ANALISI MATEMATICA B

Se f e g sono derivabili in x_0

anche $f+g$ e λf $\lambda \in \mathbb{R}$ e $f \cdot g$ sono derivabili in x_0

e vale:

(1)
$$\left[\begin{array}{l} (f+g)'(x_0) = f'(x_0) + g'(x_0) \\ (\lambda f)'(x_0) = \lambda \cdot f'(x_0) \end{array} \right]$$

Sp. vettoriale

$$\left[\begin{array}{l} D : \mathcal{D}(A) \to \mathbb{R}^A \text{ è lineare} \\ \qquad \mathcal{D}(A) = \{ f : A \to \mathbb{R} : f \text{ derivabile} \} \end{array} \right]$$

Sp. vett.

(2)
$$(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0)$$

dim (1) è banale.

$$\frac{(f+g)(x+h) - (f+g)(x)}{h} = \underbrace{\frac{f(x+h) - f(x)}{h}}_{\downarrow \; f'(x)} + \underbrace{\frac{g(x+h) - g(x)}{h}}_{\downarrow \; g'(x)} \quad h \to 0$$

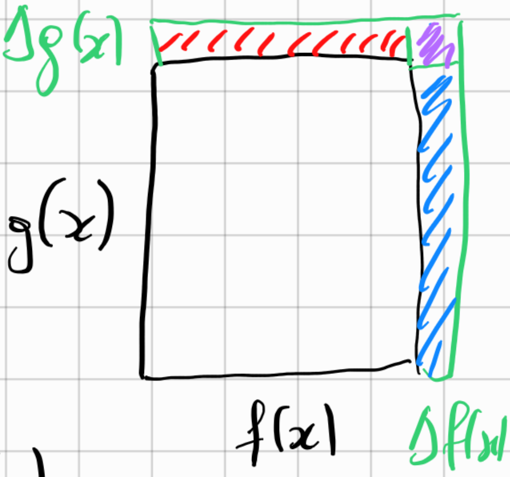
$$\frac{(\lambda f)(x+h) - (\lambda f)x}{h} = \lambda \frac{f(x+h) - f(x)}{h} \to \lambda f'(x).$$

(2)

$$\Delta\big(f(x)\cdot g(x)\big) =$$

$$\Delta g(x)\cdot f(x) + \Delta f(x)\cdot g(x)$$

$$+\ \Delta f(x)\cdot \Delta g(x)$$



$$\frac{(f\cdot g)(x+h) - (f\cdot g)(x)}{h} = \frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h} =$$

$$= \frac{[f(x+h)\cdot g(x+h) - f(x+h)\cdot g(x)] + [f(x+h)\cdot g(x) - f(x)g(x)]}{h}$$

$$= f(x+h)\cdot \frac{[g(x+h)-g(x)]}{h} + \frac{[f(x+h)-f(x)]}{h}\cdot g(x)$$

per $h \to 0$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$f(x) \cdot g'(x) + f'(x) \cdot g(x) \qquad \square$$

Derivata del rapporto $\qquad h(y) = \frac{1}{y} \qquad h'(y) = -\frac{1}{y^2}$

$$\left(\frac{f}{g}\right)'(x) = \left(f(x)\cdot\frac{1}{g(x)}\right)' = \big(f(x)\cdot h(g(x))\big)'$$

$$= f'(x)\cdot\frac{1}{g(x)} + f(x)\cdot h'(g(x))\cdot g'(x)$$

$$= \frac{f'(x)}{g(x)} - f(x)\,\frac{1}{g^2(x)}\cdot g'(x) = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g^2(x)} \qquad \square$$

DERIVATE DELLE FUNZIONI ELEMENTARI

Già visto $(mx+q)' = m$, $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$

$$\boxed{(x^n)' = n \cdot x^{n-1}} \qquad n \in \mathbb{N},\ n>0$$

dim. per induzione $n=1$ $(x)' = 1 = 1 \cdot x^0$.

$$(x^{n+1})' = (x^n \cdot x)' = n \cdot x^{n-1} \cdot x + x^n \cdot 1$$
$$= n x^n + x^n = (n+1)x^n.$$

ES $(x^2)' = (x \cdot x)' = (1 \cdot x + x \cdot 1) = 2x.$

Se $n<0$ vale la stessa formula. $(x \neq 0)$

$$(x^{-n})' = \left(\frac{1}{x^n}\right)' = -\frac{1}{(x^n)^2} \cdot n x^{n-1} = -n x^{-n-1} \quad \square$$

$(n>0)$

$$(e^x)' = e^x$$

$$\frac{e^{x+h} - e^x}{h} = e^x \cdot \frac{e^h - 1}{h} \longrightarrow e^x \cdot 1 = e^x.$$

$$(a^x)' = \left(e^{x \ln a}\right)' = e^{x \ln a} \cdot \ln a = a^x \cdot \ln a.$$

$\alpha \in \mathbb{R}$, $x>0$

$$x^\alpha = e^{\alpha \ln x} \quad \dots\dots$$

$x>0$

$$(\ln x)' = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

derivata della fn. inversa.

$$(x^\alpha)' = \left(e^{\alpha \ln x}\right)' = e^{\alpha \ln x} \cdot \alpha \cdot \frac{1}{x} = x^\alpha \cdot \alpha \cdot \frac{1}{x}$$

$$= \alpha \cdot x^{\alpha - 1}.$$

$$\left(\sqrt[n]{x}\right)' = \frac{1}{n\left(\sqrt[n]{x}\right)^{n-1}}$$

derivata delle inversa

$$\left(x^{\frac{1}{n}}\right)' = \frac{1}{n} x^{\frac{1}{n}-1} = \frac{1}{n} \frac{1}{x^{\frac{n-1}{n}}} = \frac{1}{n\left(\sqrt[n]{x}\right)^{n-1}}$$

Es $\quad (\sqrt{x})' = \frac{1}{2\sqrt{x}}$

$$(|x|)' = \frac{x}{|x|} \qquad x \neq 0$$

$$(\sin x)' : \quad \frac{\sin(x+h) - \sin x}{h} = \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \cos x \cdot \frac{\sin h}{h} - \sin x \, \frac{1 - \cos h}{h}$$

per $h \to 0$ (*)

$$\cos x \cdot 1 - \sin x \cdot 0 = \cos x$$

(*) $$\frac{1-\cos h}{h} = \frac{1-\cos^2 h}{h \cdot (1+\cos h)} = \frac{\sin^2 h}{h\,(1+\cos h)} = \frac{\sin h}{h} \cdot \frac{\sin h}{1+\cos h}$$

per $h \to 0$

$$1 \cdot \frac{0}{2} = 0$$

$(\cos x)'$:

$$\frac{\cos(x+h)-\cos x}{h} = \frac{\cos x\cos h - \sin x\sin h - \cos x}{h}$$

$$= -\sin x \cdot \frac{\sin h}{h} + \cos x \cdot \frac{\cos h - 1}{h}$$

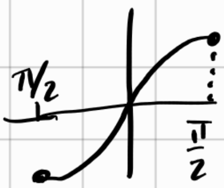
$$\downarrow \qquad\qquad \downarrow$$

$$-\sin x \cdot 1 + \cos x \cdot 0 = -\sin x.$$

$$\left[\ (e^z)' = e^z \qquad (e^{ix})' = e^{ix}\cdot i \ \right]$$

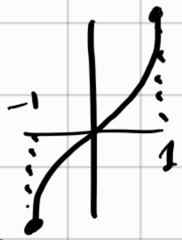
$$(\cos x + i \operatorname{sen} x)' = i\cos x - \operatorname{sen} x$$

$$(\cos x)' + i(\operatorname{sen} x)'$$

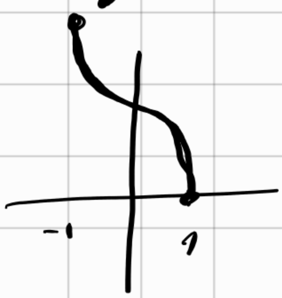


D. fn. inversa

$$(\arcsin x)' = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-x^2}}$$



$$(\arccos x)' = \frac{1}{-\sin(\arccos x)} = \frac{1}{-\sqrt{1-x^2}}$$



$$(\operatorname{tg} x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x\cdot\cos x - \sin x\cdot(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \begin{cases} \dfrac{1}{\cos^2 x} \\ 1 + \operatorname{tg}^2 x \end{cases}$$

$$\left(\arctan x\right)' = \frac{1}{1+\tan^2(\arctan x)} = \frac{1}{1+x^2}$$

Esercizio

$$\left(\frac{\sqrt{1-\sin^2 x}}{\ln x}\right)' = \frac{\left(\sqrt{1-\sin^2 x}\right)'\cdot \ln x - \sqrt{1-\sin^2 x}\cdot \frac{1}{x}}{\ln^2 x}$$

$$= \frac{\frac{1}{2\sqrt{1-\sin^2 x}}\cdot\left(1-\sin^2 x\right)'\cdot \ln x - \sqrt{1-\sin^2 x}\cdot\frac{1}{x}}{\ln^2 x}$$

$$= \frac{\frac{1}{2\sqrt{1-\sin^2 x}}\cdot\left(-2\sin x\cos x\right)\cdot \ln x - \sqrt{1-\sin^2 x}\cdot\frac{1}{x}}{\ln^2 x} \qquad \square$$

COMPITINO

1. $\displaystyle\lim_{n\to+\infty} \frac{1\pm x^n}{(1\pm x)^n}$

Facciamo solo un caso

$$\frac{1+x^n}{(1+x)^n} \qquad \text{con } x<0$$

$x<-1$ $\qquad$ $-1<x<0$ $\qquad$ $x\neq -1$

$1+x<0$

Se $x < -1$ $\qquad x = -|x|$, $\qquad |x| > 1$

$$\frac{1+x^n}{(1+x)^n} = \frac{1+(-|x|)^n}{(1-|x|)^n} = \frac{1+(-1)^n \cdot |x|^n}{(-1)^n (|x|-1)^n} =$$

$$= \frac{(-1)^n + |x|^n}{(|x|-1)^n} \sim \frac{|x|^n}{(|x|-1)^n} = \left(\frac{|x|}{|x|-1}\right)^n \to +\infty$$

$$\frac{|x|}{|x|-1} > 1$$

2.

$$\sum \frac{2n+3}{n^2+n} x^n \qquad a_n$$

convergenza assoluta:

$$\sum \frac{2n+3}{n^2+n} |x|^n \qquad \sqrt[n]{\frac{2n+3}{n^2+n}} \cdot |x| \to |x|$$

Se $|x| < 1$ c'è convergenza assoluta.

Se $|x| > 1$ la successione $|a_n| \to +\infty$

quindi non$(a_n \to 0)$

$\Rightarrow$ la serie non può convergere.

Se $x = 1$ $\qquad \sum \frac{2n+3}{n^2+n} \qquad \frac{2n+3}{n^2+n} \sim \frac{2}{n} \qquad \sum \frac{2}{n} = +\infty$

la serie diverge

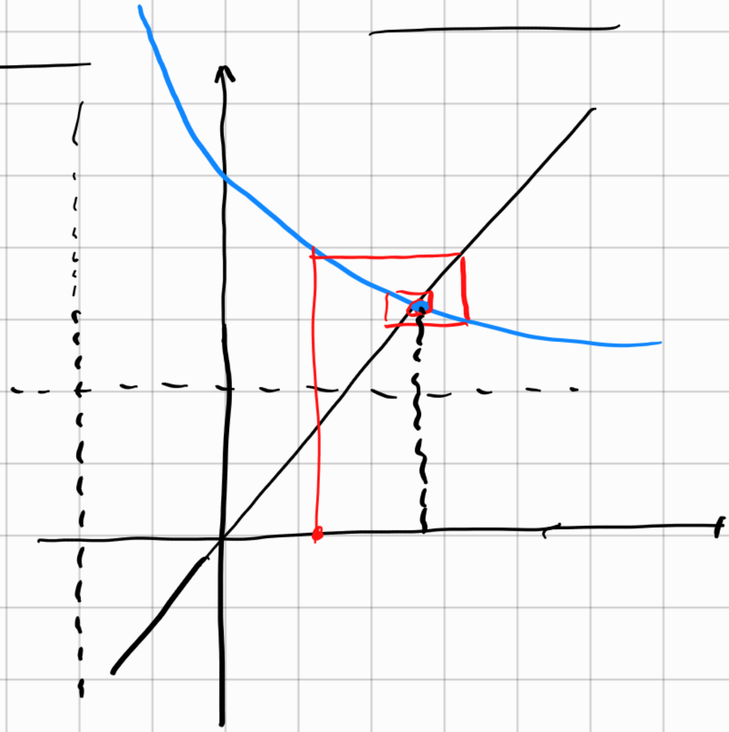
Se $x=-1$ $\quad \sum (-1)^n \frac{2n+3}{n^2+n}$

⚠ Non si può passare a $\sum (-1)^n \frac{2}{n}$

Bisogna verificare che $b_n = \frac{2n+3}{n^2+n}$ è decrescente (almeno definitivamente)

$b_{n+1} \leq b_n$

$\frac{2x+9}{x+2}$



$$\begin{cases} a_0 = \alpha \\ a_{n+1} = \frac{2a_n+9}{a_n+2} \end{cases}$$

$a_{n+1} = f(a_n)$

$$f(x) = \frac{2x+9}{x+2} = \frac{2x+4+5}{x+2} = 2 + \frac{5}{x+2}$$

f decrescente $\longrightarrow$ a_{2n} monotone $\to l_1$

a_{2n+1} $\to l_2$

l_1 e l_2 risolvono $f(f(x)) = x$

$\Downarrow$

$l_1 = l_2$.