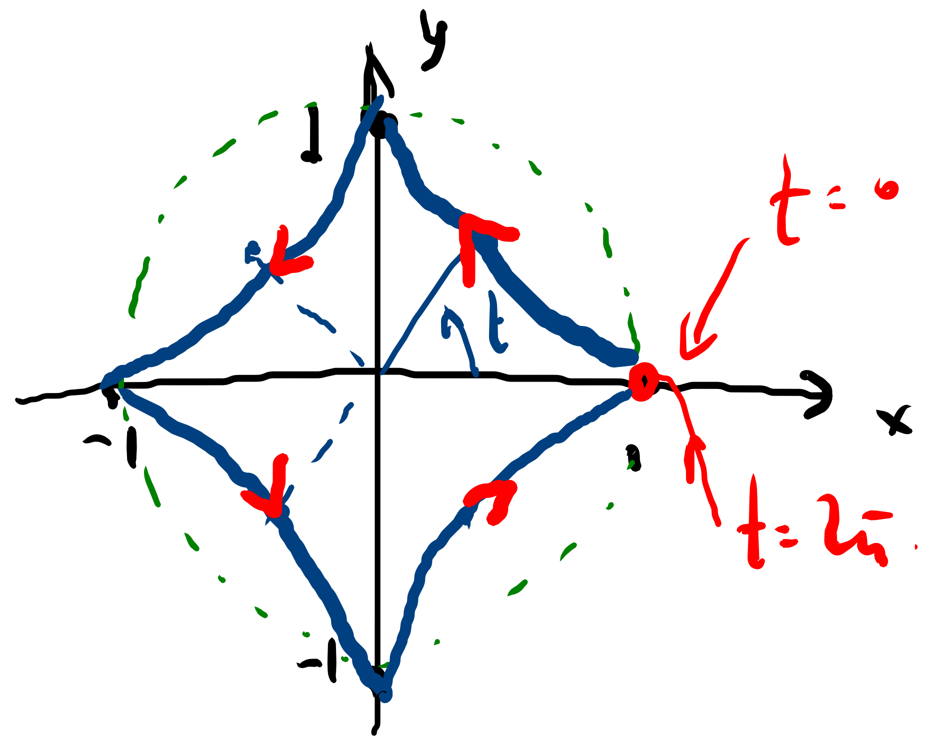
1°] L'astroide

$$A := \{(x,y) : x^{2/3} + y^{2/3} = 1\}$$



$(x,y) \in A \Rightarrow (-x,y) \in A$

$(x,-y) \in A$

$(-x,-y) \in A$.

Da confrontare:

$x^2 + y^2 = 1$ - circonferenza.

Una possibile parametrizzazione

$$\begin{cases} x = \cos^3 t \\ y = \sin^3 t \end{cases} \qquad t \in [0, 2\pi)$$

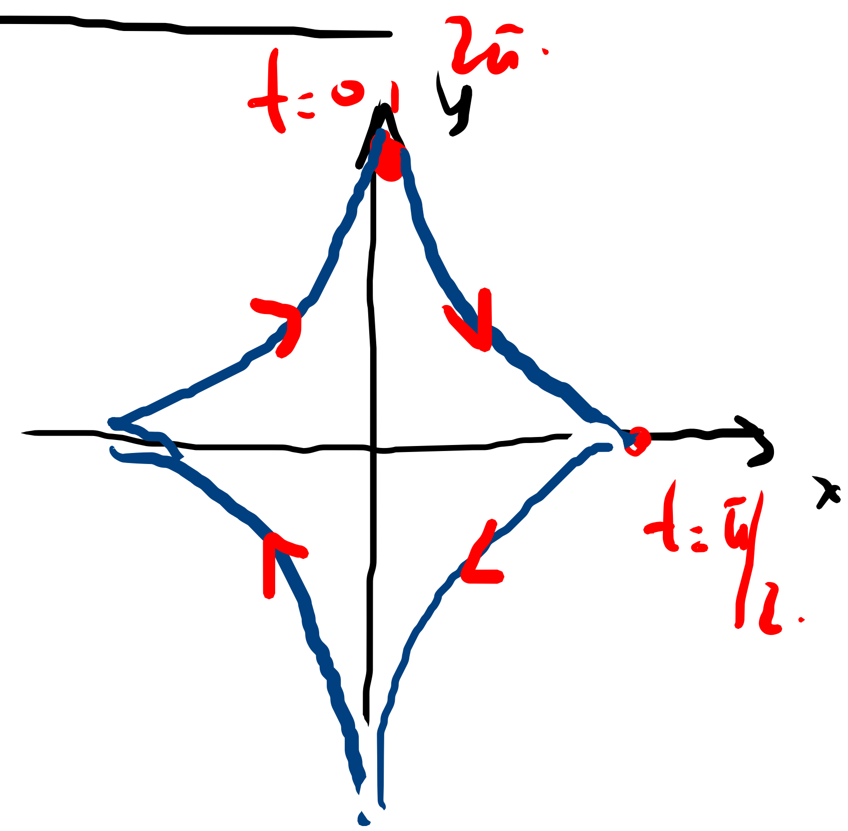
$$x^{2/3} + y^{2/3} = \cos^2 t + \sin^2 t = 1.$$

$t = 0 \Rightarrow x = 1,\ y = 0$

$t = \pi/4 \Rightarrow x = y = \left(\frac{1}{\sqrt{2}}\right)^3 = \frac{1}{2^{3/2}}$

Altre possibili parametrizzazioni:

•) $\begin{cases} x = \sin^3 t \\ y = \cos^3 t \end{cases}$



•) $\begin{cases} x = \cos^3(-t) \\ y = \sin^3(-t) \end{cases}$ $\quad\quad$ $\begin{cases} x = \cos^3 t \\ y = -\sin^3 t \end{cases}$ $\quad t \in [-2\pi, 0]$.

. . .

.) Trovare $\ell(A)$ — lunghezza

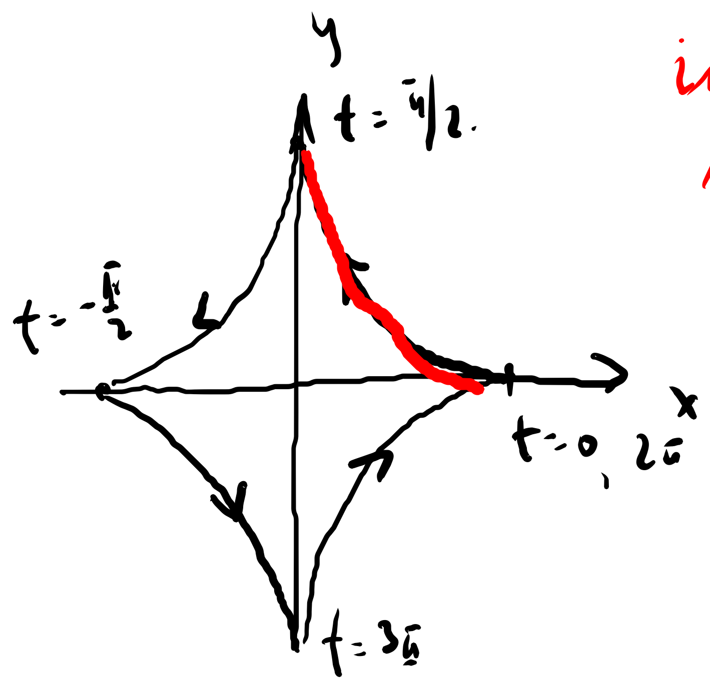
$\theta : I \to \mathbb{R}^d$, $I \subset \mathbb{R}$

$$\ell(\theta) = \int_I |\dot{\theta}(t)|\,dt$$

un caso particolare di un integrale curvilineo di 1ma specie.

$x(t) = \cos^3 t$

$y(t) = \sin^3 t$.



$$\ell(A) = 4 \int_0^{\pi/2} \left| \left(\dot{x}(t), \dot{y}(t) \right) \right| dt = 4 \int_0^{\pi/2} \left| \left(-3\cos^2 t \sin t,\ 3\sin^2 t \cos t \right) \right| dt$$

$$= 4 \int_0^{\pi/2} \sqrt{9\cos^4 t \sin^2 t + 9 \sin^4 t \cos^2 t}\ dt =$$

$$= 4 \int_0^{\pi/2} \sqrt{9\cos^2 t \sin^2 t \underbrace{(\cos^2 t + \sin^2 t)}_{1}}\ dt =$$

$$= 4 \int_0^{\pi/2} \sqrt{9 \cos^2 t \sin^2 t} \; dt = 12 \int_0^{\pi/2} \cos t \sin t \; dt =$$

$$= 6 \int_0^{\pi/2} \sin 2t \; dt = - 6 \frac{\cos 2t}{2} \Big|_0^{\pi/2} =$$

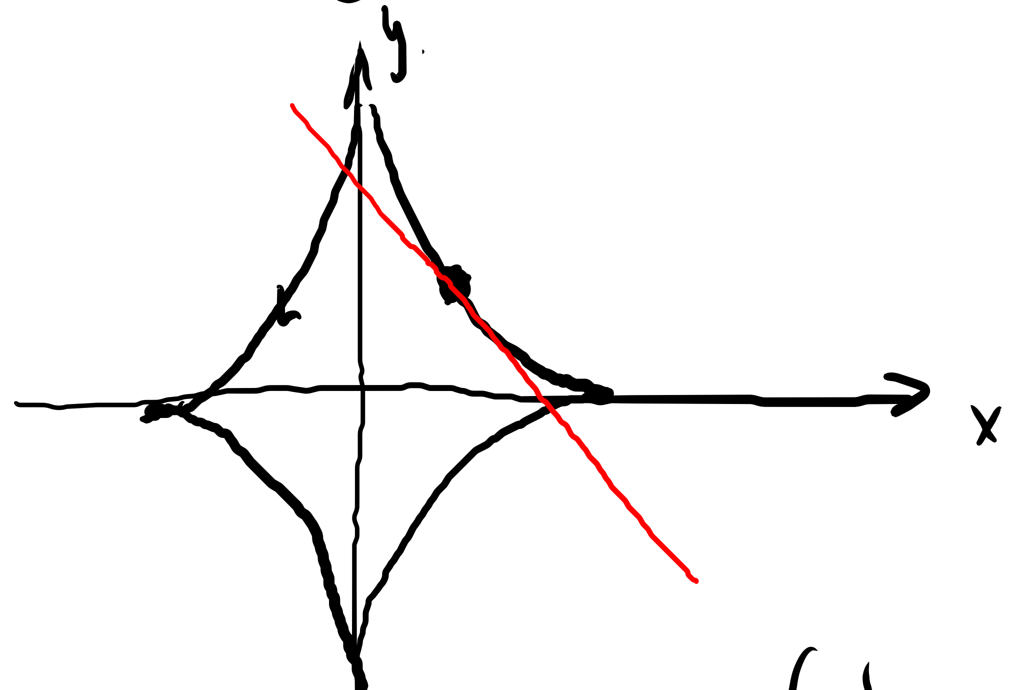
$$= -3 \cos 2t \Big|_0^{\pi/2} = 3 + 3 = 6 .$$

Risposta $\ell(A) = 6$.

•) $\left(\frac{1}{2^{3/2}}, \frac{1}{2^{3/2}}\right) \in A$.

$$\left(\frac{1}{2^{3/2}}\right)^{2/3} + \left(\frac{1}{2^{3/2}}\right)^{2/3} = \frac{1}{2} + \frac{1}{2} = 1$$

Scrivere l'equazione della retta tangente ad A in $\left(\frac{1}{2^{3/2}}, \frac{1}{2^{3/2}}\right)$



Sol 1°

$$\left(\frac{1}{2^{3/2}}, \frac{1}{2^{3/2}}\right) = \left(x\left(\frac{\pi}{4}\right), y\left(\frac{\pi}{4}\right)\right)$$

$$\begin{cases} x(t) = \cos^3 t \\ y(t) = \sin^3 t \end{cases} \qquad \begin{aligned} \dot{x}(t) &= -3\cos^2 t \sin t \\ \dot{y}(t) &= 3\sin^2 t \cos t \end{aligned}$$

$$\dot{x}\left(\frac{\pi}{4}\right) = -\frac{3}{2}\frac{1}{\sqrt{2}} = -\frac{3}{2\sqrt{2}} \qquad T = \left(-\frac{3}{2\sqrt{2}}, \frac{3}{2\sqrt{2}}\right)$$

$$\dot{y}\left(\frac{\pi}{4}\right) = \frac{3}{2}\frac{1}{\sqrt{2}} = \frac{3}{2\sqrt{2}}$$

retta : λ : $\left(\frac{1}{2^{3/2}}, \frac{1}{2^{3/2}}\right) \in \lambda$

$$J = \left(-\frac{3}{2\sqrt{2}}, \frac{3}{2\sqrt{2}}\right) \parallel \lambda$$

In forma parametrica:

$$\begin{cases} x(t) = \frac{1}{2^{3/2}} - \frac{3}{2\sqrt{2}} t \\ y(t) = \frac{1}{2^{3/2}} + \frac{3}{2\sqrt{2}} t \end{cases} \iff \begin{cases} x(t) = \frac{1}{2\sqrt{2}} (1-3t) \\ y(t) = \frac{1}{2\sqrt{2}} (1+3t) \end{cases}$$

$t \in \mathbb{R}$

<u>Ovvero:</u> $x + y = \frac{1}{\sqrt{2}}$

Sol 2°

Retta λ: $\left(\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right) \in \lambda$

e λ tangente ad A

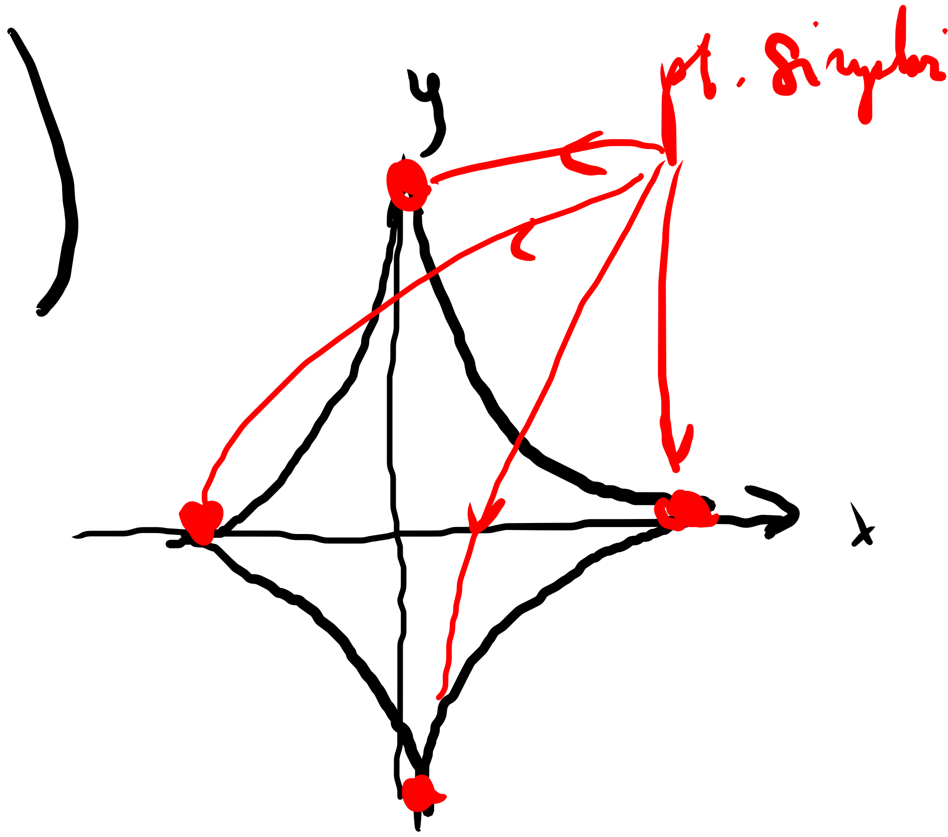
A : $x^{2/3} + y^{2/3} = 1$.

$$f(x,y) = x^{2/3} + y^{2/3}$$

$$A := \{ (x,y) : f(x,y) = 1 \}$$

$$\nabla f(x,y) = (f_x, f_y) = \left(\frac{2}{3} x^{-1/3}, \frac{2}{3} y^{-1/3} \right)$$

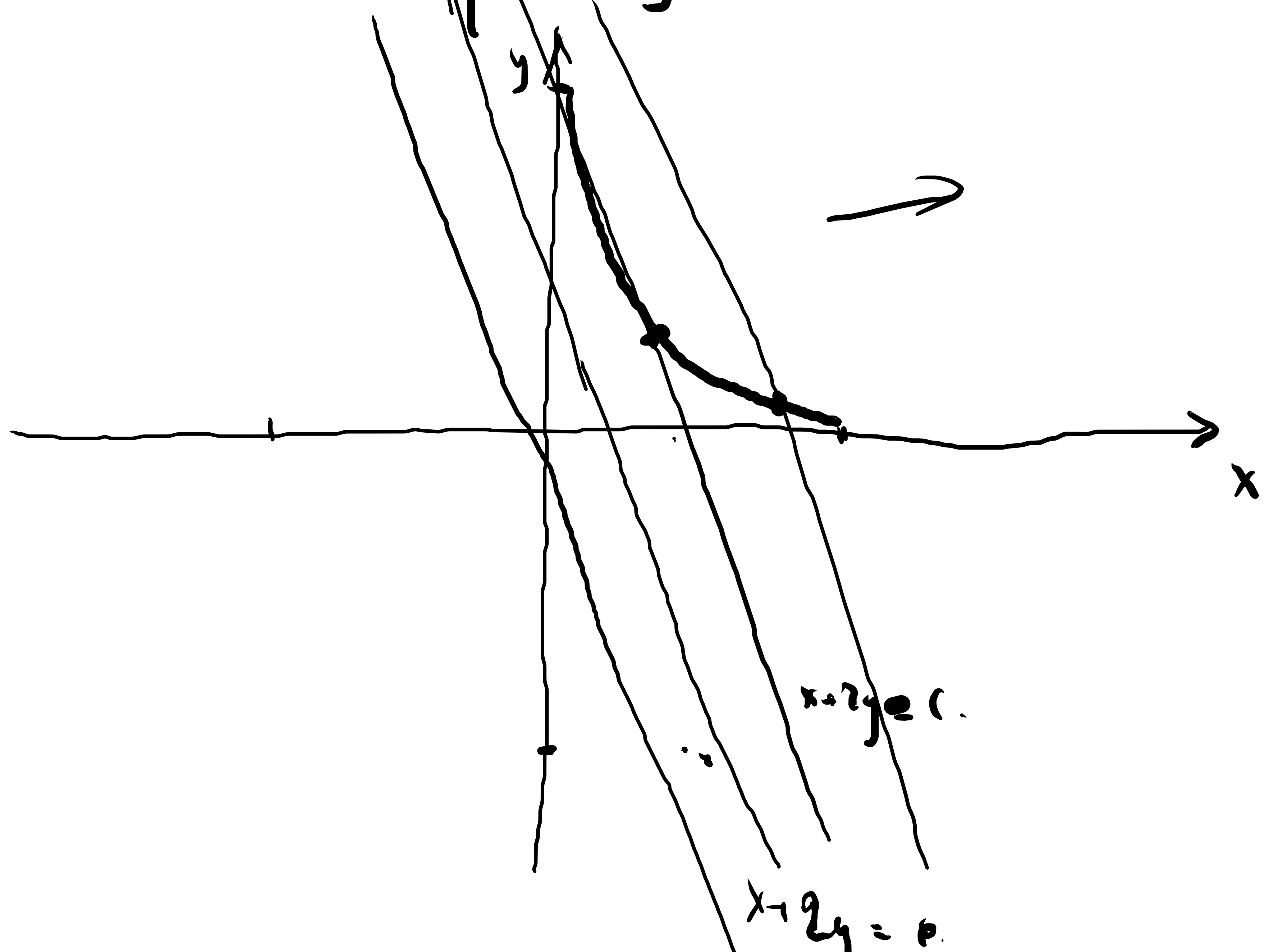
$$\nabla f\left(\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right) = \frac{2}{3}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \underbrace{\perp}_{\text{per proprietà geometrica del gradiente}} \lambda$$



$$\lambda: \quad \frac{2}{3}\cancel{\frac{1}{\sqrt{2}}}\left(x-\frac{1}{2\sqrt{2}}\right)+\frac{2}{3}\cancel{\frac{1}{\sqrt{2}}}\left(y-\frac{1}{2\sqrt{2}}\right)=0.$$

$$\boxed{x+y=\frac{1}{\sqrt{2}}.}$$

3° Trouver $\min\{x+2y : (x,y)\in A,\ x\geq 0,\ y\geq 0\}$.



$x+2y=C.$

Sol 1°

$$\begin{cases} x(t) = \cos^3 t \\ y(t) = \sin^3 t \end{cases} \qquad t \in [0, \pi/2]$$

$$f(x,y) = x + 2y = \cos^3 t + 2\sin^3 t \longrightarrow \min$$
$$t \in [0, \pi/2].$$

$$g(t) = \cos^3 t + 2\sin^3 t$$

$$g'(t) = -3\cos^2 t \sin t + 6\sin^2 t \cos t =$$

$$= 3\cos t \sin t \left(-\cos t + 2\sin t\right) = 0.$$

$$\left[\begin{array}{l} \sin t = 0 \;\; (\cos t = 1) \implies g(t) = 1. \\ \cos t = 0 \;\; (\sin t = 0) \implies g(t) = 2. \\ -\cos t + 2\sin t = 0 \implies \cos t = 2\sin t. \\ \qquad\qquad g(t) = \cos^3 t + 2\sin^3 t = 10\sin^3 t. \end{array}\right.$$

$\tan t = \frac{1}{2}$. $\qquad t \in [0, \frac{\pi}{2}]$.

$t = \arctan \frac{1}{2} \approx 0{,}46$

$g(t) = 10 \sin^3 t = 10 (\sin 0{,}46)^3 \approx 0{,}87$.

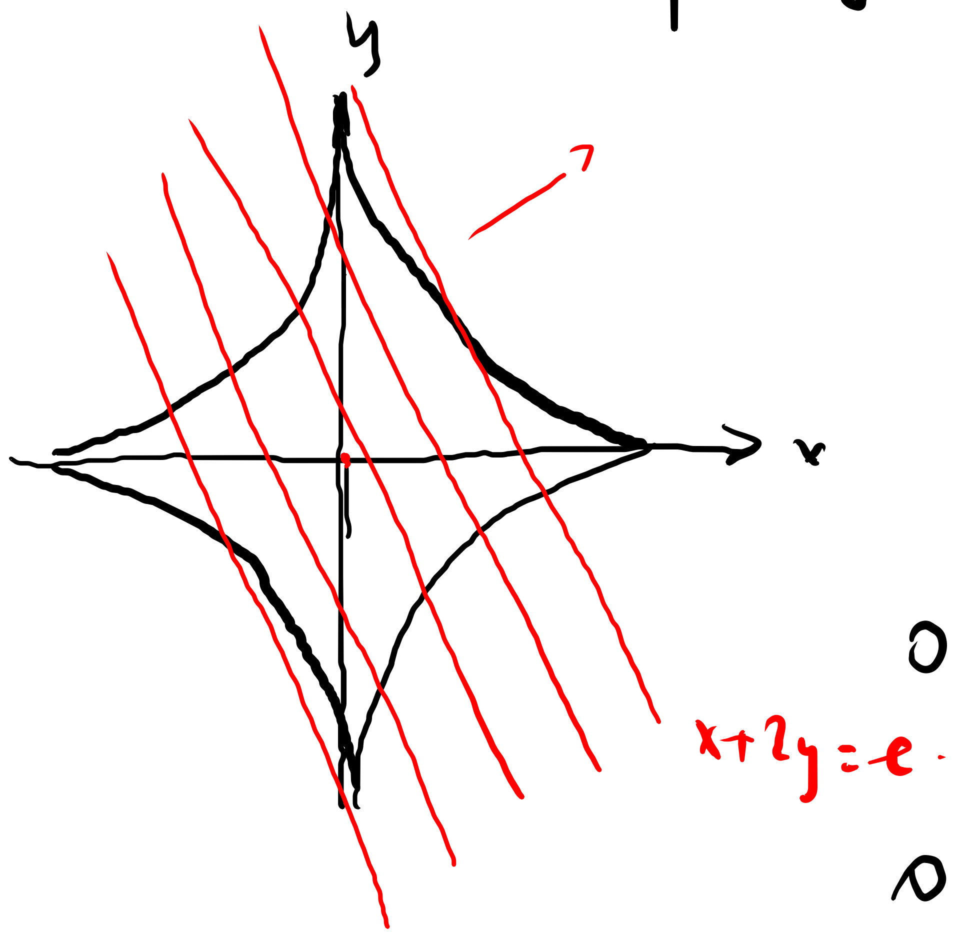
Risposta: $\min \{ x + 2y : (x, y) \in A, \, x \geq 0, \, y \geq 0 \} =$

$= 10 \sin^3 (\arctan \frac{1}{2}) \approx 0{,}87$.

Sol 2°

$$\min\{x+2y : (x,y)\in A;\ x\geq 0,\ y\geq 0\} =$$

$$= \min\{x+2y : \underline{x^{2/3}+y^{2/3}=1},\ \underline{x\geq 0,\ y\geq 0}\}.$$



$$G(x,y) := x+2y+\lambda\left(\underline{x^{2/3}+y^{2/3}-1}\right)$$

$$0 = G_x = 1+\tfrac{2}{3}\lambda x^{-1/3} = 0.$$

$$0 = G_y = 2+\tfrac{2}{3}\lambda y^{-1/3} = 0.$$

$$\begin{cases} x^{-1/3} = -\dfrac{3}{2\lambda} \\ y^{-1/3} = -\dfrac{3}{\lambda} \end{cases} \qquad \begin{aligned} x^{1/3} &= -\frac{2\lambda}{3} \\ y^{1/3} &= -\frac{\lambda}{3} \end{aligned}$$

$$1 = x^{2/3}+y^{2/3} = \frac{4}{9}\lambda^2+\frac{1}{9}\lambda^2 = \frac{5}{9}\lambda^2$$

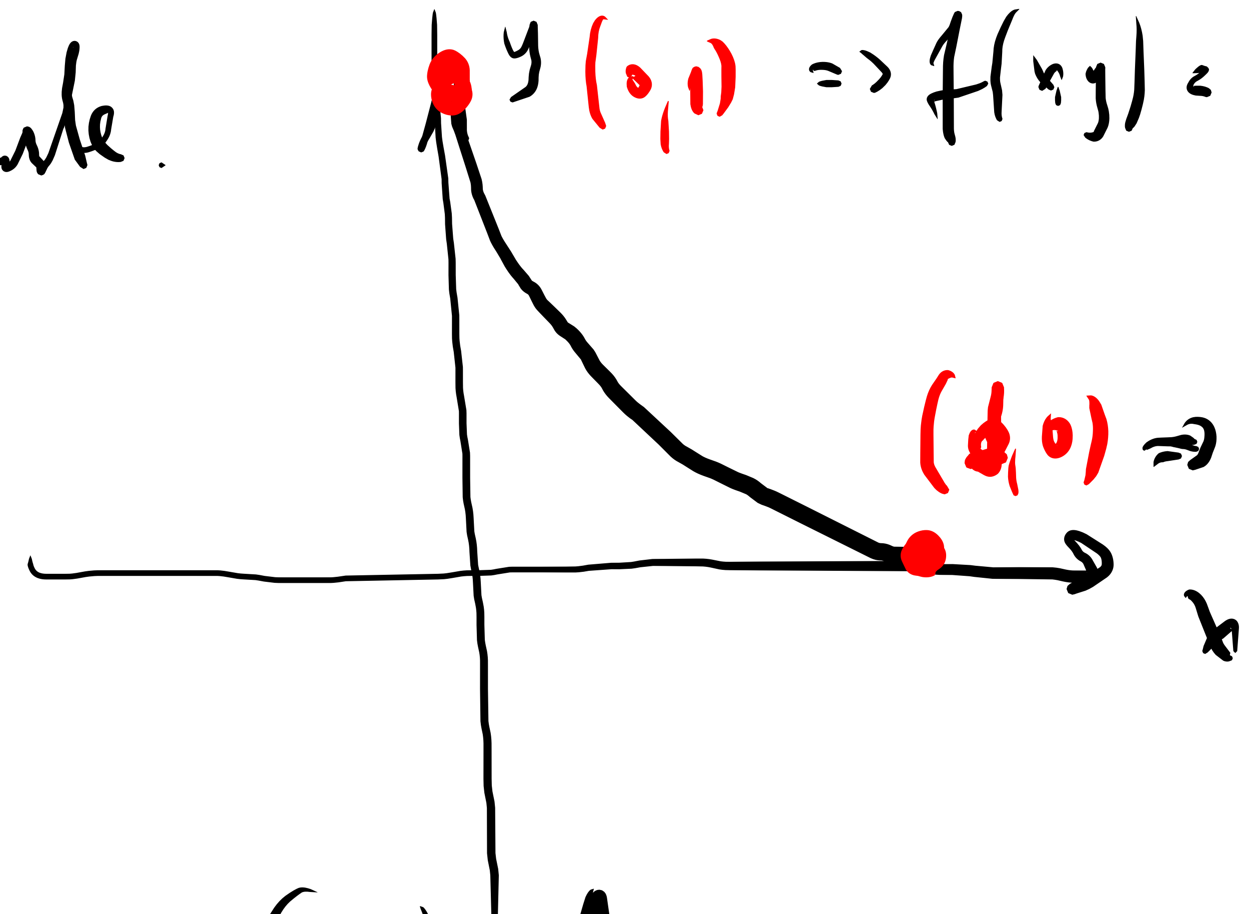
$$\Rightarrow \lambda^2 = \frac{9}{5} \Rightarrow \lambda = \pm\frac{3}{\sqrt{5}}$$

$$\lambda = -\frac{3}{\sqrt{5}},\quad x^{1/3} = \frac{2}{\sqrt{5}},\quad y^{1/3} = \frac{1}{\sqrt{5}}.$$

$$f(x,y) = x+2y = \left(\frac{2}{\sqrt{5}}\right)^3 + 2\left(\frac{1}{\sqrt{5}}\right)^3 = \frac{1}{5\sqrt{5}}\left(2^3+2\right) =$$

$$= \frac{10}{5\sqrt{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5} = 0{,}4\cdot\sqrt{5} \approx \underline{0{,}85}$$

Da verificare separatamente.



$(0,1) \Rightarrow f(x,y) = x+2y = 2$

$(1,0) \Rightarrow f(x,y) = x+2y = 1$

Risposta: $\min\left\{x+2y : (x,y)\in A,\ x\geq 0,\ y\geq 0\right\} = \frac{2\sqrt{5}}{5} \approx$

$\approx 0{,}85.$

2°

$$\begin{cases} x(t,\theta) = t\cos\theta \\ y(t,\theta) = t\sin\theta \\ z(t,\theta) = mt \end{cases}$$

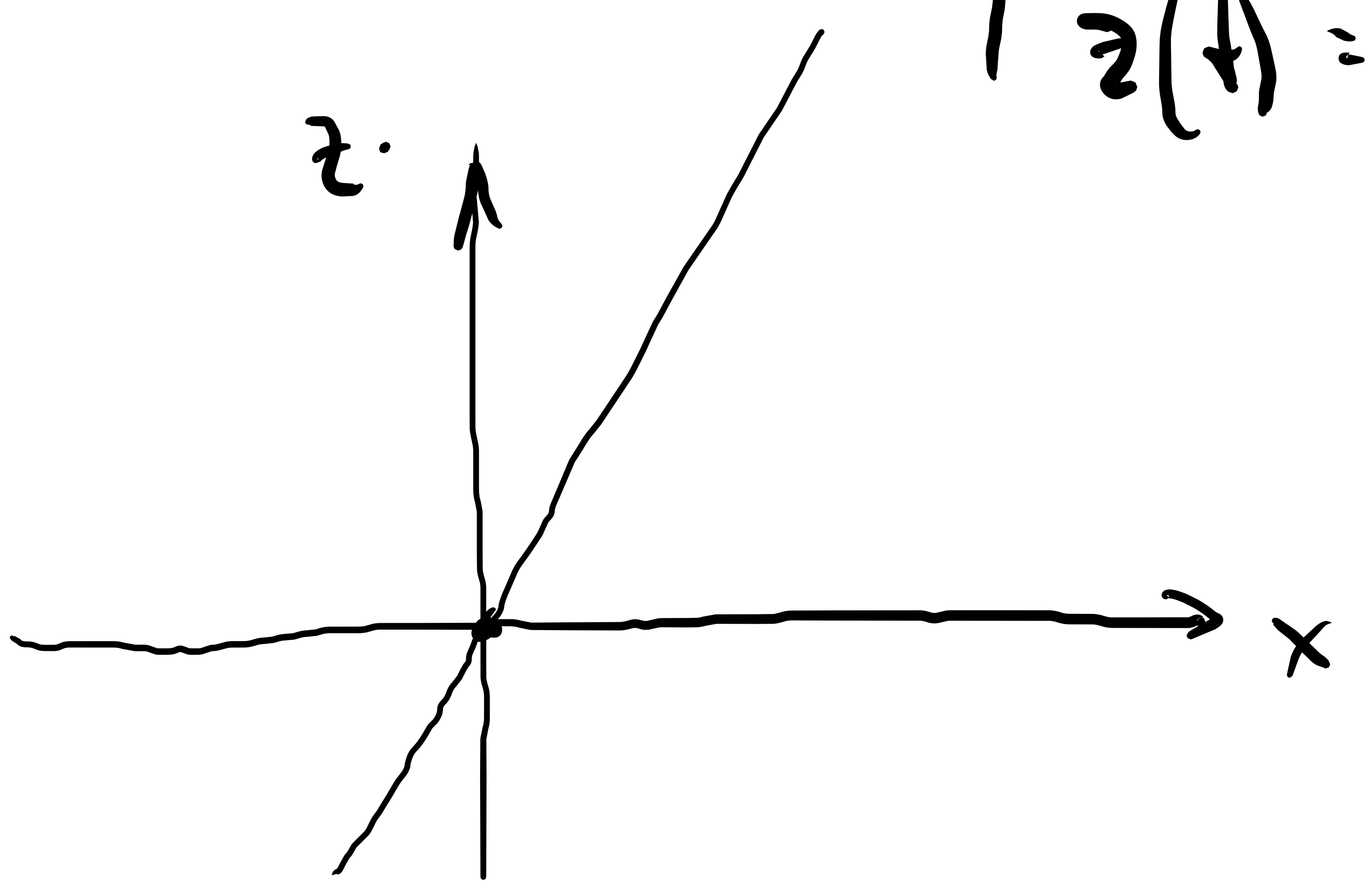
$m > 0$ dato.

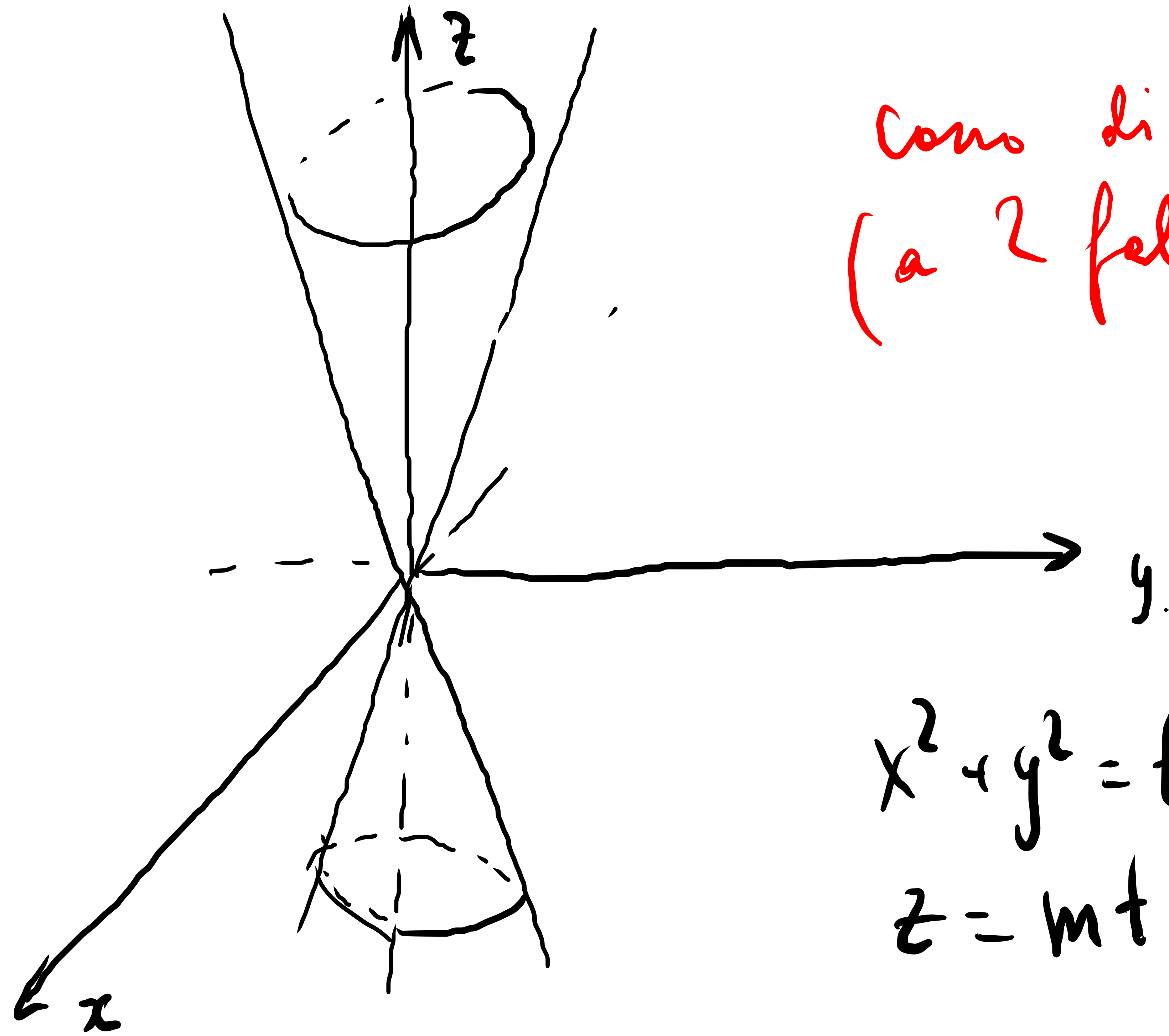
$t \in \mathbb{R}, \quad \theta \in [0, 2\pi)$

superficie (parametrizzata) in $\mathbb{R}^3$.

$$\theta = 0 \implies \begin{cases} x(t) = t \\ y = 0 \\ z(t) = mt \end{cases} \qquad t \in \mathbb{R}$$

curva parametrica in (x, z)





cono di rotazione
(a 2 falde)

$\underline{m > 0}$

$$\left.\begin{array}{l} x^2+y^2=t^2 \\ z=mt \end{array}\right| \Rightarrow \quad x^2+y^2=\frac{z^2}{m^2}$$

$$\boxed{z^2=m^2\left(x^2+y^2\right)}$$

$$z=m\sqrt{x^2+y^2}$$

$$z=-m\sqrt{x^2+y^2}$$

·) $m = 2$.

$$C: \begin{cases} x = t\cos\theta, \\ y = t\sin\theta, \\ z = 2t. \end{cases} \qquad t \in \mathbb{R},\ \theta \in [0, 2\pi)$$

$$t = 1,\ \theta = \frac{\pi}{4} \Rightarrow x = \frac{1}{\sqrt{2}},\ y = \frac{1}{\sqrt{2}},\ z = 2.$$

Scrivere una eq. del piano tangente al cono C nel punto $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 2\right)$

$$\begin{cases} x_t = \cos\theta \\ y_t = \sin\theta \\ z_t = 2. \end{cases} \qquad \begin{cases} x_\theta = -t\sin\theta \\ y_\theta = t\cos\theta \\ z_\theta = 0 \end{cases}$$

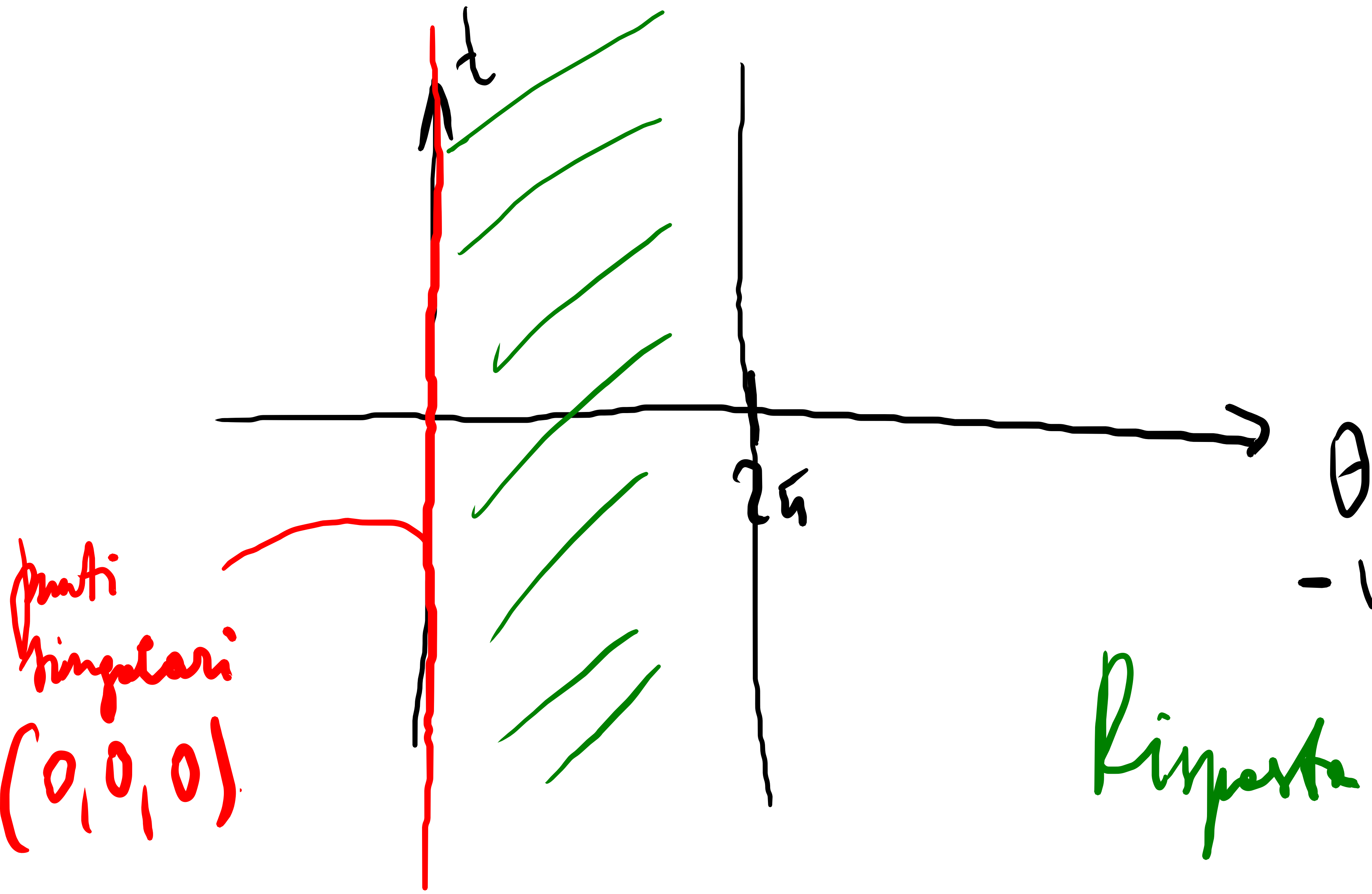
$$(x_t, y_t, z_t) \times (x_\theta, y_\theta, z_\theta) = \begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ \cos\theta & \sin\theta & 2 \\ -t\sin\theta & t\cos\theta & 0 \end{vmatrix} =$$

$$= \bar{e}_1(-2t\cos\theta) - \bar{e}_2(2t\sin\theta) + \bar{e}_3(t\cos^2\theta + t\sin^2\theta) =$$

$$= -(2t\cos\theta)\bar{e}_1 - (2t\sin\theta)\bar{e}_2 + t\bar{e}_3$$

$$\bar N\left(-2t\cos\theta,\ -2t\sin\theta,\ t\right)$$

$$\bar N = 0 \iff t = 0.$$



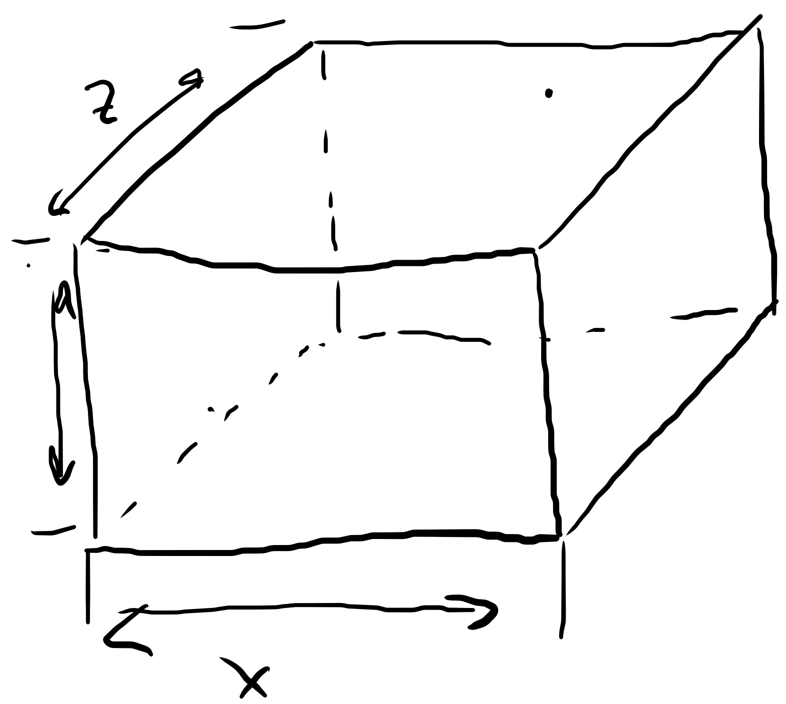
$$\bar N\left(1, \frac{\pi}{4}\right) = \left(-\sqrt{2}, -\sqrt{2}, 1\right).$$

$$-\sqrt{2}\left(x - \frac{1}{\sqrt{2}}\right) - \sqrt{2}\left(y - \frac{1}{\sqrt{2}}\right) + (z-2) = 0$$

Risposta $-\sqrt{2}\,x - \sqrt{2}\,y + z = 0.$

$A > 0$

3°)



$x \geq 0,\ y \geq 0,\ z \geq 0.$

$2(xy + yz + xz) = A.$

$Vol(x,y,z) = xyz \to \max.$

$$Vol(x,y,z) \to \max : \quad xy + yz + xz = A/2.$$
$$(x \geq 0,\ y \geq 0,\ z \geq 0).$$

$$G(x,y,z) = Vol(x,y,z) + \lambda\,(xy + yz + xz - A/2)$$

$$g(x,y,z) = xy + yz + xz = A.$$

$$\nabla g = (y+z,\ x+z,\ x+y) = 0 \iff \begin{cases} x + z = 0 \\ x + y = 0. \\ z + y = 0 \end{cases}$$

$$z = -x \Rightarrow \begin{cases} x + y = 0 \\ y - x = 0 \end{cases} \Rightarrow 2y = 0 \Rightarrow y = 0$$
$$\Downarrow \quad x = 0 \Rightarrow z = 0$$

$$0 = xz + yz + xy \neq A/2,\ \text{se } A > 0$$

$$G(x,y,z) = xyz + \lambda\left(xy + xz + yz - A/2\right)$$

$$\begin{cases} 0 = G_x = yz + \lambda(y+z) = 0 & \cdot x \\ 0 = G_y = xz + \lambda(x+z) = 0 & \cdot y \\ 0 = G_z = xy + \lambda(x+y) = 0. & \cdot z. \\ xy + xz + yz = A/2. \end{cases}$$

$$\lambda x(y+z) - \lambda y(x+z) = 0. \Leftrightarrow \begin{cases} \lambda x(y+z) = \lambda y(x+z) \\ \lambda y(x+z) = \lambda z(x+y) \end{cases}$$

$$\lambda y(x+z) - \lambda z(x+y) = 0$$

$$\begin{cases} \cancel{\lambda xy} + \lambda xz = \cancel{\lambda xy} + \lambda yz \\ \lambda xy + \cancel{\lambda yz} = \lambda xz + \cancel{\lambda yz} \end{cases} \Leftrightarrow \begin{cases} \lambda xz = \lambda yz \\ \lambda xy = \lambda xz \end{cases}$$

~~Caso 1~~. $\lambda = 0 \Rightarrow xy = xz = yz = 0 \Rightarrow$

$$0 = xy + xz + yz \neq A/2.$$

Caso 2 $\lambda \neq 0$.
$$\begin{cases} xz = yz \\ xy = xz \end{cases} \iff \begin{cases} \left[\begin{array}{l} \cancel{z = 0} \\ x = y \end{array}\right. \\ \left[\begin{array}{l} \cancel{x = 0} \\ y = z \end{array}\right. \end{cases}$$

Ma se $z = 0 \Rightarrow xy = 0 \Rightarrow 0 = xy + xz + yz \neq A/2$

— " — $x = 0 \Rightarrow yz = 0 \Rightarrow 0 = xy + xz + yz \neq A/2$

Conclusione: $x = y = z$.

$$xy + yz + xz = A/2$$

$$\|$$

$$3x^2 = x^2 + x^2 + x^2$$

$$x^2 = A/6$$

$$x = \sqrt{A/6} = y = z.$$

$$\mathrm{Vol}(x,y,z) = xyz = \left(\sqrt{A/6}\right)^3$$

(!) Resta da dimostrare l'esistenza della "scatola massima"

$$B_R(0) := \{x^2+y^2+z^2 \le R^2\}.$$

Se $x > R$ $\Longrightarrow$

$$y \le A/2R$$

$$z \le A/2R.$$

$$xy+yz+xz = A/2.$$

$$\Downarrow$$

$$\begin{cases} xy \le A/2. \Rightarrow y \le A/2x \\ yz \le A/2 \\ xz \le A/2. \Rightarrow z \le A/2x \end{cases}$$

$$\mathrm{Vol}(x,y,z) = xyz \le \cancel{x} \cdot \frac{A}{2\cancel{x}} \cdot A/2x = \frac{A^2}{4x} \le \frac{A^2}{4R}.$$

Se $R > \tilde{R}$, allora $\frac{A^2}{4R} < \left(\sqrt{A/6}\right)^3$

$$/\ \tilde{R} := \frac{A^2}{4\left(\sqrt{A/6}\right)^3}\ /$$

Conclusione: se $x > \tilde{R}$ oppure $y > \tilde{R}$ oppure $z > \tilde{R}$

$$\Rightarrow \quad \mathrm{Vol}(x,y,z) < \left(\sqrt{A/6}\right)^3.$$

$$\sup\left\{ \mathrm{Vol}(x,y,z) : \begin{array}{l} x \geq 0,\ y \geq 0,\ z \geq 0 \\ xy + yz + xz = A/2 \end{array} \right\} =$$

$$= \sup\left\{ \underset{\substack{\| \\ xyz}}{\mathrm{Vol}(x,y,z)} : \begin{array}{l} 0 \leq x \leq \tilde{R},\ 0 \leq y \leq \tilde{R},\ 0 \leq z \leq \tilde{R} \\ xy + yz + xz = A/2 \end{array} \right\}$$

$$= \max \quad \text{——}\ ''\ \text{——}\ ''\ \text{——}\ ''\ \text{——}\ ''\ \text{——}$$

Risposta: scatola massima

$$x = y = z = \sqrt{A/6}$$

$$\mathrm{Vol} = \left(\sqrt{A/6}\right)^3$$