ANALISI MATEMATICA B
LEZIONE 23 - 14.11.2022

limite per $n \to +\infty$ **ES. 6 TEST SETTIMANALE**

$$\frac{\ln\left[\left(4^{\ln n}+\sqrt{n}\right)^n+n^n\right]}{\ln\left(n!+n^n\right)} = (*)$$

$$4^{\ln n} = e^{\ln 4 \cdot \ln n} = n^{\ln 4}$$

$$\sqrt{n} = n^{1/2}$$

ORDINI DI INFINITO

$$\left[\log_a n \ll n^\alpha \ll a^n \ll n! \ll n^n\right] \quad n \to +\infty$$

OVVIAMENTE: $n^p \ll n^q$ se $p < q$

$$\left.\begin{array}{l} \ln 4 = \ln 2^2 = 2\ln 2 > \frac{1}{2} \\ \ln 4 > \ln e = 1 > \frac{1}{2} \end{array}\right\} \sqrt{n} \ll n^{\ln 4}$$

$$(*) = \frac{\ln\left[n^{n\ln 4}\left(1+\frac{\sqrt{n}}{n^{\ln 4}}\right)^n + n^n\right]}{\ln\left(n^n\left(\frac{n!}{n^n}+1\right)\right)}$$

$$\left[\left(1+\frac{\sqrt{n}}{n^{\ln 4}}\right)^{\frac{n^{\ln 4}}{\sqrt{n}}}\right]^{\frac{\sqrt{n}\cdot n}{n^{\ln 4}}} \qquad \left[\left(1+\frac{\sqrt{n}}{n^{\ln 4}}\right)^{\frac{n^{\ln 4}}{\sqrt{n}}}\right] \to e \qquad \frac{\sqrt{n}\cdot n}{n^{\ln 4}} \to \infty$$

forse non serve

$$\left[\begin{array}{l} \sqrt{n}\cdot n = n^{3/2} \\ \ln 4 < \frac{3}{2} \end{array}\right.$$

$$\sqrt{n}\cdot n \gg n^{\ln 4}$$

$$(*) = \frac{\ln\left[n^{n\ln 4}\cdot\left(1+\frac{\sqrt{n}}{n^{\ln 4}}\right)^{n}\cdot\left(1+\frac{n^{4}}{\%}\right)\right]}{n\ln n + \ln\left(1+\frac{n!}{n^{n}}\right)}$$

$$= \frac{n\ln 4\cdot\ln n + n\ln\left(1+\frac{\sqrt{n}}{n^{\ln 4}}\right) + \ln(\%)}{n\ln n + \ln\left(1+\frac{n!}{n^{n}}\right)}$$

$$= \frac{\cancel{n}\cdot\ln 4\cdot\cancel{\ln n}}{\cancel{n}\,\cancel{\ln n}}\cdot\frac{1+\ldots\ldots}{1+\ldots\ldots}$$

$$\to \ln 4$$

Teoremi di confronto (LEZIONE SCORSA)

1. $a_n \le b_n$ $\quad \begin{cases} a_n \to \ell \\ b_n \to m \end{cases} \Rightarrow \ell \le m$

2. $a_n \le b_n$ $\quad a_n \to +\infty \Rightarrow b_n \to +\infty$

3. $a_n \le b_n \le c_n$ $\quad \begin{cases} a_n \to \ell \\ c_n \to \ell \end{cases} \Rightarrow b_n \to \ell$

Corollario (limitata per infinitesima)

$$\begin{cases} a_n \text{ è limitata } \quad (\exists M : |a_n| \le M) \\ b_n \to 0 \quad (\text{è infinitesima}) \end{cases}$$

Allora $a_n \cdot b_n \to 0$

ES $\lim_{n \to +\infty} \frac{\sin n}{n} = 0$

dim

$$|a_n \cdot b_n| \le |a_n| \cdot |b_n| \le M \cdot |b_n|$$

$$\underset{\downarrow \atop 0}{-M \cdot |b_n|} \le \underset{\downarrow \atop 0}{a_n b_n} \le \underset{\downarrow \atop 0}{M \cdot |b_n|}$$

$\square$

Def (LEZIONE SCORSA)

$a_n \ll b_n$ se $\frac{a_n}{b_n} \to 0$ per $n \to +\infty$

$a_n \gg b_n$ se $\frac{b_n}{a_n} \to 0$

$a_n \sim b_n$ se $\frac{a_n}{b_n} \to 1$

asintoticamente equivalenti.

limiti notevoli

$\boxed{e^x - 1 \sim x}$ per $x \to 0$

significa $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

$\boxed{\ln(1+x) \sim x}$ per $x \to 0$

perché $\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$.

CRITERIO del RAPPORTO e della RADICE

$a_n = q^n$ $\qquad a_0 = 1,\ a_1 = q,\ a_2 = q^2,\ a_3 = q^3, \dots,$

Se $q > 1$ $\quad a_n \to +\infty$

Se $q < 1$ $\quad a_n \to 0$

$q^n = \left(\frac{1}{q}\right)^{-n}$

Se $q = 1$ $\quad a_n = 1 \to 1$

$\frac{a_{n+1}}{a_n} = q$ $\qquad \sqrt[n]{a_n} = q$

$\boxed{\text{Criterio del rapporto}}$ Se $a_n > 0$ e $\frac{a_{n+1}}{a_n} \to q \in [0, +\infty]$, $q \neq 1$

se $q > 1$ allora $a_n \to +\infty$

se $q < 1$ allora $a_n \to 0$

$\boxed{\text{Criterio della radice}}$ Se $a_n \geq 0$ e $\sqrt[n]{a_n} \to q \in [0, +\infty]$, $q \neq 1$

se $q > 1$ $a_n \to +\infty$ $\quad$ se $q < 1$ $a_n \to 0$.

ES CATTIVO

$$\begin{cases} a_n = n^2 & \dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)^2}{n^2} \to 1 \\[2ex] a_n = \dfrac{1}{n^2} & \dfrac{a_{n+1}}{a_n} = \dfrac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} \to 1. \end{cases}$$

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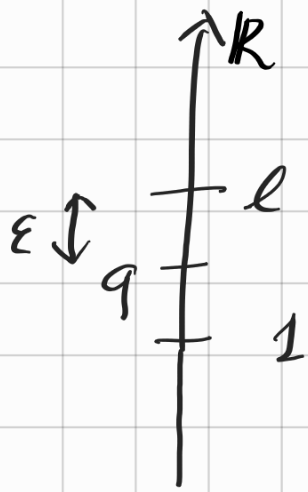
dim (rapporto) $\quad \dfrac{a_{n+1}}{a_n} \to \ell \neq 1.$



Supponiamo $\ell > 1$. $\exists q : 1 < q < \ell$

Per la def. di limite con $\varepsilon = \ell - q > 0$

$\exists N$ t.c. $\forall n \geq N \quad \dfrac{a_{n+1}}{a_n} > q$

$$\frac{a_{N+1}}{a_N} \geq q \qquad a_{N+1} \geq q \cdot a_N.$$

$$\frac{a_{N+2}}{a_{N+1}} \geq q \qquad a_{N+2} \geq q \cdot a_{N+1} \geq q^2 a_N$$

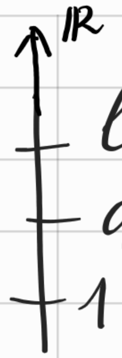
$$\vdots \qquad\qquad \vdots$$

$$\frac{a_{N+k+1}}{a_{N+k}} \geq q \qquad a_{N+k} \geq q^k \cdot a_N \xrightarrow[k\to+\infty]{} +\infty$$

$$\lim_{k\to+\infty} a_{N+k} = +\infty$$

$$\lim_{n\to+\infty} a_n = +\infty \qquad \square$$

dim (radice) $\sqrt[n]{a_n} \to \ell$



Se $\ell > 1$

$\exists N : \forall n > N \quad \sqrt[n]{a_n} \geq q$

$$a_n \geq q^n \to +\infty$$

$$a_n \to +\infty \qquad \square$$

$\ell < 1$ $\quad \exists N : \forall n > N \quad \sqrt[n]{a_n} \leq q \quad , \quad q < 1$

$$\underset{\downarrow\atop 0}{0} \leq \underset{\downarrow\atop 0}{a_n} \leq \underset{\downarrow\atop 0}{q^n} \qquad \square$$

ORDINI DI INFINITO

Teo $\boxed{n^\alpha << a^n}$ per $n \to +\infty$ $\quad [\alpha > 0 , \ a > 1]$

dim dobbiamo mostrare che $\frac{n^\alpha}{a^n} \to 0$

usiamo il criterio del rapporto

$$a_n = \frac{n^\alpha}{a^n} \qquad \frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^\alpha}{a^{n+1}}}{\frac{n^\alpha}{a^n}} = \frac{(n+1)^\alpha}{n^\alpha} \cdot \frac{a^n}{a^{n+1}}$$

$$= \left(1+\frac{1}{n}\right)^\alpha \cdot \frac{1}{a} \to \frac{1}{a} < 1$$

$\square$

Teo $\boxed{a^n << n!}$, $a>1$.

dim

$$a_n = \frac{a^n}{n!}$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{a^{n+1}}{(n+1)!}}{\frac{a^n}{n!}} = \frac{a^{n+1}}{a^n} \cdot \frac{n!}{(n+1)!} =$$

$$= \frac{a}{n+1} \longrightarrow 0 \quad \text{per } n \to +\infty$$

$0<1$ $\square$

Teo $n! << n^n$

dim $a_n = \frac{n!}{n^n}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \frac{(n+1)!}{n!} \frac{n^n}{(n+1)^{n+1}}$$

$$= \cancel{(n+1)} \cdot \frac{n^n}{(n+1)^n \cdot \cancel{(n+1)}} = \frac{1}{\left(\frac{n+1}{n}\right)^n} =$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n} \longrightarrow \frac{1}{e} \quad \square$$

Teo $\boxed{x^\alpha << a^x}$ per $x \to +\infty$ $\quad x \in \mathbb{R}$ $\quad \alpha>0,\ a>1$.

dim $\lfloor x \rfloor \leq x \leq \lfloor x \rfloor + 1$ $\quad x \to \infty \Rightarrow \lfloor x \rfloor \to +\infty$

$$\frac{x^\alpha}{a^x} \leq \frac{(\lfloor x \rfloor + 1)^\alpha}{a^{\lfloor x \rfloor}} = \frac{\lfloor x \rfloor^\alpha}{a^{\lfloor x \rfloor}} \cdot \left(1+\frac{1}{\lfloor x \rfloor}\right)^\alpha \longrightarrow 0$$

$\frac{\lfloor x \rfloor^\alpha}{a^{\lfloor x \rfloor}} \to 0$, $\quad \left(1+\frac{1}{\lfloor x \rfloor}\right)^\alpha \to 1$ $\quad \square$

$n = \lfloor x \rfloor$ $\quad \frac{n^\alpha}{a^n} \to 0$

Teo $\boxed{\log_a x << x^\alpha}$ per $x \to +\infty$

dim $y = \log_a x$ $\quad y \to +\infty$

$x = a^y$ $\quad x^\alpha = a^{\alpha y}$

$$\frac{\log_a x}{x^\alpha} = \frac{y}{a^{\alpha y}} = \frac{y}{(a^\alpha)^y} \longrightarrow 0 \quad \text{per il caso precedente}$$

$\square$

$$\log_a n << n^\alpha$$

Cosa succede per $x \to 0$?

$$\boxed{\underset{+\infty}{\underbrace{-\log_a x}} << \underset{+\infty}{\underbrace{\frac{1}{x^\alpha}}} \quad \text{per } x \to 0^+}$$

$-\infty$

COSA STRANA

$$-\frac{1}{n} >> \frac{1}{n^2} \quad \text{per } n \to +\infty$$

$$\frac{\frac{1}{n^2}}{-\frac{1}{n}} = -\frac{n}{n^2} = -\frac{1}{n} \to 0$$

$$y = \frac{1}{x}$$

$$-\log_a x = -\log_a \frac{1}{y} = \log_a y$$

$$\frac{1}{x^\alpha} = \left(\frac{1}{x}\right)^\alpha = y^\alpha$$

$$\frac{-\log_a x}{\frac{1}{x^\alpha}} = \frac{\log_a y}{y^\alpha} \xrightarrow{y \to +\infty} 0$$

Cosa succede a $-\infty$?

$$\boxed{a^x << \frac{1}{(-x)^\alpha} \quad \text{per } x \to -\infty}$$

$$\overset{+\infty}{\uparrow}$$
$$y = -x$$

$$a^x = a^{-y} = \frac{1}{a^y} \qquad \frac{1}{(-x)^\alpha} = \frac{1}{y^\alpha}$$

$$\frac{a^x}{\frac{1}{|-x|^\alpha}} = \frac{\frac{1}{a^y}}{\frac{1}{y^\alpha}} = \frac{y^\alpha}{a^y} \longrightarrow 0 \quad \text{per } y \to +\infty$$

$\square$