$f: A \subset \mathbb{R}^m \to \mathbb{R}^n$

<u>Esempi</u>. 1). $m = 1$, $A = I$ - intervallo $\subset \mathbb{R}$

$f: I \to \mathbb{R}$ è una curva parametrizzata..

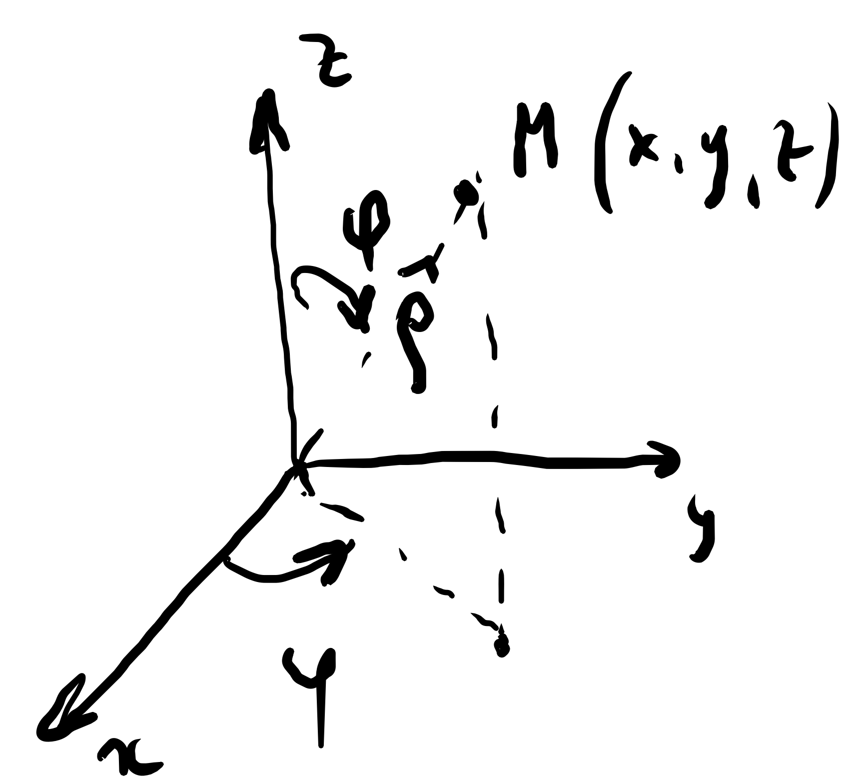
2). $n = 1$ $\quad f: A \subset \mathbb{R}^m \to \mathbb{R}$

3) $m = n$ "cambio di coordinate"

•) $f: (\rho, \theta, \varphi) \longmapsto (x, y, z)$

$\rho \in [0, +\infty)$, $\theta \in [0, 2\pi)$

$\varphi = [0, \pi]$



$x = \rho \sin\varphi \cos\theta$

$y = \rho \sin\varphi \sin\theta$

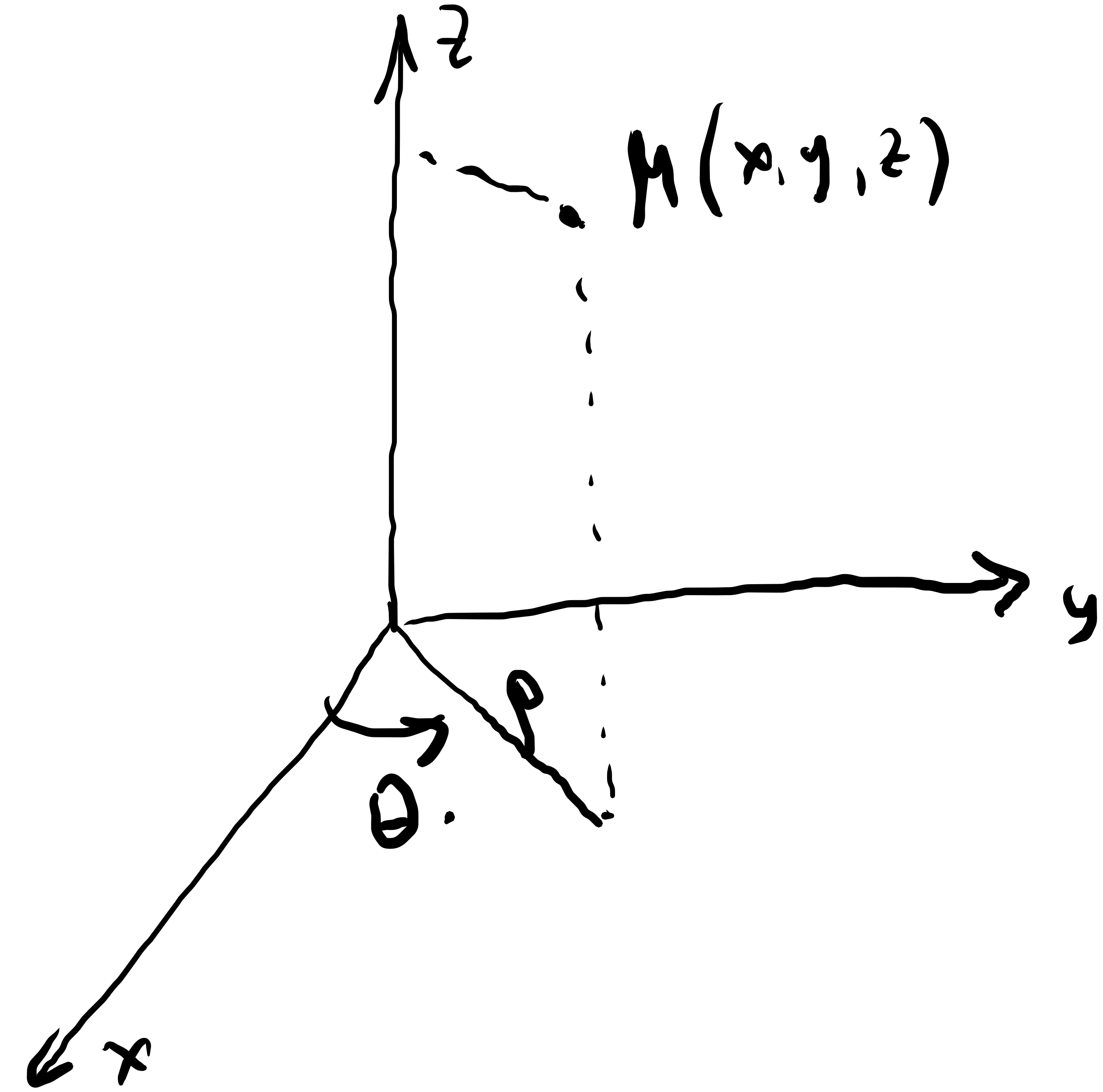
$z = \rho \cos\varphi$.

a) $f : (\rho, \theta, z) \longmapsto (x, y, z)$

$\rho \in [0, +\infty)$

$\theta \in [0, 2\pi)$

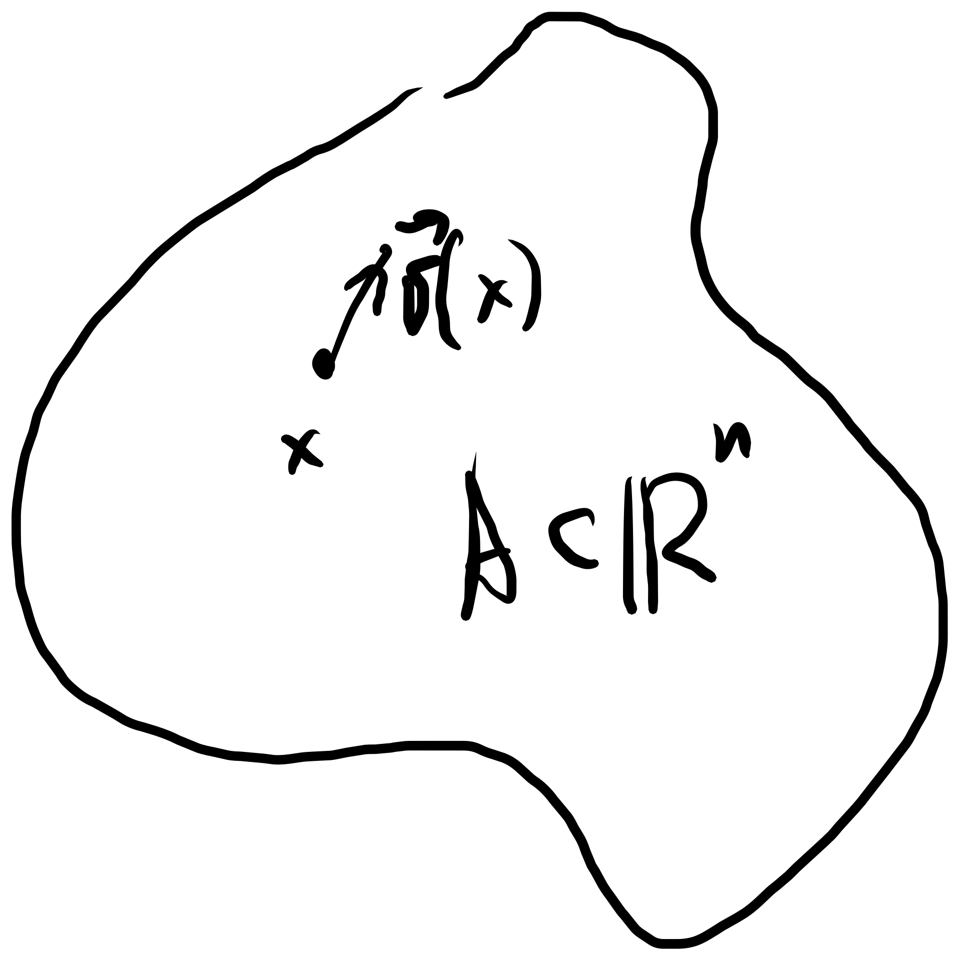
$z \in \mathbb{R}$



$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}$$

$$f(\rho, \theta, z) := (\rho\cos\theta, \rho\sin\theta, z)$$

•) Campi vettoriali



$$(x_1, \dots, x_n) = \big(V_1(x_1, \dots, x_n), \dots, V_n(x_1, \dots, x_n)\big)$$

$$V = (V_1, \dots, V_n)$$

$$V : A \subset \mathbb{R}^n \to \mathbb{R}^n$$

4). $m \leq n$

$f : A \subset \mathbb{R}^m \to \mathbb{R}^n$ varietà parametrizzata (parametrica) m dimensionale in $\mathbb{R}^n$

$m = 1$ - curva

$m = 2$ - superficie

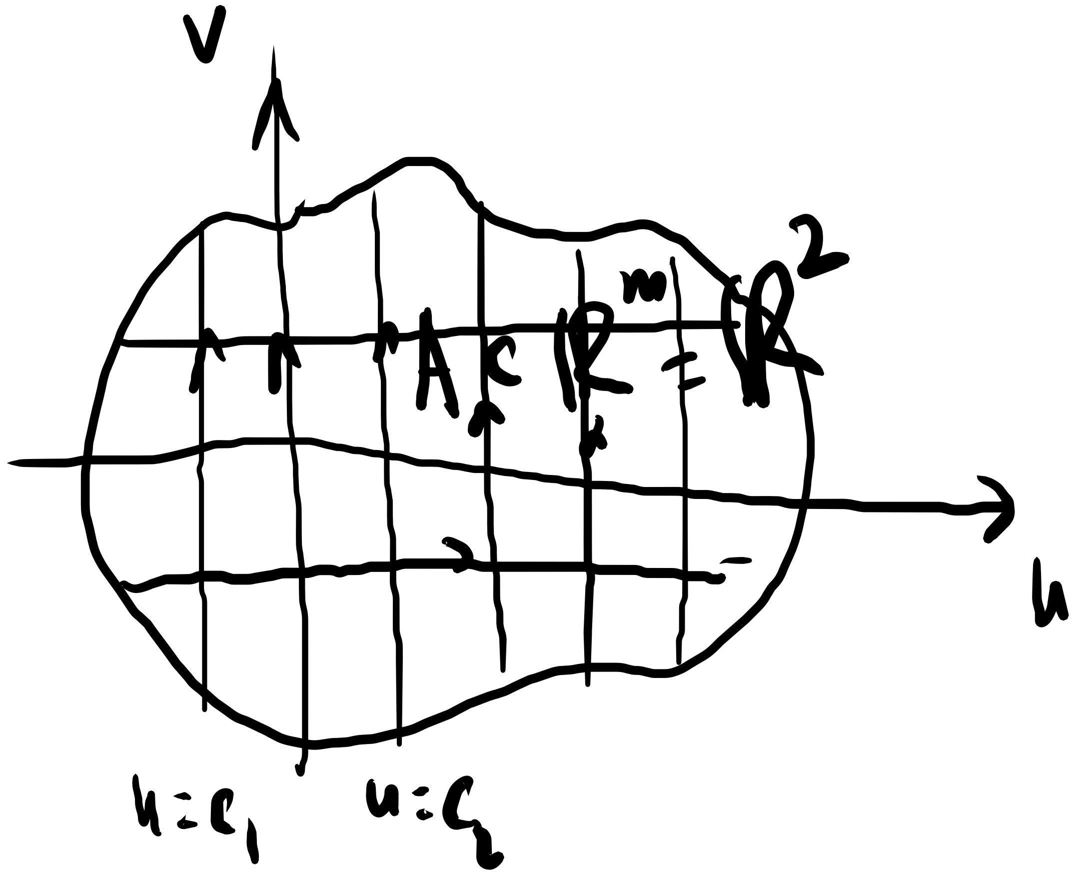
·) $\{(x,y,z) : x^2 + y^2 + z^2 = R^2\} \subset \mathbb{R}^3$

$$f : (\theta, \varphi) \in \underbrace{[0, 2\pi) \times [0, \pi]}_{A} \subset \mathbb{R}^2 \longmapsto (x, y, z)$$

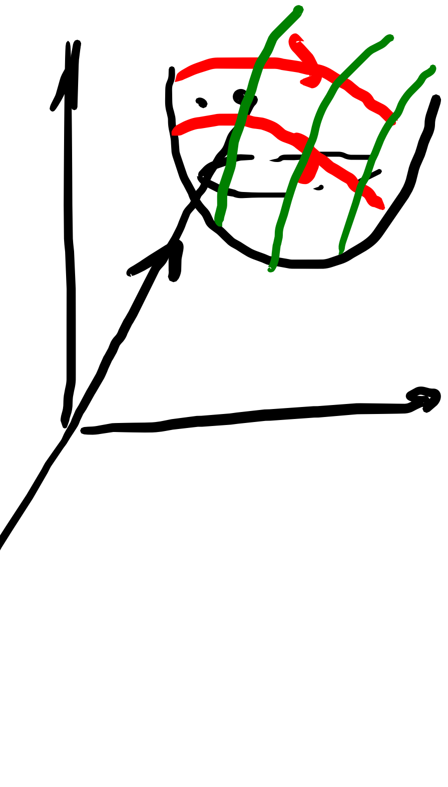
$$\begin{cases} x = R \sin\varphi \cos\theta \\ y = R \sin\varphi \sin\theta \\ z = R \cos\varphi . \end{cases}$$

$$f(\theta, \varphi) = (R \sin\varphi \cos\theta, R \sin\varphi \sin\theta, R \cos\varphi)$$

$n = 2$.



$r = r(u,v)$

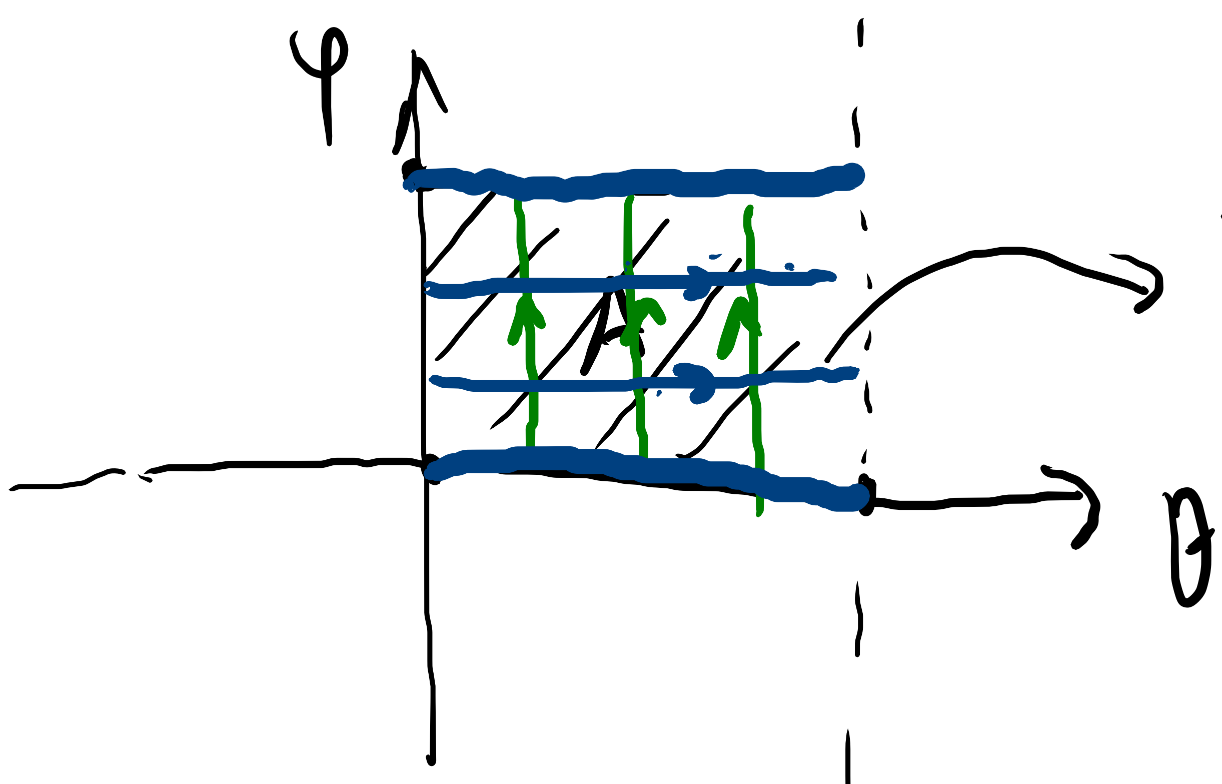


$r : A \subset \mathbb{R}^2 \to \mathbb{R}^3$

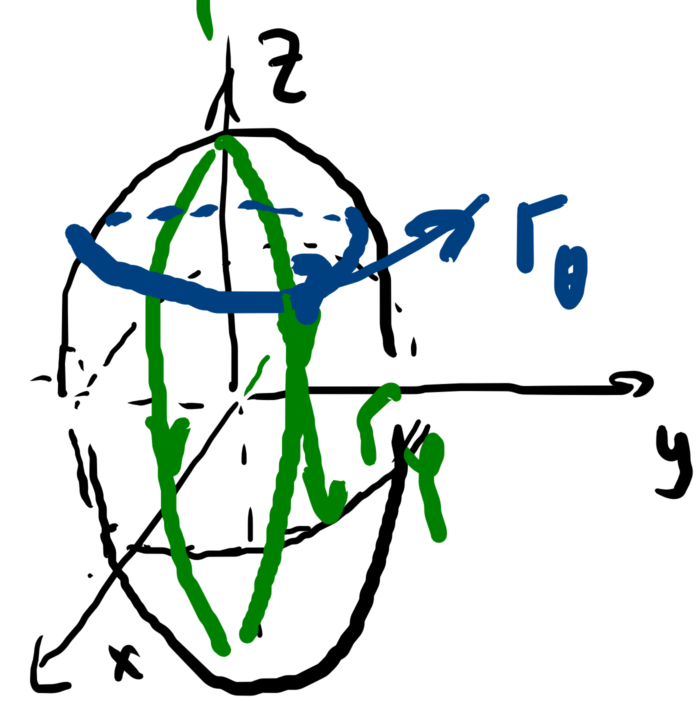
$$r = r(u,v) = \big(x(u,v),\ y(u,v),\ z(u,v) \big)$$

Esempio sfera

$$x = R \sin\varphi \cos\theta$$
$$y = R \sin\varphi \sin\theta$$
$$z = R \cos\varphi$$



$r = r(\varphi, \theta)$



$z = z(u,v)$

$r_u(u_0,v_0)$, $r_v(u_0,v_0)$ determinano il "piano tangente" quando sono linearmente indipendenti

$r_u(u_0,v_0)$ e $r_v(u_0,v_0)$ linearm. indipendenti

$\Updownarrow$

$$Dr(u_0,v_0) = \begin{pmatrix} x_u(u_0,v_0) & x_v(u_0,v_0) \\ y_u(-''-) & y_v(-''-) \\ z_u(-''-) & z_v(-''-) \end{pmatrix}$$

superfici
$r : A \subset \mathbb{R}^2 \to \mathbb{R}^3$

$\operatorname{rank} Dr(u_0,v_0) = 2$

$\Updownarrow$

$n(u_0,v_0) := r_u(u_0,v_0) \times r_v(u_0,v_0) \neq 0.$

vettore normale

(u_0,v_0) è punto regolare

Es (sfera) $\quad r(\theta,\varphi) = (R\sin\varphi\cos\theta, R\sin\varphi\sin\theta, R\cos\varphi)$

$$r_\theta = (-R\sin\varphi\sin\theta, R\sin\varphi\cos\theta, 0)$$

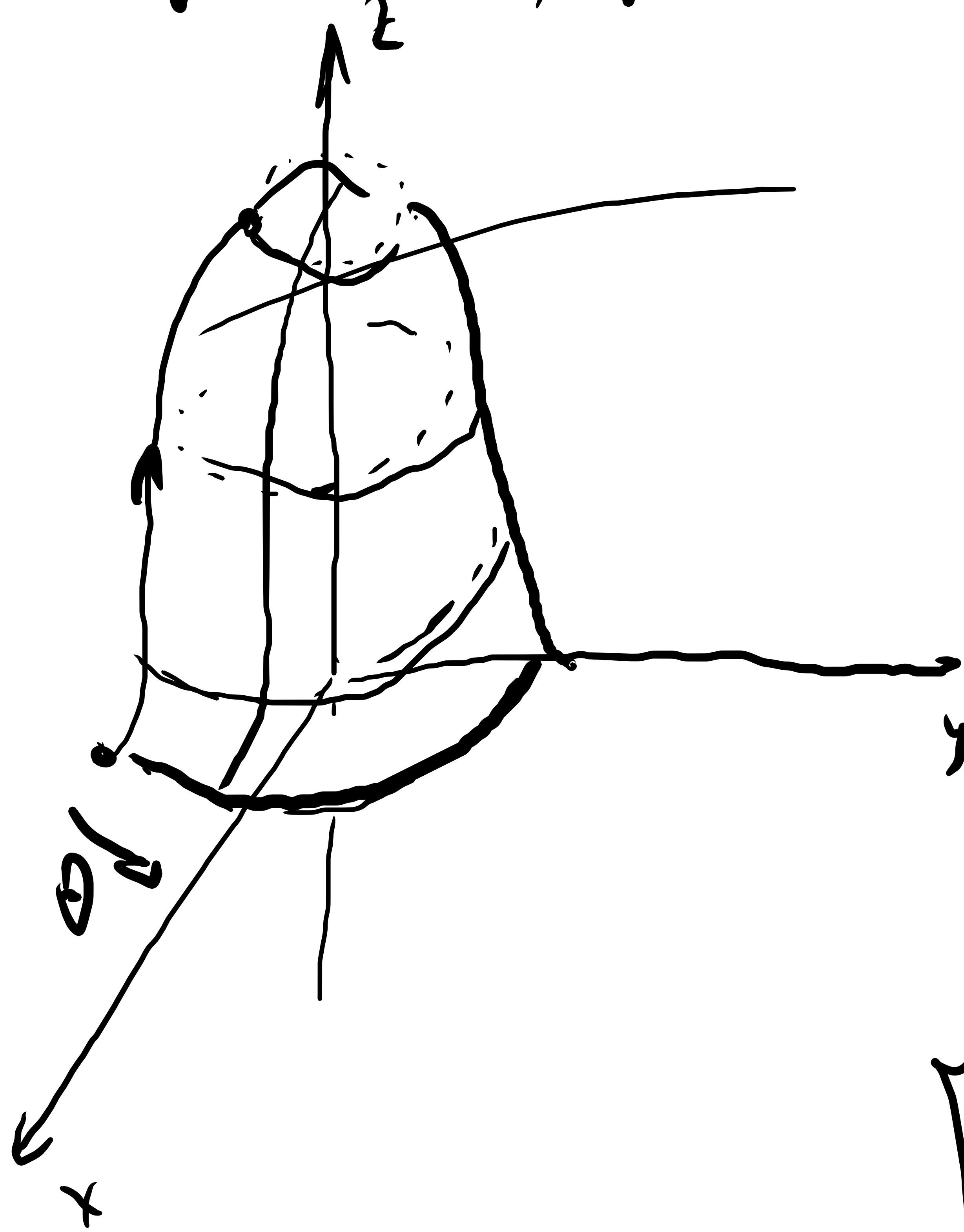
$$r_\varphi = (R\cos\varphi\cos\theta, R\cos\varphi\sin\theta, -R\sin\varphi)$$

$$n = r_\theta \times r_\varphi = \det\begin{pmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ -R\sin\varphi\sin\theta & R\sin\varphi\cos\theta & 0 \\ R\cos\varphi\cos\theta & R\cos\varphi\sin\theta & -R\sin\varphi \end{pmatrix}$$

$$= -R^2\sin^2\varphi\cos\theta\,\bar{e}_1 - R^2\sin^2\varphi\sin\theta\,\bar{e}_2 +$$
$$- R^2\sin\varphi\cos\varphi\,\bar{e}_3$$

$$(\theta,\varphi)\text{ singolare} \Leftrightarrow \begin{cases} \sin^2\varphi\cos\theta = 0 \\ \sin^2\varphi\sin\theta = 0 \\ \sin\varphi\cos\varphi = 0 \end{cases} \Leftrightarrow \operatorname{sen}\varphi \Leftrightarrow \left[\begin{array}{l} \varphi = 0 \\ \varphi = \pi \end{array}\right.$$

Esempio superficie di rotazione in $\mathbb{R}^3$.



$$x = \tilde{x}(t) \qquad t \in I \subset \mathbb{R}$$
$$z = \tilde{z}(t)$$

$$\underline{r} = \underline{r}(t, \theta) = (x(t,\theta), y(t,\theta), z(t,\theta))$$

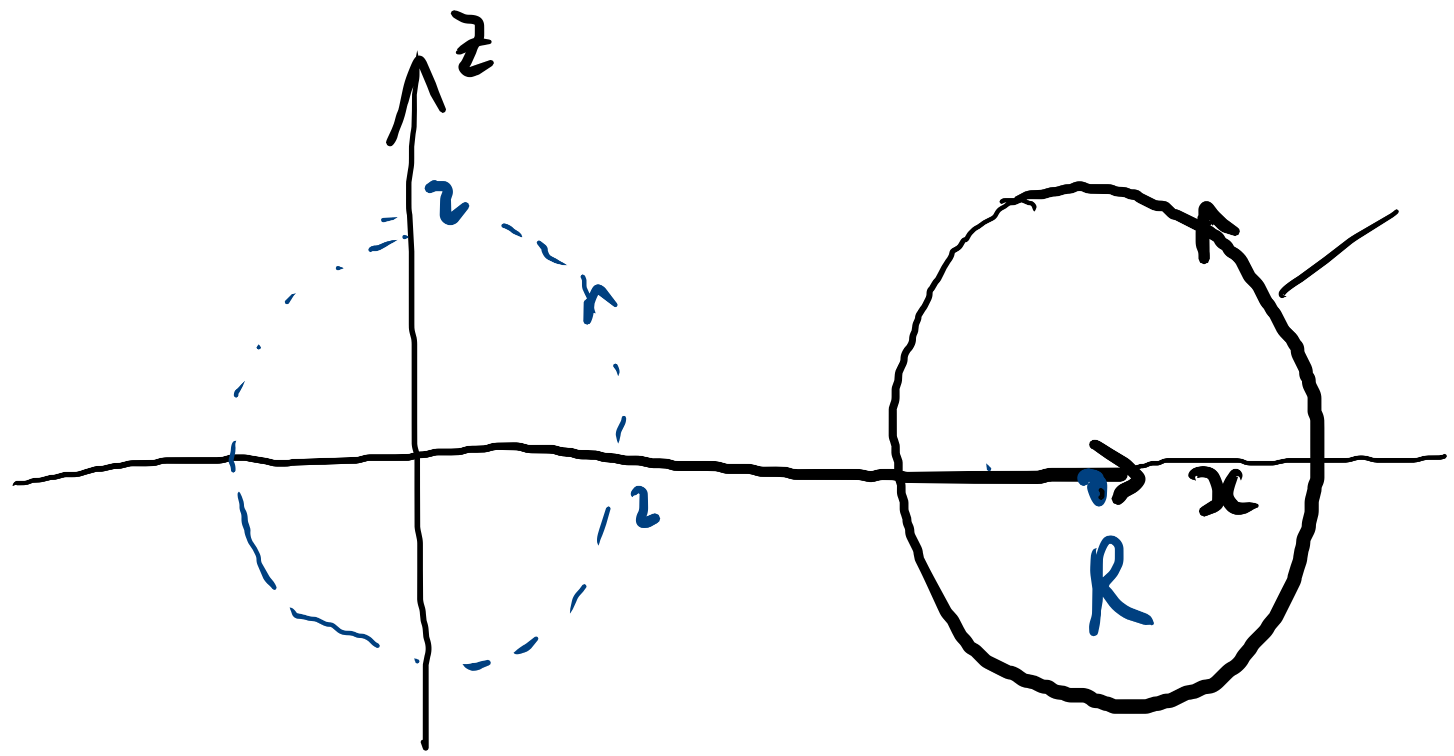
$$\begin{cases} x(t,\theta) = \tilde{x}(t)\cos\theta & t \in I \\ y(t,\theta) = \tilde{x}(t)\sin\theta & \theta \in [0, 2\pi) \\ z(t,\theta) = \tilde{z}(t) \end{cases}$$

Esempio particolare

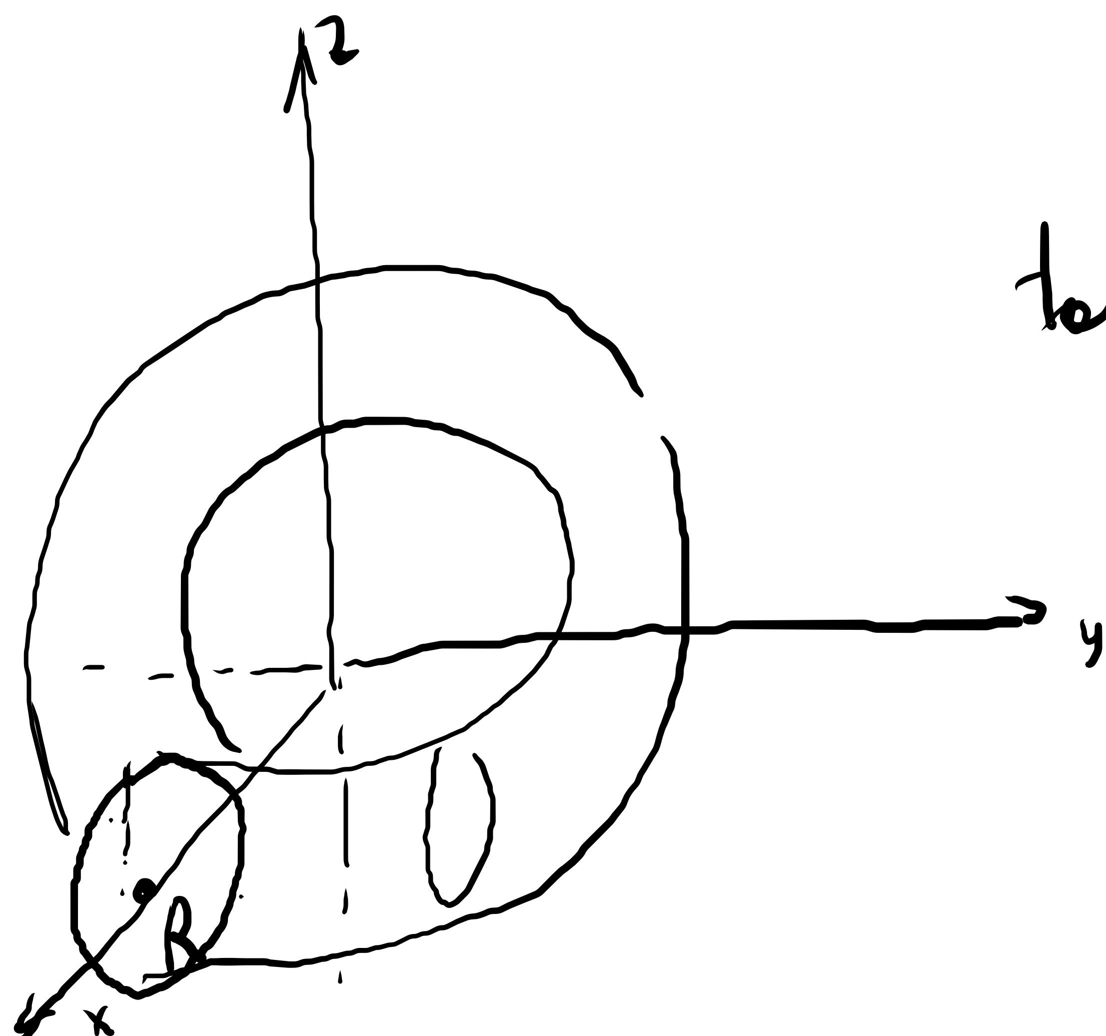
$R > r > 0$ – numeri dati

$$x = R + r\cos\varphi$$
$$z = r\sin\varphi$$

$\varphi \in [0, 2\pi)$.



circonf. di raggio r centrata in $(R, 0)$



toro di rotazione

$$x = \tilde{x}(\varphi) = R + 2\cos\varphi \qquad \varphi \in [0, 2\pi)$$
$$z = \tilde{z}(\varphi) = 2\sin\varphi$$

$$r = (x, y, z) \qquad \begin{cases} x = x(\varphi, \theta) = (R + 2\cos\varphi)\cos\theta \\ y = y(\varphi, \theta) = (R + 2\cos\varphi)\sin\theta \\ z = z(\varphi, \theta) = 2\sin\varphi \end{cases}$$

$$r_\varphi = (-2\sin\varphi\cos\theta, -2\sin\varphi\sin\theta, 2\cos\varphi)$$
$$r_\theta = (-(R + 2\cos\varphi)\sin\theta, (R+2\cos\varphi)\cos\theta, 0)$$

$$n = r_\varphi \times r_\theta = \det\begin{pmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ -2\sin\varphi\cos\theta & -2\sin\varphi\sin\theta & 2\cos\varphi \\ -(R+2\cos\varphi)\sin\theta & (R+2\cos\varphi)\cos\theta & 0 \end{pmatrix} =$$

$$= \bar{e}_1\left(-2(R+2\cos\varphi)\cos\varphi\cos\theta\right) + \bar{e}_2\left(2(R+2\cos\theta)\cos\varphi\sin\theta\right)$$
$$+ \bar{e}_3\left(-2(R+2\cos\varphi)\sin\varphi\sin^2\theta - 2(R+2\cos\varphi)\sin\varphi\cos^2\theta\right) =$$

$$= 2(R+2\cos\varphi)\left(-\cos\varphi\cos\theta\,\bar{e}_1 + \cos\varphi\sin\theta\,\bar{e}_2 - \sin\varphi\,\bar{e}_3\right)$$

$$\begin{cases} 2(R+2\cos\varphi)(-\cos\varphi\cos\theta) = 0 \\ (\text{---}\ "\ \text{---})(\cos\varphi\sin\theta) = 0 \\ (\text{---}\ "\ \text{---})(-\sin\varphi) = 0 \end{cases} \iff \begin{cases} \cos\varphi\cos\theta = 0 \\ \cos\varphi\sin\theta = 0 \\ \sin\varphi = 0 \end{cases}$$

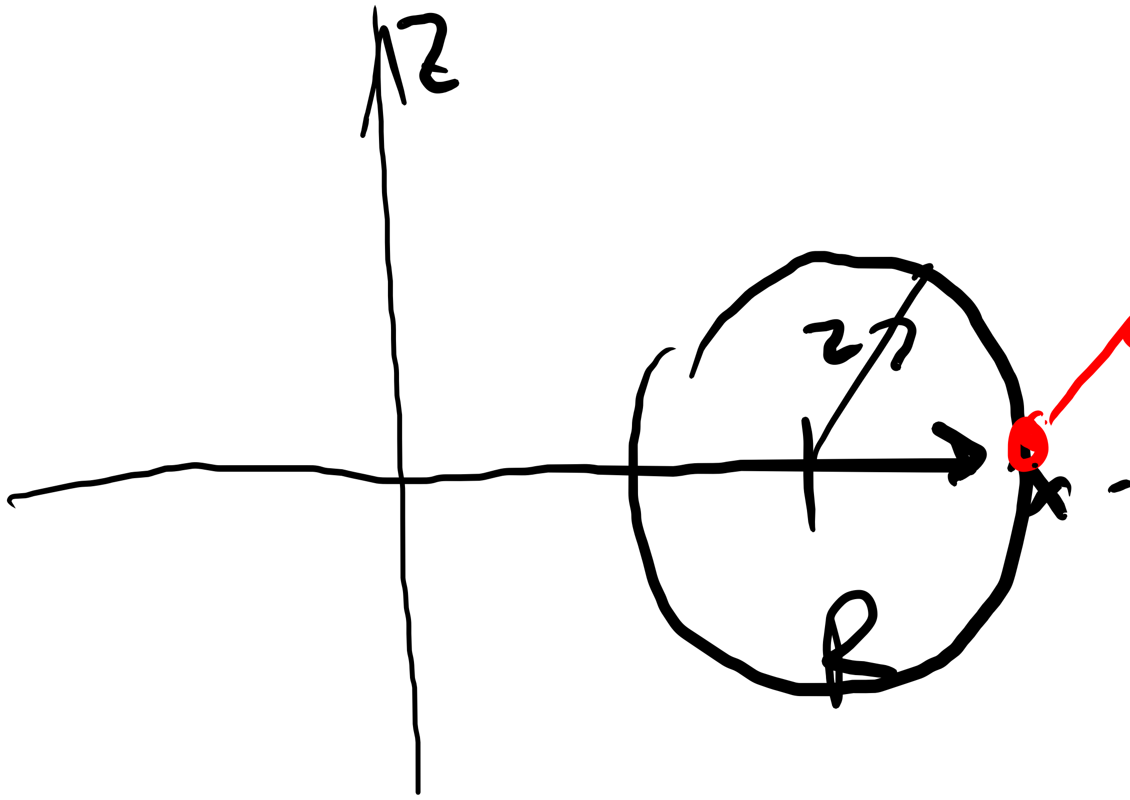
Conclusione: la superficie del toro di rotazione parametrizzata con

$$r(\varphi, \theta) = \Big(\big(R + 2\cos\varphi\big)\cos\theta, \big(R + 2\cos\varphi\big)\sin\theta, 2\sin\varphi \Big)$$

è regolare

$$\varphi \in [0, 2\pi)$$
$$\theta \in [0, 2\pi).$$

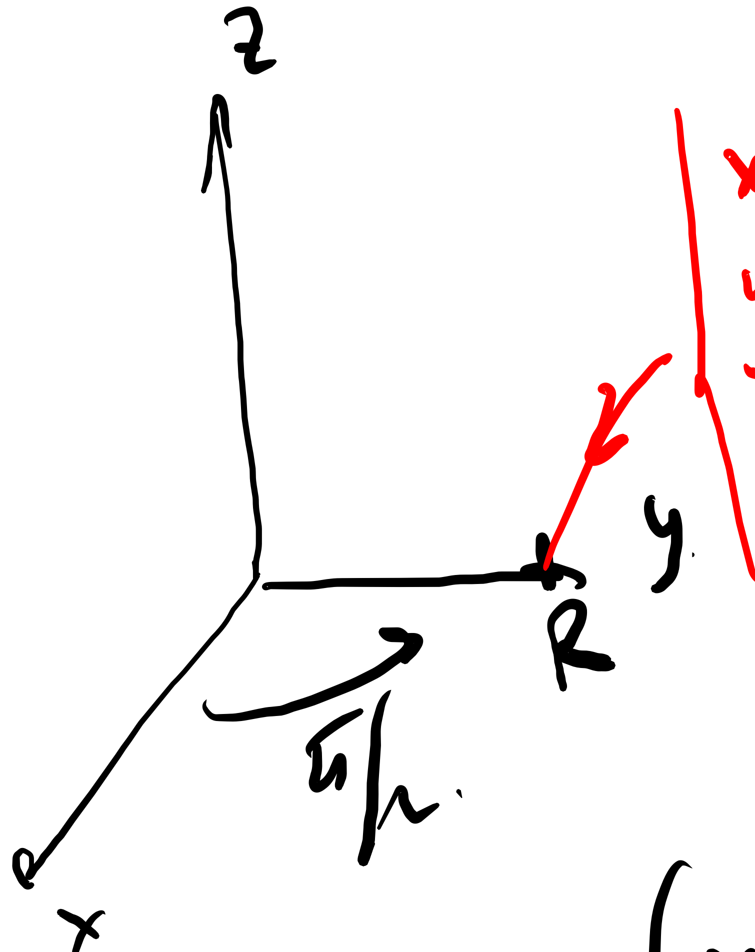
Esercizio Scrivere una eq. equazione del piano tangente al toro nel punto $\varphi = 0$, $\theta = \pi/2$.



$\varphi = 0$

$\begin{pmatrix} x = R+2 \\ z = 0 \end{pmatrix}$

$$\bar{n}(\varphi,\theta) = 2(R+2\cos\varphi)(-\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi)$$



$x = 0$

$y = R+2$

$z = 0.$

$$\bar{n}\left(0,\frac{\pi}{2}\right) = 2(R+2)(0,1,0)$$

$$(x-0,\ y-(R+2),\ z-0)\cdot\bar{n} = 0.$$

$$(x,\ y-(R+2),\ z)\cdot(0,1,0) = 0$$

$$\boxed{y = R+2.}$$

Derivate di una funzione composta
("regola della catena" / Chain rule)

Da ricordare: $f: A \subset \mathbb{R} \to \mathbb{R}$

$$\begin{cases} g: B \subset \mathbb{R} \to \mathbb{R} \\ h(x) := g(f(x)) \qquad x_0 \in A \\ \boxed{h'(x_0) = g'(f(x_0))\, f'(x_0)} \end{cases}$$

Caso generale:

$$f : A \subset \mathbb{R}^m \to \mathbb{R}^k$$
$$g : B \subset \mathbb{R}^k \to \mathbb{R}^n$$

$$x_0 \in \mathring{A}$$

$h(x) := g(f(x))$ definita almeno in un intorno di x_0

Th. $\begin{cases} h : \cancel{A} \subset \mathbb{R}^m \to \mathbb{R}^n \\ (Dh)(x_0) = \underbrace{Dg(f(x_0))}_{n \times k} \cdot \underbrace{Df(x_0)}_{k \times m} \end{cases}$

Casi particolari

$\boxed{1°}$ $m = 1$, $A = I \subset \mathbb{R}$ intervallo

$f : I \to \mathbb{R}^k$ curva parametrica

$h = 1$. $g : \mathbb{R}^k \to \mathbb{R}$

$h(t) := g(f(t))$, $h : I \to \mathbb{R}$

$$h'(t_0) = \underbrace{\nabla g(f(t_0))}_{\text{riga}} \cdot \underbrace{\dot{f}(t_0)}_{\text{colonna}}$$

Corollario

$g = g(x,y)$ $\quad g : \mathbb{R}^2 \to \mathbb{R}$

"curva di livello" $\{(x,y) : g(x,y) = c\}$.

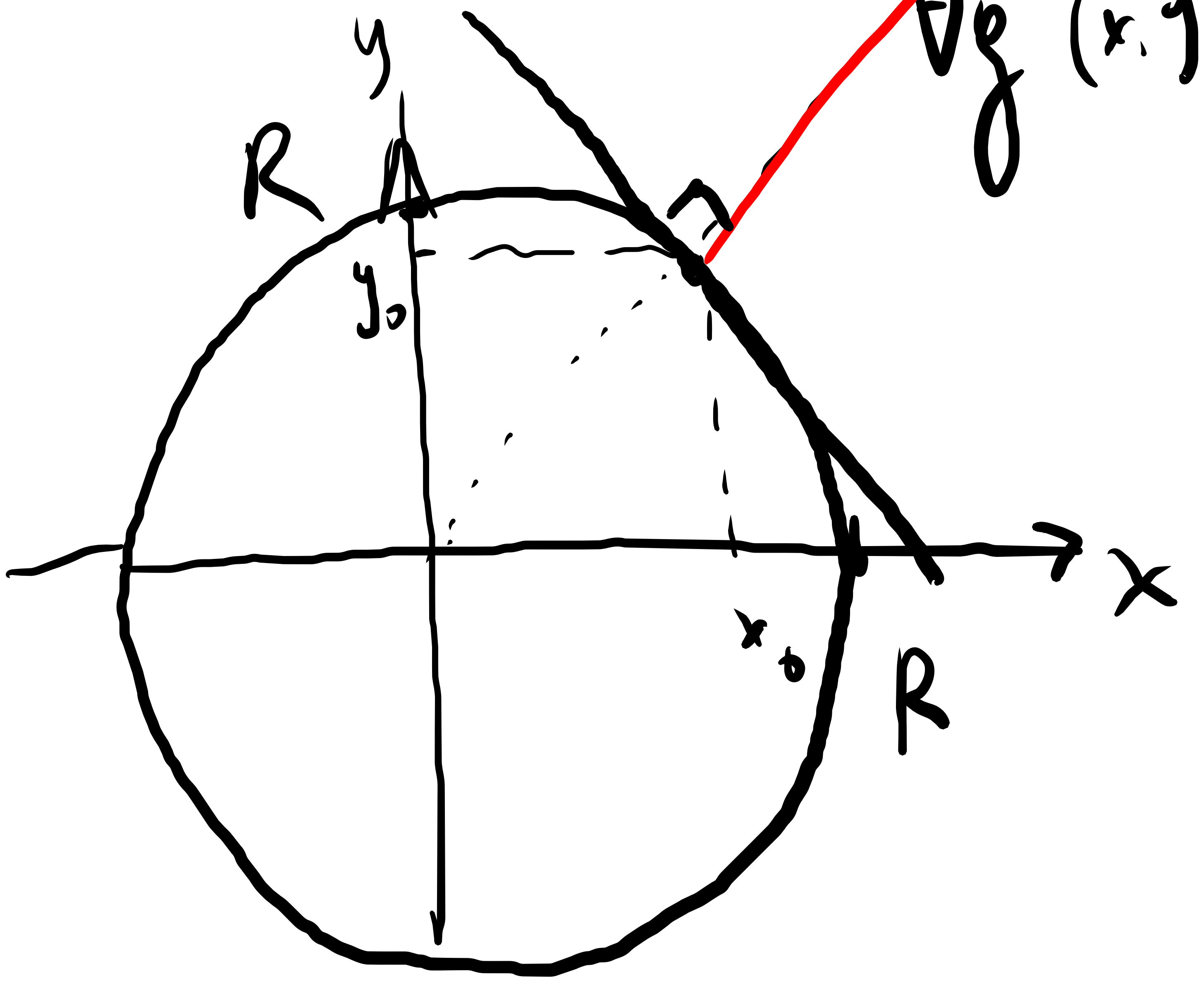
parametrizzata con $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ $t \in I$ Allora $g(x(t), y(t)) = c$ $\forall t \in I$.

$h(t) = g(x(t), y(t)) = c \Rightarrow \dot{h}(t) = \nabla g(x(t), y(t)) \cdot (\dot{x}(t), \dot{y}(t)) = 0$

Ossia, $\nabla g(x(t), y(t)) \perp (\dot{x}(t), \dot{y}(t))$

Es. $g(x,y) = x^2 + y^2$ $\quad \{(x,y) : g(x,y) = c\}$ circonferenza di raggio $\sqrt{c}$ centrata in 0, se $c > 0$.

$x^2 + y^2 = R^2$



$g(x,y) = x^2 + y^2$

$\nabla g(x,y) = (2x, 2y) = 2(x,y)$

$|\nabla g(x_0,y_0)|^2 =$

$= 4(x^2+y^2) = 4R^2$

Esercizio. $g(x,y) = 2x^2 + 3y^2 - y$

$$g(x,y) = 2x^2 + 3\left(y^2 - \frac{1}{3}y + \frac{1}{36}\right) - \frac{1}{12} =$$

$$= 2x^2 + 3\left(y - \frac{1}{6}\right)^2 - \frac{1}{12}.$$

L'insieme di livello $A := \{(x,y) : g(x,y) = c\}$

$$2x^2 + 3\left(y - \frac{1}{6}\right)^2 - \frac{1}{12} = c. \iff$$

$$\iff 2x^2 + 3\left(y - \frac{1}{6}\right)^2 = c + \frac{1}{12}$$

$A = \emptyset$ se $c + \frac{1}{12} < 0$

$A = \left(0, \frac{1}{6}\right)$, se $c + \frac{1}{12} = 0$ (ovvero $c = -\frac{1}{12}$)

A è un ellisse, se $c + \frac{1}{12} > 0$

$$\frac{x^2}{\left(c + \frac{1}{12}\right)/2} + \frac{\left(y - \frac{1}{6}\right)^2}{\left(c + \frac{1}{12}\right)/3} = 1.$$

Semiassi $\sqrt{\left(c + \frac{1}{12}\right)/2}$

$\sqrt{\left(c + \frac{1}{12}\right)/3}$

Consideriamo $c = \frac{1}{6}$

$$A = \left\{ (x,y) : \ 2x^2 + 3\left(y - \frac{1}{6}\right)^2 = \frac{1}{4}. \right.$$

$$y = \frac{1}{6} \qquad 2x^2 = \frac{1}{4} \Rightarrow x^2 = \frac{1}{8} \Rightarrow x = \pm \frac{1}{2\sqrt{2}}$$

$$\left(\frac{1}{2\sqrt{2}}, \frac{1}{6} \right) \in A \qquad \left(-\frac{1}{2\sqrt{2}}, \frac{1}{6} \right) \in A.$$

Scrivere l'eq della retta tangente all'ellisse A nel punto (,)

$$\nabla g\left(\frac{1}{6}, \frac{1}{2\sqrt{2}}\right) = \nabla \left(2x^2 + 3y^2 - y\right) \Big|_{(x,y) = \left(\frac{1}{2\sqrt{2}}, \frac{1}{6}\right)} =$$

$$= \left(4x, 6y - 1\right) \Big|_{(x,y) = (\quad)} =$$

$$= \left(\sqrt{2}, 0\right)$$

Eq della retta tangente ad A nel punto $\left(\frac{1}{2\sqrt{2}}, \frac{1}{6}\right)$ è

$$\left(x - \frac{1}{2\sqrt{2}}, y - \frac{1}{6}\right) \cdot \left(\sqrt{2}, 0\right) = 0.$$

$$\sqrt{2}\left(x - \frac{1}{2\sqrt{2}}\right) = 0 \Rightarrow \boxed{x = \frac{1}{2\sqrt{2}}}$$

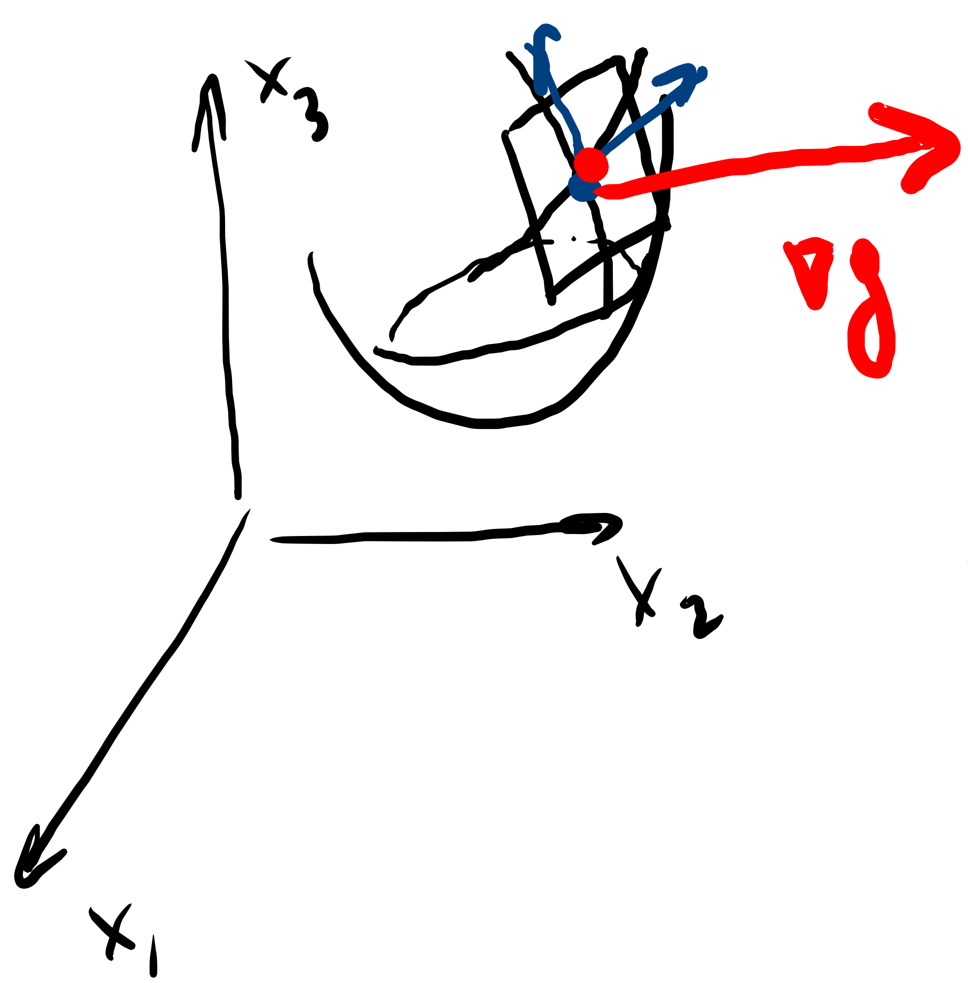
Caso più generale.

$$g = g(x_1, \dots, x_m) \qquad g : \mathbb{R}^m \to \mathbb{R}$$

L'insieme di livello $A := \{ (x_1, \dots, x_m) \in \mathbb{R}^m : g(x_1, \dots, x_m) = c \}$

(Esempio: $g = g(x, y, z) = x^2 + y^2 + z^2$
$m = 3$

$\{ x^2 + y^2 + z^2 = c \}$
è sfera di raggio $\sqrt{c}$, se $c > 0$.)



$$\nabla g(x_0) \perp \underbrace{T_{x_0} A}$$

piano tangente ad A nel punto x_0

2°) L'Altro caso particolare della regola della catena.

$$f : \mathbb{R}^m \to \mathbb{R} \qquad (k=1)$$

$$g : \mathbb{R} \to \mathbb{R}. \qquad h(x) = g(f(x)).$$

x_0

$$\underbrace{\nabla h(x_0)}_{\text{riga}} = Dh(x_0) = g'(f(x_0))\, Df(x_0) =$$

$$= g'(f(x_0))\, \underbrace{\nabla f(x_0)}_{\text{riga.}}$$

Esempio

$$f(x,y) = x^2+y^2$$

$$g(u) = \sin u$$

$$h(x,y) = g(f(x,y)) = \sin(x^2+y^2)$$

$$\nabla h(x,y) = \cos(x^2+y^2)\cdot(2x, 2y)$$

Forme quadratiche $\quad h \in \mathbb{R}^n$

$$g(h) := \sum_{i,j=1}^{n} a_{ij} h_i h_j$$

$a_{ij} \in \mathbb{R}$

Es. $n = 2$

$$g(h) = h_1^2 - 2 h_1 h_2 + h_2^2 .$$

$$\{a_{ij}\} = \begin{pmatrix} 1 & \cancel{-1} \\ \cancel{-1} & 1 \end{pmatrix} \begin{matrix} -2 \\ \\ \end{matrix}$$

$$\begin{matrix} 0 \end{matrix}$$

$$a_{ij} \rightsquigarrow \frac{a_{ij} + a_{ji}}{2}$$

$$g(h) = h^T A h = \qquad A = \{a_{ij}\}$$

$$= (Ah) \cdot h \qquad \text{simmetrica.}$$

$$= (Ah, h)$$

prodotto scalare

Def. La forma quadratica g / la matrice A si dice

$\lambda_j > 0 \ \forall j \iff$ •) definita positiva, se $g(h) > 0 \ \forall h \neq 0.$

$\lambda_j \geq 0 \ \forall j \iff$ •) semi — " — " —, se $g(h) \geq 0 \ \forall h$

$\lambda_j < 0 \ \forall j \iff$ •) definita negativa, se $g(h) < 0 \ \forall h \neq 0.$

$\lambda_j \leq 0 \ \forall j \iff$ •) semi — " — " —, se $g(h) \leq 0 \ \forall h.$

•) indefinita altrimenti.

$$\sigma(A) = \{\lambda_1, \dots, \lambda_n\}$$

Th. (Criterio di Sylvester)

A è definita positiva, se e solo se

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix} \qquad M_1 = a_{11} \quad M_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \quad \vdots$$

$$\det M_j > 0 .$$