Esercizî di Algebra Lineare (Petronio 12/13)

19 ottobre 2012

Esercizio 1 Esibire una base dello spazio vettoriale assegnato:

(a) $\{x \in \mathbb{R}^3 :\ 7x_1 + 5x_2 - 4x_3 = 0\}$

(b) $\left\{ x \in \mathbb{R}^3 :\ \begin{array}{l} 3x_1 + 2x_2 - 5x_3 = 0 \\ -4x_1 + x_2 + 7x_3 = 0 \end{array} \right\}$

(c) $\left\{ x \in \mathbb{R}^3 :\ \begin{array}{l} -4x_1 + 2x_2 + 5x_3 = 0 \\ 3x_1 + 7x_2 - 2x_3 = 0 \\ x_1 + 27x_2 + 4x_3 = 0 \end{array} \right\}$

(d) $\{x \in \mathbb{R}^4 :\ 5x_1 - 8x_2 + 2x_3 - 3x_4 = 0\}$

(e) $\{x \in \mathbb{R}^4 :\ 9x_1 + 4x_3 - 5x_4\}$

(f) $\left\{ x \in \mathbb{R}^4 :\ \begin{array}{l} 5x_1 + 3x_2 + 7x_3 - 2x_4 = 0 \\ 8x_1 - 7x_2 + 4x_3 - 5x_4 = 0 \end{array} \right\}$

(g) $\left\{ x \in \mathbb{R}^4 :\ \begin{array}{l} 2x_1 + 4x_2 + 7x_3 - 9x_4 = 0 \\ -5x_1 + 6x_2 + 3x_3 + 2x_4 = 0 \\ 9x_1 + 2x_2 + 11x_3 - 20x_4 = 0 \end{array} \right\}$

(h) $\left\{ x \in \mathbb{R}^n :\ \sum_{j=1}^{n} (-1)^{j+1} x_j = 0 \right\}$

(j) $\left\{ x \in \mathbb{R}^n :\ \sum_{j=1}^{n} (-1)^j x_j = \sum_{j=1}^{n} \left(1 + (-1)^{j+1}\right) x_j = 0 \right\}$

(k) $\{A \in \mathcal{M}_{3\times 2}(\mathbb{R}) :\ a_{11} + a_{32} - 2a_{21} = a_{12} - a_{31} + 5a_{21} = 0\}$

(l) $\left\{ A \in \mathcal{M}_{2\times 3}(\mathbb{R}) : \begin{array}{l} a_{11} - 2a_{12} + 3a_{23} = 0 \\ -a_{11} + a_{12} + 2a_{13} = 0 \\ a_{12} + a_{21} + a_{23} = 0 \\ a_{21} + 4a_{23} + 2a_{13} = 0 \end{array} \right\}$

(m) $\{A \in \mathcal{M}_{n\times n}(\mathbb{R}) : \ \mathrm{tr}(A) = 0\}$

(n) $\mathcal{S}_n = \{A \in \mathcal{M}_{n\times n}(\mathbb{R}) : \ {}^{\mathrm{t}}A = A\}$

(o) $\mathcal{A}_n = \{A \in \mathcal{M}_{n\times n}(\mathbb{R}) : \ {}^{\mathrm{t}}A = -A\}$

(p) $\{p(t) \in \mathbb{R}_{\leqslant d}[t] : \ p(1) = 0\}$

(q) $\{p(t) \in \mathbb{R}_{\leqslant 4}[t] : \ 2p(1) + p''(-1) = p(2) - 3p'(1) = 0\}$

(r) $\{p(t) \in \mathbb{R}_{\leqslant d}[t] : \ p(-t) = p(t)\}$

(s) $\{p(t) \in \mathbb{R}_{\leqslant d}[t] : \ p(-t) = -p(t)\}$

(t) $\{p(t) \in \mathbb{R}_{\leqslant 3}[t] : \ 6p(t) - 3tp'(t) + 2p''(t) = 0\}$

(u) $\mathcal{F}(\{a, b, c\}, \mathbb{R})$

(v) $\{f \in \mathcal{F}(\{a, b, c\}, \mathbb{R}) : \ 3f(a) - 2f(b) + 5f(c) = 0\}$

(w) $\left\{ f \in \mathcal{F}(\{a, b, c\}, \mathbb{R}) : \begin{array}{l} 2f(a) + 5f(b) + 3f(c) = 0 \\ 7f(a) - 3f(b) + 2f(c) = 0 \end{array} \right\}$

Esercizio 2. Completare i vettori $v_1, \ldots, v_k$ assegnati a una base dello spazio vettoriale V dato.

(a) $V = \mathbb{R}^4, \ v_1 = \begin{pmatrix} 4 \\ 5 \\ -6 \\ 1 \end{pmatrix}, \ v_2 = \begin{pmatrix} -3 \\ 4 \\ 1 \\ 7 \end{pmatrix}$

(b) $V = \mathbb{R}^4, \ v_1 = \begin{pmatrix} 3 \\ -1 \\ 2 \\ 5 \end{pmatrix}, \ v_2 = \begin{pmatrix} 4 \\ 2 \\ -1 \\ 3 \end{pmatrix}, \ v_3 = \begin{pmatrix} 2 \\ -4 \\ 5 \\ 7 \end{pmatrix}$

(c) $V = \{x \in \mathbb{R}^4 :\ 4x_1 + 3x_2 + 5x_3 - 2x_4 = 0\}$, $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 6 \end{pmatrix}$

(d) $V = \{x \in \mathbb{R}^4;\ 3x_1 - 5x_2 + x_3 - 2x_4 = 0\},\ v_1 = \begin{pmatrix} 1 \\ 1 \\ 4 \\ 1 \end{pmatrix},\ v_2 = \begin{pmatrix} 2 \\ -1 \\ -1 \\ 5 \end{pmatrix}$

(e) $V = \{x \in \mathbb{R}^4 :\ 7x_1 + 2x_2 - 4x_3 + 5x_4 = 0\}$,
$v_1 = \begin{pmatrix} 1 \\ -4 \\ 1 \\ 1 \end{pmatrix},\ v_2 = \begin{pmatrix} 2 \\ 1 \\ -3 \\ 4 \end{pmatrix}$

(f) $V = \left\{ x \in \mathbb{R}^4 :\ \begin{array}{c} x_1 + x_2 + x_3 - x_4 = 0 \\ 2x_1 + 5x_2 - x_3 + x_4 = 0 \end{array} \right\},\ v_1 = \begin{pmatrix} 4 \\ -2 \\ 1 \\ 3 \end{pmatrix}$

(g) $V = \mathcal{S}_2,\ v_1 = \begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix},\ v_2 = \begin{pmatrix} -1 & 1 \\ 1 & 4 \end{pmatrix}$

(h) $V = \mathbb{R}[t],\ v_1 = -4 + 3t + 5t^4,\ v_2 = 3 + t^2 - t^7,\ v_3 = 5 + t + t^2 - t^3$

(i) $V = \mathbb{R}_{\leqslant 3}[t],\ v_1 = -4 + 3t + 5t^3,\ v_2 = 3 + t^2 - t^3,\ v_3 = 5 + t + t^2 - t^3$

(j) $V = \{p(t) \in \mathbb{R}_{\leqslant 4}[t] :\ p'(2) = p(-1)\},\ v_1 = 5 + \frac{1}{2}t + 4t^2 - 3t^3 + t^4$

Esercizio 3. Estrarre dai vettori $v_1, \ldots, v_k$ assegnati una base dello spazio vettoriale V dato.

(a) $V = \mathbb{R}^3,\ v_1 = \begin{pmatrix} 4 \\ 6 \\ -10 \end{pmatrix},\ v_2 = \begin{pmatrix} -6 \\ -9 \\ 15 \end{pmatrix},\ v_3 = \begin{pmatrix} 5 \\ -7 \\ 2 \end{pmatrix}$
$v_4 = \begin{pmatrix} -1 \\ 13 \\ -12 \end{pmatrix},\ v_5 = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix},\ v_6 = \begin{pmatrix} \pi \\ \sqrt{17} \\ e \end{pmatrix}$

(b) $V = \{x \in \mathbb{R}^4 : \ x_1 + x_2 = x_3 + x_4\},\ v_1 = \begin{pmatrix} 3 \\ -7 \\ 1 \\ -5 \end{pmatrix},\ v_2 = \begin{pmatrix} 2 \\ 6 \\ 9 \\ -1 \end{pmatrix},$

$v_3 = \begin{pmatrix} 0 \\ 32 \\ 25 \\ 7 \end{pmatrix},\ v_4 = \begin{pmatrix} 6 \\ -1 \\ 1 \\ 4 \end{pmatrix},\ v_5 = \begin{pmatrix} 7 \\ 8 \\ -5 \\ 20 \end{pmatrix}$

(c) $V = \mathbb{R}_{\leqslant 3}[t],\ v_1 = -t,\ v_2 = 1 - t + t^2,\ v_3 = 5 + 2t - t^3,\ v_4 = 6 + t^2 - t^3,$ $v_5 = -8t + 5t^2 + t^3,\ v_6 = 2 - t + 3t^2 - 4t^3$

(d) $V = \mathcal{A}_3,\ v_1 = \begin{pmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ 3 & 2 & 0 \end{pmatrix},\ v_2 = \begin{pmatrix} 0 & -7 & 4 \\ 7 & 0 & -1 \\ -4 & 1 & 0 \end{pmatrix},$

$v_3 = \begin{pmatrix} 0 & 9 & -10 \\ -9 & 0 & 3 \\ 10 & -3 & 0 \end{pmatrix},\ v_4 = \begin{pmatrix} 0 & -1 & 3 \\ 1 & 0 & 2 \\ -3 & -2 & 0 \end{pmatrix},$

$v_5 = \begin{pmatrix} 0 & -7 & 11 \\ 7 & 0 & 3 \\ -11 & -3 & 0 \end{pmatrix}$

(e) $V = \{x \in \mathbb{R}^3 : \ 2x_1 + 5x_2 + 7x_3 = 0\},$

$v_1 = \begin{pmatrix} -13 \\ 1 \\ 3 \end{pmatrix},\ v_2 = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix},\ v_3 = \begin{pmatrix} -1 \\ -8 \\ 6 \end{pmatrix},\ v_4 = \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}$

Esercizio 4. Nello spazio V dato considerare i sottospazî W e Z assegnati e calcolare le dimensioni di $W,\ Z,\ W \cap Z,\ W + Z$, verificando la formula di Grassmann.

(a) $V = \mathbb{R}^4,\ W = \text{Span}\left(\begin{pmatrix} -1 \\ 4 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \\ 3 \end{pmatrix}\right),$

$Z = \text{Span}\left(\begin{pmatrix} 3 \\ 2 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} -9 \\ 3 \\ 0 \\ 8 \end{pmatrix}\right)$

(b) $V = \mathbb{R}^4,\ W = \operatorname{Span}\left(\left(\begin{array}{c} 1 \\ 2 \\ -1 \\ 3 \end{array}\right), \left(\begin{array}{c} 3 \\ -4 \\ 5 \\ 1 \end{array}\right), \left(\begin{array}{c} 7 \\ 2 \\ -1 \\ -1 \end{array}\right)\right\},$
$Z = \left\{ x \in \mathbb{R}^4 : \begin{array}{l} 2x_1 + x_2 - 3x_3 + 5x_4 = 0 \\ 3x_1 - x_2 + 2x_3 + x_4 = 0 \end{array} \right\}$

(c) $V = \mathbb{R}^4,\ W = \left\{ x \in \mathbb{R}^4 : \begin{array}{l} 2x_1 - x_2 + 3x_3 + 4x_4 = 0 \\ 3x_1 + 2x_2 - 5x_3 - x_4 = 0 \\ -5x_1 - 8x_2 + 21x_3 + 11x_4 = 0 \end{array} \right\},$
$Z = \left\{ x \in \mathbb{R}^4 : \begin{array}{l} x_1 + 3x_2 - 2x_3 + 5x_4 = 0 \\ 3x_1 + 2x_2 - 2x_3 + 4x_4 = 0 \end{array} \right\}.$

(d) $V = \mathbb{R}^5,\ W = \operatorname{Span}(2e_1 + e_2 - e_3 + 3e_4 + 4e_5,\ 5e_1 + 2e_2 - e_3 + e_4 - e_5),$
$Z = \left\{ x \in \mathbb{R}^5 : \begin{array}{l} x_2 + x_3 - x_4 + x_5 = 0 \\ x_1 - x_2 + 3x_3 - x_4 + 2x_5 = 0 \end{array} \right\}$

(e) $V = \mathbb{R}_{\leqslant 4}[t],\ W = \operatorname{Span}(1 - t + t^2 + t^3,\ 2 + t - 2t^3 + 3t^4),$
$Z = \left\{ p(t) \in V :\ p(-1) = p''\left(\frac{1}{6}\right) = 0 \right\}$