0}^{n} k \cdot (\lambda_1 a_k + \lambda_2 b_k)\, t^{k-1}$$

$$\parallel$$

$$\left.\begin{aligned} p_1'(t) &= \sum_{k=0}^{n} k\, a_k\, t^{k-1} \\ p_2'(t) &= \sum_{k=0}^{n} k\, b_k\, t^{k-1} \end{aligned}\right\} \Rightarrow \lambda_1 p_1'(t) + \lambda_2 p_2'(t)$$

$$(\lambda_1 p_1 + \lambda_2 p_2)(-3) + (\lambda_1 p_1 + \lambda_2 p_2)''(2) =$$

$$= \lambda_1 p_1(-3) + \lambda_2 p_2(-3) + (\lambda_1 p_1'' + \lambda_2 p_2'')(2)$$

$$= \lambda_1 p_1(-3) + \lambda_2 p_2(-3) + \lambda_1 p_1''(2) + \lambda_2 p_2''(2)$$

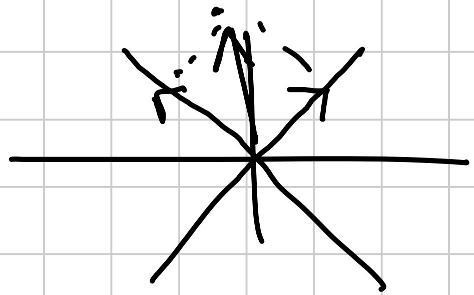
$$= \lambda_1 \Big(\underbrace{p_1(-3) + p_1''(2)}_{=\,0}\Big) + \lambda_2 \Big(\underbrace{p_2(-3) + p_2''(2)}_{=\,0}\Big) = 0$$

(b) $p(1) \cdot p'''(-1) = 0$

non è (evidentemente) un'eq.ne lineare.

<u>OSS</u>: $W = \{p \mid p(1) = 0\} \cup \{p \mid p'''(-1) = 0\}$

Ma in generale l'unione di s.s. <u>non è</u> un s.s.:



$\mathbb{R}^2$

$p(t) = 1 \in W \ni q(t) = (t-1)^3 \quad q'''(t) = 6, \; q(1) = 0$

$$p(t) + q(t) = 1 + (t-1)^3$$

$$(p(t) + q(t))(\underline{1}) = p(1) + q(1) = 1 + 0 \neq 0$$

$$(p(t) + q(t))'''(-1) = \underbrace{p'''(-1)}_{\parallel \atop 0} + \underbrace{q'''(-1)}_{\parallel \atop 6} = 6 \neq 0$$

$\Longrightarrow p(t) + q(t) \notin W \Longrightarrow W$ non è s.s.

(e) $\deg\left(2p'(t) - 3t^2 \cdot p'''(t)\right) \leq 5$

$p_1, p_2 \in W \quad , \quad \lambda_1, \lambda_2 \in \mathbb{R}$

$$2\cdot(\lambda_1 p_1 + \lambda_2 p_2)'(t) - 3t^2(\lambda_1 p_1 + \lambda_2 p_2)'''(t)$$

$$2\cdot\left(\lambda_1 p_1'(t) + \lambda_2 p_2'(t)\right) - 3t^2\cdot\left(\lambda_1 p_1'''(t) + \lambda_2 p_2'''(t)\right)$$

$$= \lambda_1\left(2p_1'(t) - 3t^2 p_1'''(t)\right) + \lambda_2\left(2p_2'(t) - 3t^2 p_2'''(t)\right)$$

ha deg ≤ 5 perché $p_1 \in W$

ha deg ≤ 5 perché $p_2 \in W$

$$\| \\ (*)$$

$\Rightarrow$ (*) ha deg ≤ 5

$\Rightarrow$ W chiuso rispetto alle combinazioni lineari $\Rightarrow$ W è un s.s.

Esercizio 3

(b), (c) : sono s.s. perché definiti da eq.ni lineari omogenee nei coefficienti

(a) $(A)_{1,1} \cdot (A)_{m,n} = 0$

$$W = \left\{ \begin{pmatrix} 0 & \\ & * \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} * & \\ & 0 \end{pmatrix} \right\}$$

$$M_1 = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}$$

$$M_1 , M_2 \in W$$

$$M_1 + M_2 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 1 \end{pmatrix} \notin W$$

$\Longrightarrow$ W non è chiuso rispetto alla somma $\Rightarrow$ non è S.S.

(d) $\displaystyle\sum_{j=1}^{\min(m,n)} (A)_{j,j}^2 = 0 \iff$

$$(A)_{11} = (A)_{22} = \dots = (A)_{\min(m,n),\min(m,n)} = 0$$

(insieme) sistema di eq.ni lineari omogenee $\Rightarrow$ W S.S.

(e) non è chiuso rispetto alla moltiplicazione per scalari

ad es. $A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \cdots & & 1 \end{pmatrix} \in W$,

$10 \cdot A = \begin{pmatrix} 10 & \cdots & 10 \\ \vdots & & \\ 10 & \cdots & 10 \end{pmatrix} \notin W$.

Esercizio 4

(a) $f(c) \geq 2 f(a)$ ← non è una condizione lineare

$f \in \mathcal{F}(a,b,c)$, $f(a) = 1$

$f \in V$ $\quad f(b) = 0$

$\quad f(c) = 2$

$(-f)(a) = -1$

$(-f)\cdot(b) = 0$

$(-f)\cdot(c) = -2$

$$\underset{-2}{\overset{\shortparallel}{(-f)(c)}} \overset{?}{\geq} \underset{-1}{\overset{\shortparallel}{(-f)(a)}}$$

no

moltiplicazione per -1 non conserva

$W \Rightarrow W$ non è chiuso rispetto alla moltiplicazione per scalare

$\Rightarrow$ non è s.s.

(b) $3f(a) - 7f(b) = 0$

eq. lin. omogenea $\Rightarrow W$ s.s.

$f_1, f_2 \in W$, $\lambda_1, \lambda_2 \in \mathbb{R}$

$$3(\lambda_1 f_1 + \lambda_2 f_2)(a) - 7(\lambda_1 f_1 + \lambda_2 f_2)(b) =$$

$$3\cdot(\lambda_1 f_1(a) + \lambda_2 f_2(a)) - 7(\lambda_1 f_1(b) + \lambda_2 f_2(b))$$

$$= \lambda_1 \big(\underbrace{3 f_1(a) - 7 f_1(b)}_{=0 \quad f_1 \in W}\big) + \lambda_2 \big(\underbrace{3 f_2(a) - 7 f_2(b)}_{=0 \quad f_2 \in W}\big)$$

$$= 0 \Rightarrow W \text{ è s.s.}$$

Es. 5

c) $v \overset{?}{\in} \mathrm{Span}(w_1, w_2)$

$$a w_1 + b w_2 = v \quad ?$$

$$a \cdot \begin{pmatrix} 4 \\ -11 \end{pmatrix} + b \cdot \begin{pmatrix} 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix} \iff$$

$$\parallel$$

$$\begin{pmatrix} 4a + 3b \\ -11a + 8b \end{pmatrix} \qquad \begin{cases} 4a + 3b = 3 \\ -11a + 8b = 7 \end{cases}$$

$\rightarrow$ si ricavano a, b unici che risolvono il sistema $\Rightarrow$ $v \in \operatorname{Span}(w_1, w_2)$ e si scrive in modo unico.