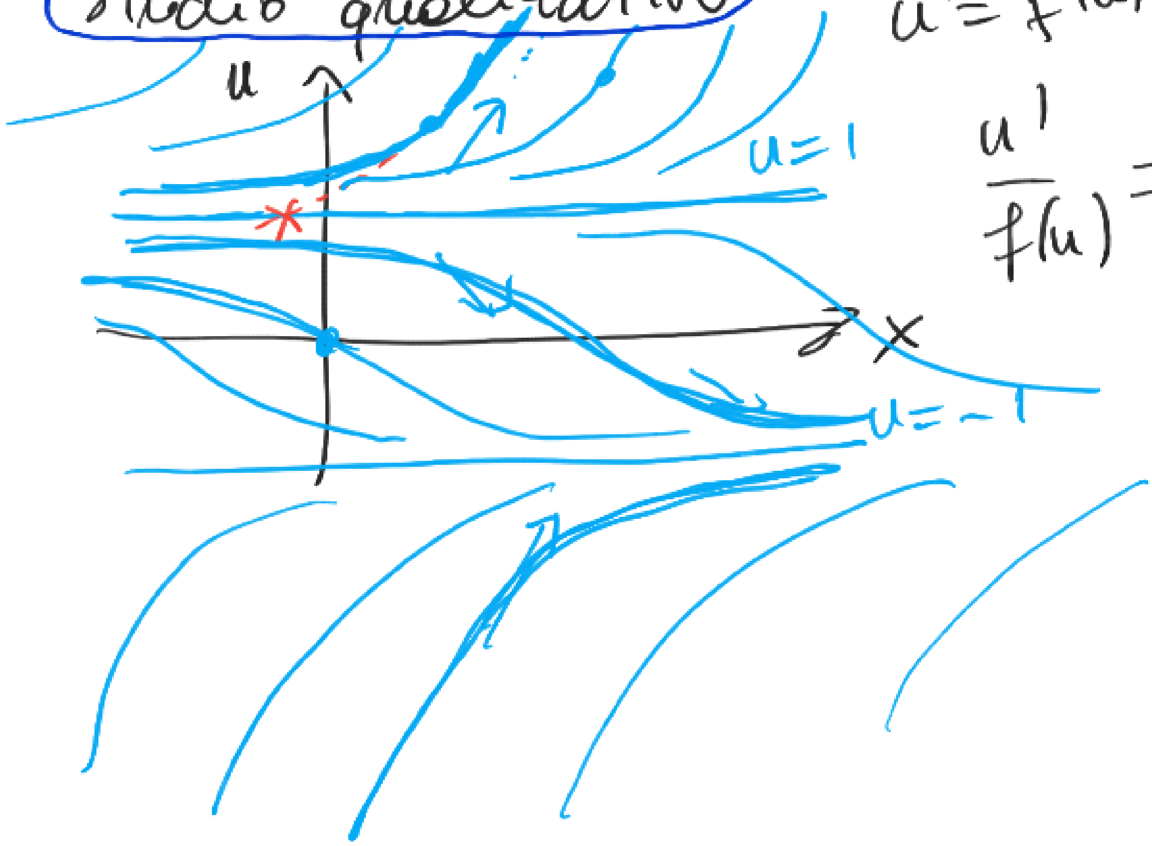
ANALISI MATEMATICA (B)
LEZIONE 77
(8.4.2020)

ES $u' = \dfrac{u^2-1}{u^2+1}$ Eq. autonoma $u=u(x)$.

Studio qualitativo

$u' = f(u)$

$\dfrac{u'}{f(u)} = 1.$



Gli zeri di f corrispondono a soluzioni stazionarie (costanti) ||

$$\frac{u^2-1}{u^2+1}=0 \qquad \begin{cases} u=1 \\ u=-1 \end{cases}$$

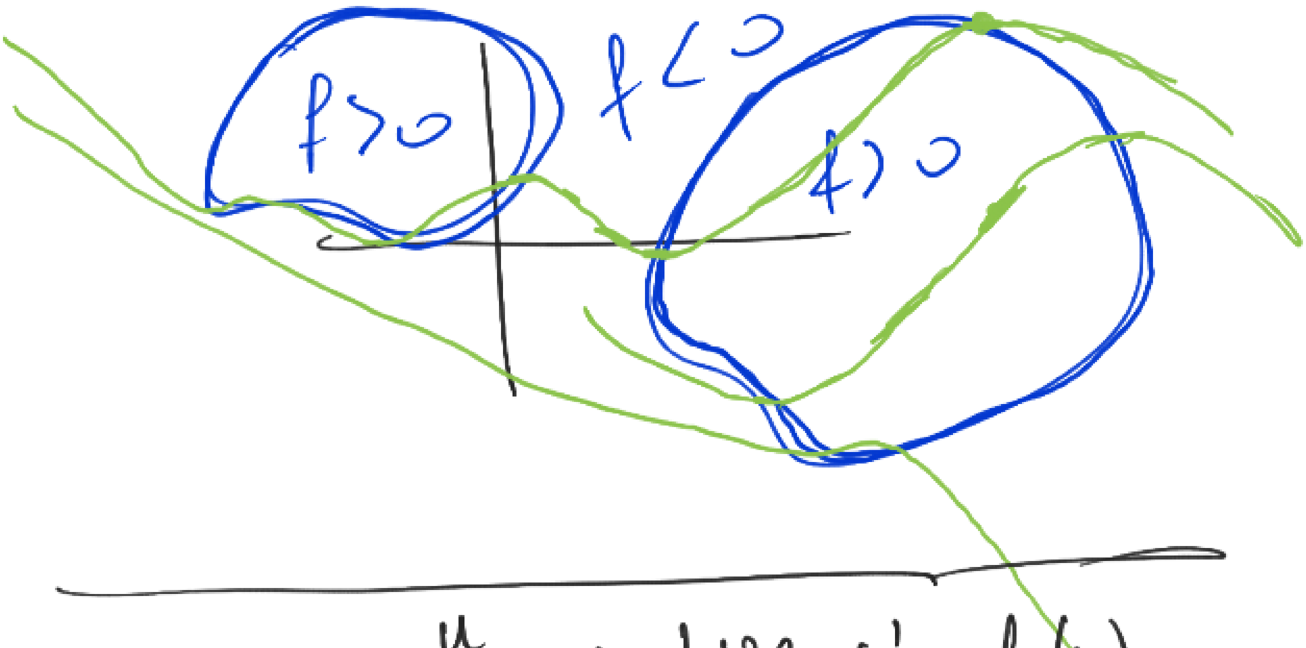
$$u=u(x) \qquad u'(x)=\frac{u^2(x)-1}{u^2(x)+1}$$

ES Se $u(x)$ è una soluzione $u(x)>1$ è crescente

$\lim_{x\to-\infty} u(x)=\ell$ esiste

Dimostrare che $\ell=1$

Se $u'(x)=f(x,u(x))$



$u'=f(u)$

$f(u)=0$

Stazionari

stabile

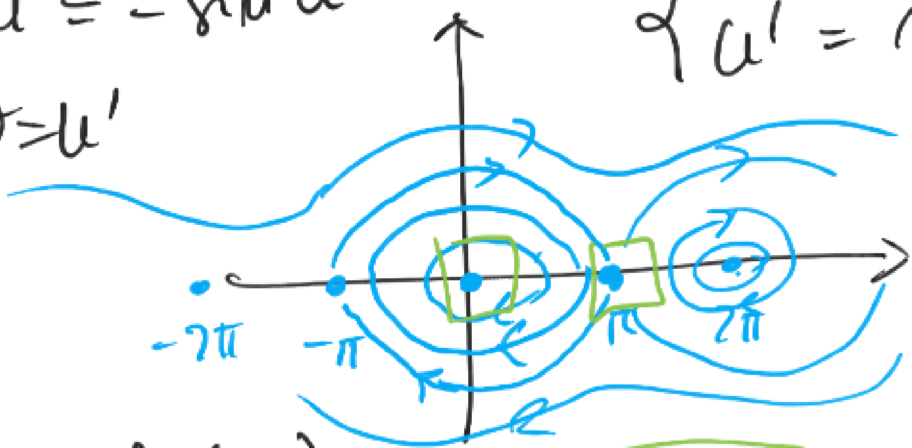
$f(u) = 0$

Sistemi di equazioni del I ordine

$u'' = -\sin u$

$$\begin{cases} v' = -\sin u \\ u' = v \end{cases}$$

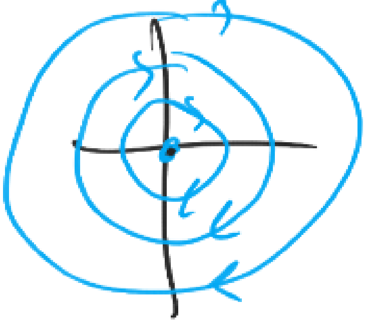
$v = u'$



$\underline{u}' = \underline{f}(\underline{u})$

$$\boxed{\begin{cases} v' = -u \\ u' = v \end{cases}}$$

linearizz.



Sistemi lineari del I ordine

a coefficienti costanti (omogeneo)

$$\underline{u}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

2 equazioni
2 incognite

$$\boxed{\begin{cases} x'(t) = a \cdot x(t) + b \cdot y(t) \\ y'(t) = c \cdot x(t) + d \cdot y(t) \end{cases}} \leftarrow \underline{u} \in \mathbb{R}^2$$

$$\boxed{\underline{u}'(t) = A \cdot \underline{u}(t)}$$

$\underline{u} \in \mathbb{R}^n$

sistema $n \times n$

$$\underline{u}'(t) - A \cdot \underline{u}(t) = 0$$

$$e^{-At} \cdot \underline{u}'(t) - e^{-At} \cdot A \cdot \underline{u}(t) = 0$$

$$\left(e^{-At} \cdot \underline{u}(t)\right)' = 0$$

$$e^{-At} \cdot \underline{u}(t) = \underline{u}_0$$

$$\underline{u}(t) = e^{At} \cdot \underline{u}_0$$

$$e^{At} = \sum_{k=0}^{+\infty} \frac{(At)^k}{k!}$$

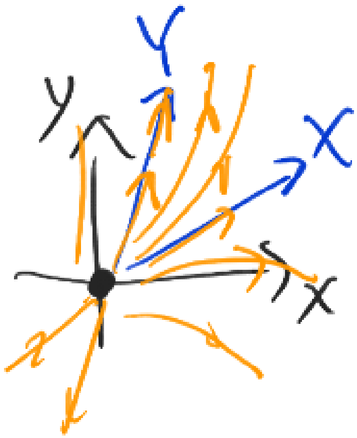
Sistemi 2x2

$$\begin{cases} x'(t) = a\,x(t) + b\,y(t) \\ y'(t) = c\,x(t) + d\,y(t). \end{cases}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}'(t) = A \cdot \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Conviene scegliere delle variabili "migliori".



$$\begin{pmatrix} X \\ Y \end{pmatrix} = M \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} X \\ Y \end{pmatrix}' = M \begin{pmatrix} x \\ y \end{pmatrix}'$$

$$= M A \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \underbrace{M A M^{-1}}_{B} \begin{pmatrix} X \\ Y \end{pmatrix}.$$

$$\begin{pmatrix} X \\ Y \end{pmatrix}' = B \begin{pmatrix} X \\ Y \end{pmatrix} \qquad B \sim A.$$

Siano λ, μ gli autovalori di A.

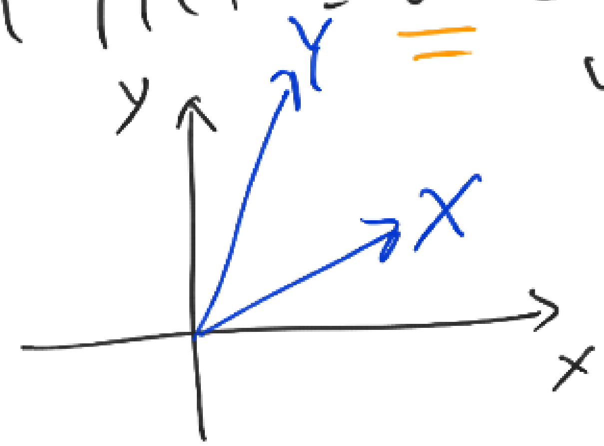
(soluzioni di $\det(A - tI) = 0$)

① Se λ, μ sono reali distinte

$$A \sim \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

$$\begin{cases} X' = \lambda X \\ Y' = \mu Y \end{cases}$$

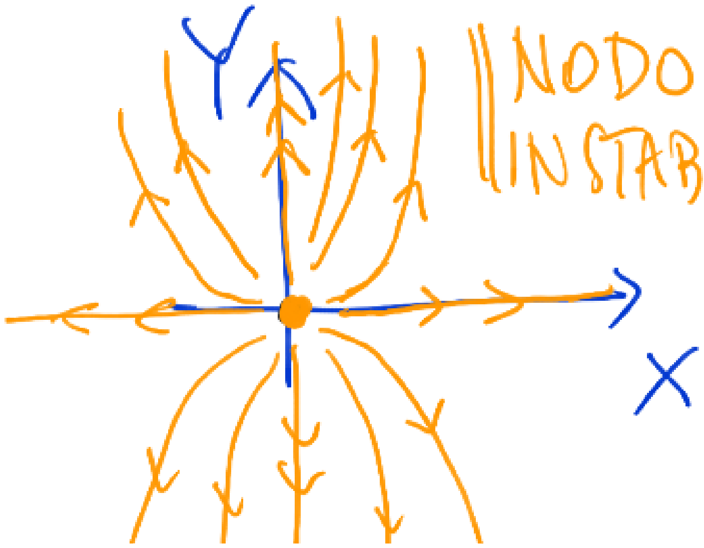
$$\begin{cases} X(t) = a \cdot e^{\lambda t} \\ Y(t) = b \cdot e^{\mu t} \end{cases}$$



$$Y(t) = b \cdot \left(e^{\lambda t}\right)^{\frac{\mu}{\lambda}}$$

$$Y = \pm b \left(\frac{|X|}{|a|}\right)^{\frac{\mu}{\lambda}}$$

$$Y = k \cdot |X|^{\frac{\mu}{\lambda}}$$



NODO INSTABILE $\mu, \lambda > 0$, $\mu > \lambda$

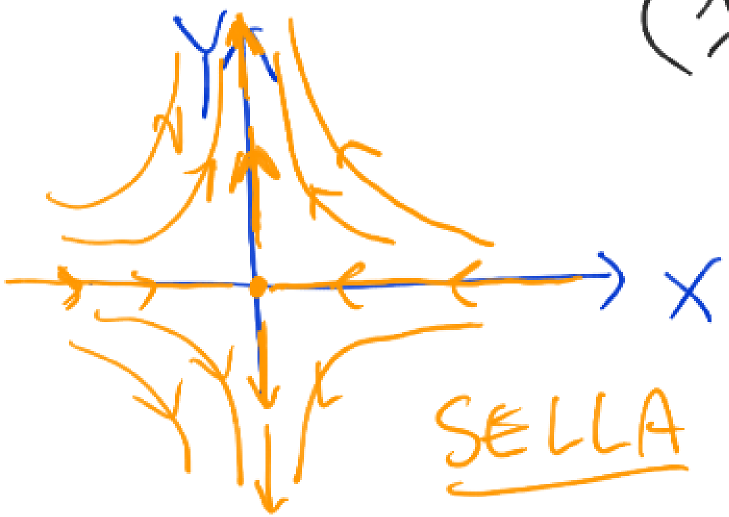
$\mu, \lambda < 0$

NODO

INSTABILE

$\lambda < 0 < \mu$

($\lambda = 0$ o $\mu = 0$ CASI DEGENERI)



SELLA (INSTABILE)

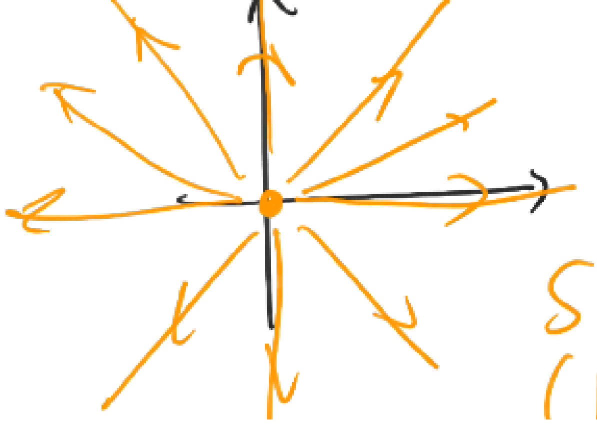
(2) Autovalori reali coincidenti

$\lambda = \mu$.

(2a) se A è diag.bile:

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$\begin{cases} x' = \lambda x \\ y' = \lambda y \end{cases} \qquad \begin{cases} x(t) = a e^{\lambda t} \\ y(t) = b e^{\lambda t} \end{cases}$$



$\lambda > 0$

$\lambda < 0$

STELLA (INSTABILE)

(26) se A non è diag.bile:

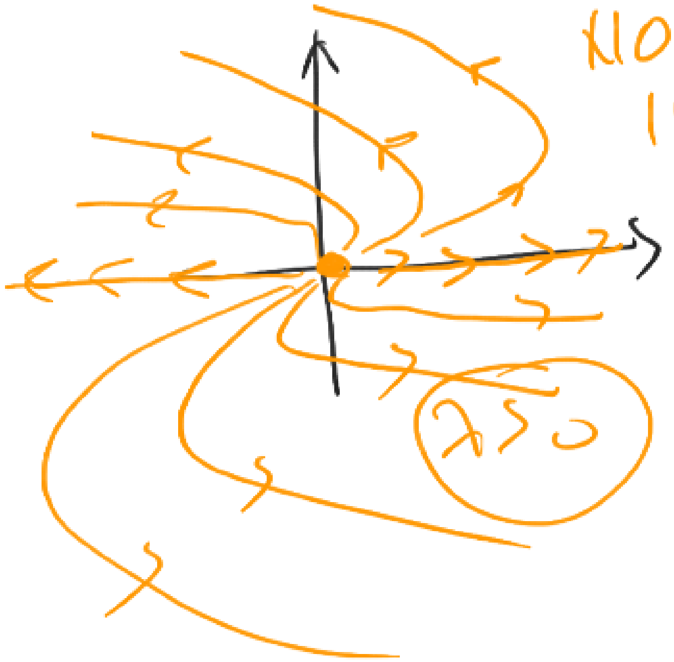
$$A \sim \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$$

autovettore v

$A v = \lambda v$

autovettore generalizzato w

$A w = \lambda w + v$



NODO (INSTABILE) IMPROPRIO

$$\begin{cases} X' = \lambda X + Y \\ Y' = \lambda Y \end{cases}$$

$$Y = b \cdot e^{\lambda t}$$

$$\boxed{b = 0 \quad Y = 0 \quad X = a \cdot e^{\lambda t}}$$

3. Autovalori complessi coniugati

$\lambda = \alpha - i\beta$, $\quad \underline{v} + i\underline{w}$ autovettore complesso

$\mu = \alpha + i\beta$, $\quad \underline{v} - i\underline{w}$

$$A(v + iw) = (\alpha - i\beta)(v + iw)$$

Allora $A(\underline{v}-i\underline{w}) = (\alpha+i\beta)(\underline{v}-i\underline{w})$

$$A \sim \begin{pmatrix} \alpha-i\beta & 0 \\ 0 & \alpha+i\beta \end{pmatrix} \sim \begin{pmatrix} \rho\cos\theta & -\rho\sin\theta \\ \rho\sin\theta & \rho\cos\theta \end{pmatrix}$$

(Forma di Jordan reale)

.....

$$\begin{cases} A\underline{v} = \alpha\underline{v} + \beta\underline{w} \\ A\underline{w} = -\beta\underline{v} + \alpha\underline{w} \end{cases}$$

$\beta Y = \alpha X - X'$

$$A \sim \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

$$\begin{cases} X' = \alpha X - \beta Y \\ Y' = \beta X + \alpha Y \end{cases}$$

$\alpha, \beta \in \mathbb{R}$

$\alpha \pm i\beta$ autovalori

Riconduco il sistema ad una equazione del

1º secondo ordine.

$$X'' = \alpha X' - \beta Y' = \alpha X' - \beta(\beta X + \alpha Y)$$

$$= \alpha X' - \beta^2 X - \alpha\beta Y$$

$$= \alpha X' - \beta^2 X - \alpha(\alpha X - X')$$

$$= 2\alpha X' - \beta^2 X - \alpha^2 X$$

$$X'' - 2\alpha X + (\alpha^2 + \beta^2) X = 0$$

$$P(\lambda) = \lambda^2 - 2\alpha\lambda + (\alpha^2 + \beta^2)$$

$$\lambda_{12} = \alpha \pm \sqrt{\alpha^2 - (\alpha^2 + \beta^2)}$$

$$= \alpha \pm \sqrt{-\beta^2} = \alpha \pm i\beta.$$

autovalori!

$$X(t) = e^{\alpha t}(a \cdot \cos\beta t + b \cdot \sin\beta t).$$

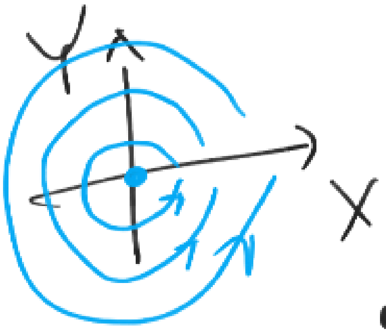
⋮ ← coniugati

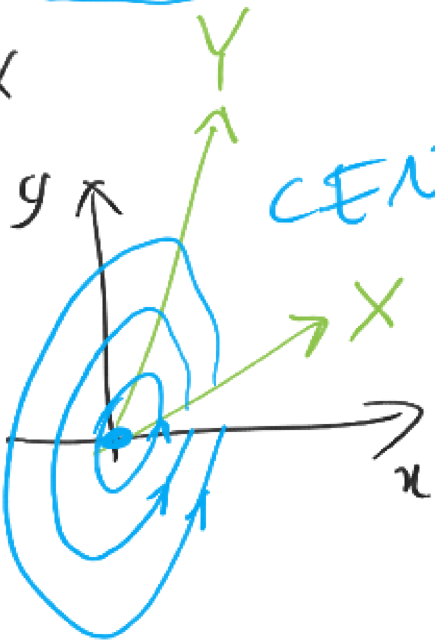
$$Y(t) = \ldots = e^{\alpha t}(a \sin\beta t - b\cos\beta t)$$

$a + ib$

$$\begin{cases} X(t) = \rho e^{\alpha t} \cos(\beta t + \theta) \\ Y(t) = \rho e^{\alpha t} \sin(\beta t + \theta) \end{cases}$$

$\boxed{\alpha = 0}$ ← Re λ

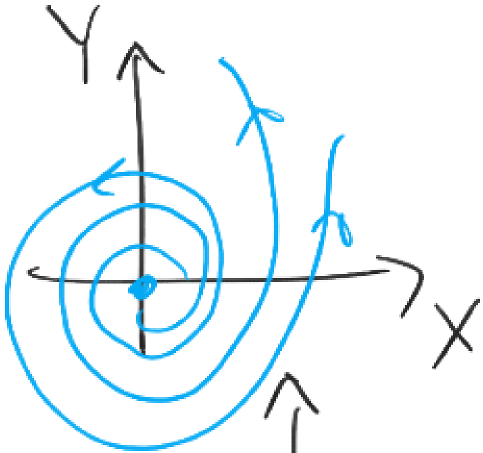


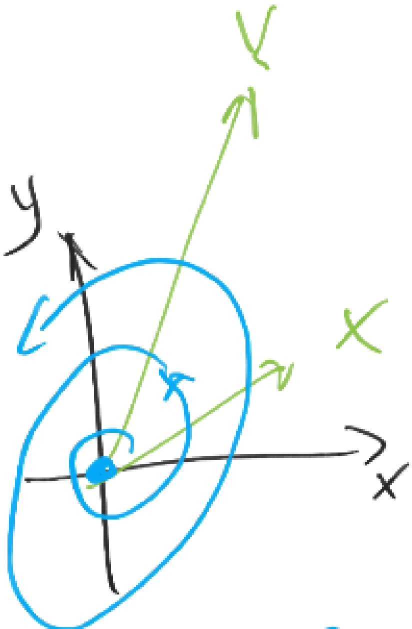


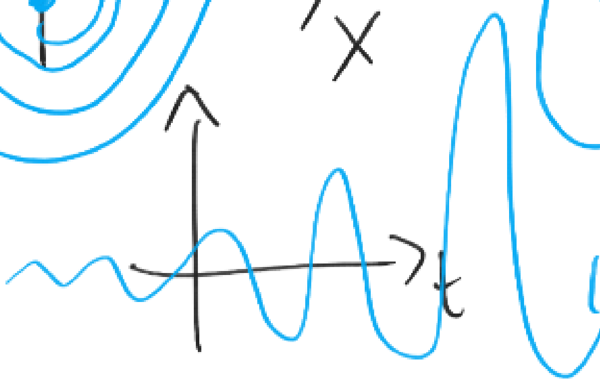
CENTRO STABILE

(NON ASINTOTICAMENTE)

$\boxed{\alpha > 0}$







FUOCO INSTABILE

$\alpha < 0$

FUOCO STABILE