APPROXIMATE EXTENSION OF PARTIAL ε-CHARACTERS OF ABELIAN GROUPS TO CHARACTERS WITH APPLICATION TO INTEGRAL POINT LATTICES

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Prologue

The results presented in this talk grew out of long-year unsuccessful attempts to prove (or disprove) Gordon's conjecture [1991], concerning a natural nonstandard version of the Pontryagin-van Kampen (PvK) duality for locally compact abelian groups.

Given a hyperfinite abelian group G (in an ω_1-saturated nonstandard universe) with two distinguished subgroups $G_0 \subseteq G_\omega \subseteq G$, where G_0 is a Π^0_1 subgroup of infinitesimals and G_ω is a Σ^0_1 subgroup of finite elements, such that $\# S / \# R$ is finite for any internal sets $G_0 \subseteq R \subseteq S \subseteq G$, one can form a (classical) locally compact, metrizable and σ-compact group as the quotient G_ω / G_0.

One can also form the *internal* dual $G^\wedge = {}^*\mathrm{Hom}(G, {}^*\mathbb{T})$ of G (where $\mathbb{T}$ denotes the multiplicative group of complex units) and consider the infinitesimal annihilators

$$
\begin{aligned}
G_0^\perp &= \{\alpha \in G^\wedge;\ (\forall\, x \in G_0)(\alpha(x) \approx 1)\} \quad (S\text{-continuous internal characters}),\\
G_\omega^\perp &= \{\alpha \in G^\wedge;\ (\forall\, x \in G_\omega)(\alpha(x) \approx 1)\} \quad (\text{internal characters infinitesimal on } G_\omega),
\end{aligned}
$$

of G_0 and G_ω, respectively. Then the triple $\left(G^\wedge, G_\omega^\perp, G_0^\perp\right)$ satisfies the same above mentioned conditions as the original triple (G, G_0, G_ω).

Gordon's conjecture states that the canonic map Φ from $G_0^\perp/G_\omega^\perp$ to the (classical) dual group $\widehat{G_\omega/G_0}$ of G_ω/G_0, making the diagram

$$\begin{array}{ccc} G_\omega & \xrightarrow{\alpha\restriction G_\omega} & {}^*\mathbb{T} \\ \downarrow & & \downarrow{\scriptstyle\circ} \\ G_\omega/G_0 & \xrightarrow[\Phi(\alpha)]{} & \mathbb{T} \end{array}$$

commute for each $\alpha \in G_0^\perp$ (with $G_\omega \to G_\omega/G_0$ denoting the restriction of the canonic projection $G \to G/G_0$ to G_ω and ${}^\circ\colon {}^*\mathbb{T} \to \mathbb{T}$ denoting the standard part map), is indeed a continuous homomorphism of topological groups. As it is an isomorphisms of $G_0^\perp/G_\omega^\perp$ onto a closed subgroup in $\widehat{G_\omega/G_0}$, the only problem is the surjectivity of Φ.

It is known that Gordon's conjecture is true whenever there is an internal subgroup K such that $G_0 \subseteq K \subseteq G_\omega$. In particular, this is the case for triples of the form $(G, \{1\}, G_\omega)$ with countable discrete $G_\omega = G_\omega/\{1\}$, and (G, G_0, G) with compact G/G_0.

We will use the countable discrete special case to derive certain almost-near theorems in the sense of Anderson [1986], with rather strong uniformity properties, for (partial almost) homomorphisms of abelian groups into the group $\mathbb{T}$ and for dual lattices of integral point lattices.

Let G, H be groups, the latter endowed with a (left) invariant metric ρ, and $\varepsilon > 0$. A mapping $f\colon S \to H$, where $S \subseteq G$, is called a *partial ε-homomorphism* if $\rho(f(xy), f(x)f(y)) \leq \varepsilon$ for all $x, y \in S$ such that $xy \in S$. If $S = G$ then f is called an *ε-homomorphism.*

If $f\colon S \to H$ satisfies the homomorphy condition $f(xy) = f(x)f(y)$ whenever $x, y, xy \in S$, then f is called a *partial homomorphism.*

Two mappings $f\colon U \to H$, $g\colon V \to H$, where $U, V \subseteq G$, are said to be *ε-close on a set $S \subseteq U \cap V$* if $\rho(f(x), g(x)) \leq \varepsilon$ for each $x \in S$.

The topic can be traced back to Ulam. Some conditions under which a (continuous) δ-homomorphism $f\colon G \to H$ is ε-close on G to a (continuous) homomorphism $\varphi\colon G \to H$ were studied, e.g., by Kazhdan [1982], Grove, Karcher and Ruh [1974], Alekseev, Glebskii and Gordon [1999], Špakula and Zlatoš [2004]; extensive reference lists in can be found in Hyers and Rassias [1992], and Székelyhidi [2000].

In this talk we will examine the problem when a partial δ-homomorphism $f\colon R \to \mathbb{T}$ from a finite subset R of an abelian group G to the multiplicative group of all complex units $\mathbb{T}$ is ε-close to a homomorphism $\varphi\colon G \to \mathbb{T}$ on a set $S \subseteq R$. Alternatively, we will use terms like *(partial) ε-character* and *(partial) character.*

Not even all partial homomorphisms can be extended to homomorphisms. The necessary and sufficient condition can easily be stated:

A partial homomorphism $f\colon S \to H$, defined on a subset S of a group G extends to a homomorphism $\varphi\colon \langle S\rangle \to H$ if and only if, for any integer $n > 0$ and all $x_1, \dots, x_n \in S$, the equality $x_1 \dots x_n = 1$ in G implies the equality $f(x_1) \dots f(x_n) = 1$ in H, or, equivalently, if f extends to a partial homomorphism $\langle S\rangle_n \to H$ for each $n > 0$, where

$$\langle S\rangle_n = \left(S \cup \{1\} \cup S^{-1}\right)^n \qquad \text{and} \qquad \langle S\rangle = \bigcup_{n\in\mathbb{N}} \langle S\rangle_n$$

is the subgroup of G generated by S. For G abelian and $H = \mathbb{T}$, this automatically implies the extendability of f to a character $\varphi\colon G \to \mathbb{T}$.

As a finite set $S \subseteq G$ may contain elements of arbitrarily big order, there seems to be no reason for the existence of an integer n, depending uniformly just on the number $\#S$ of elements of S, such that the extendability of $f\colon S \to \mathbb{T}$ to a partial character $\langle S\rangle_n \to \mathbb{T}$ would guarantee its extendability to a character $\varphi\colon G \to \mathbb{T}$ for *all* G abelian, S and f. Therefore it is perhaps surprising that the approximative version of this statement is true.

Kazhdan's theorem

An *amenable group* G is a locally compact group, endowed with an *invariant mean* M; i.e., $M\colon L_\infty(G) \to \mathbb{C}$ is a (left) invariant positive linear functional, assigning the value 1 to the constant function $1\colon G \to \mathbb{C}$.

Theorem 1. (Kazhdan [1982] *Let G be an amenable group, $H = U(X)$ be the group of all unitary operators on some Hilbert space X with the usual operator norm, and $\varepsilon < 1/200$. Then any (continuous) ε-homomorphism $f\colon G \to H$ is 2ε-close to a (continuous) homomorphism $\varphi\colon G \to H$.*

A more elementary proof, working for amenable G and finite dimensional compact Lie group H, was given by Alekseev, Glebskii and Gordon [1999].

For $H = \mathbb{T} = U(\mathbb{C})$ one can give even a more elementary proof, under a considerably weaker restriction on ε and a better estimation of the distance of both maps.

We use the *arc* or *angular metric* $|\arg(a/b)|$ on $\mathbb{T}$, instead of the euclidean metric $|a-b|$.

Theorem 2. *Let G be an amenable locally compact group, and $0 < \varepsilon < \frac{\pi}{2}$. Then for every ε-homomorphism $f\colon G \to \mathbb{T}$ there exists a homomorphism $\varphi\colon G \to \mathbb{T}$ such that*

$$\left|\arg \frac{\varphi(x)}{f(x)}\right| \le \varepsilon$$

for each $x \in G$. Moreover, if f is continuous then one can assume the same for φ.

Sketch of proof. Let $0 < \varepsilon < \frac{\pi}{2}$, and $f\colon G \to \mathbb{T}$ be an ε-homomorphism. Define $\varphi\colon G \to \mathbb{T}$ by

$$\varphi(x) = f(x) \exp\left(\mathrm{i} M_t \left[\arg \frac{f(xt)}{f(x)f(t)}\right]\right),$$

where M_t denotes the invariant mean M on G with the argument regarded as a function of t. Then φ obviously is ε-close to f and continuous if f is. Its homomorphy can be established by a fairly straightforward computation.

Gordon's theorem

We will actually need a special case of one of Gordon's results, only, formulated in terms of ultraproducts of abelian groups with respect to a nontrivial (hence countably incomplete) ultrafilter over the set $\mathbb{N}$. On the other hand, we will slightly generalize this result from hyperfinite to all internal groups. This could be done just by an inspection of Gordon's proof, or by proving the ultraproduct version directly.

Theorem 3. (Gordon [1991]) *Let $G = \prod_{i\in\mathbb{N}} G_i/D$ be an ultraproduct of a system of abelian groups G_i with respect to a nontrivial ultrafilter D on $\mathbb{N}$, and X be a countable subgroup of G. Then for each character $g\colon X \to \mathbb{T}$ there exists an internal character $\gamma\colon G \to {}^*\mathbb{T}$ such that*

$$g(x) = {}^\circ\gamma(x),$$

for each $x \in X$.

Sketch of proof. Let $\mathit{\Gamma}_i = \widehat{G}_i = \mathrm{Hom}(G_i, \mathbb{T})$ denote the dual group of G_i,

$$\mathit{\Gamma} = \prod_{i\in\mathbb{N}} \mathit{\Gamma}_i/D \qquad \text{and} \qquad {}^*\mathbb{T} = \mathbb{T}^{\mathbb{N}}/D$$

Thus the elements of $\mathit{\Gamma}$ are exactly all the *internal* characters $\gamma\colon G \to {}^*\mathbb{T}$, and (neglecting topology) $\mathit{\Gamma}$ plays the role of the dual group of G within the "world of internal objects." Similarly, $\widehat{X} = \mathrm{Hom}(X, \mathbb{T})$ denotes the (usual) dual group of the discrete abelian group X. Thus $\widehat{X}$ is a compact metrizable topological group.

Consider the map $\mathit{\Phi}\colon \mathit{\Gamma} \to \widehat{X}$ given by $\mathit{\Phi}(\gamma) = {}^\circ\gamma \restriction X$, i.e., $\mathit{\Phi}(\gamma)(x) = {}^\circ\gamma(x)$ for $\gamma \in \mathit{\Gamma}$, $x \in X$. Obviously, $\mathit{\Phi}$ is a group homomorphism. The proof will be complete once we show that $\mathit{\Phi}$ is onto. To this end it is enough to prove that $\mathit{\Phi}[\mathit{\Gamma}]$ is both closed and dense in $\widehat{X}$, i.e., it separates points in X.

Theorem 4. *Let $0 < \delta < \varepsilon \leq \frac{\pi}{2}$ and $1 \leq q \in \mathbb{N}$. Then there exists a positive integer $n \in \mathbb{N}$ (depending just on δ, ε and q) such that for any abelian group G, a set $S \subseteq G$, satisfying $\#S \leq q$, and a partial δ-character $f\colon \langle S \rangle_n \to \mathbb{T}$ there is a character $\varphi\colon G \to \mathbb{T}$ such that*

$$\left| \arg \frac{\varphi(x)}{f(x)} \right| < \varepsilon,$$

for each $x \in S$.

Sketch of proof. Assume the contrary and choose a δ, ε and q witnessing it. Then there is a sequence G_i of abelian groups with subsets $S_i \subseteq G_i$, $\#S_i \leq q$, and partial δ-characters $f_i \colon \langle S_i \rangle_i \to \mathbb{T}$, such that for each genuine character $\varphi_i \colon G_i \to \mathbb{T}$ there is an $x_i \in S_i$ subject to

$$\left| \arg \frac{\varphi_i(x_i)}{f_i(x_i)} \right| \geq \varepsilon.$$

Let D be any nontrivial ultrafilter on the set $\mathbb{N}$ and $G = \prod_{i \in \mathbb{N}} G_i / D$. Put

$$S_{ik} = \langle S_i \rangle_k$$

for $i, k \in \mathbb{N}$, and denote

$$X_k = \prod_{i \in \mathbb{N}} S_{ik}/D \subseteq G \qquad \text{and} \qquad X = \bigcup_{k \in \mathbb{N}} X_k.$$

Then X is a countable subgroup of G, and the restriction of the internal map $f = (f_i)/D$ to X gives rise to a δ-character ${}^\circ f \restriction X$ of X such that

$$\left| \arg \frac{\gamma(x)}{{}^\circ f(x)} \right| \geq \varepsilon.$$

for any internal character $\gamma \in \widehat{G}$ and some $x = (x_i)/D \in X$, which can be shown to contradict the conjunction of Theorems 2 and 3.

Application to duals of integral point lattices

A *point lattice* in $\mathbb{R}^q$ is a discrete subgroup of $\mathbb{R}^q$; an *integral point lattice* in $\mathbb{R}^q$ is a subgroup of $\mathbb{Z}^q$. A point lattice $H \subseteq \mathbb{R}^q$ has *full rank* if its linear span $[H]$ equals $\mathbb{R}^q$.

The dual group $\widehat{\mathbb{Z}^q} = \mathrm{Hom}(\mathbb{Z}^q, \mathbb{T})$ of $\mathbb{Z}^q$ is canonically isomorphic to $\mathbb{T}^q$; the action of an $\alpha = (\alpha_1, \dots, \alpha_q) \in \mathbb{T}^q$ on $\mathbb{Z}^q$ is given by

$$x \mapsto \alpha^x = \alpha_1^{x_1} \dots \alpha_q^{x_q},$$

for $x = (x_1, \dots, x_q) \in \mathbb{Z}^q$. For any set $X \subseteq \mathbb{Z}^q$ we denote by

$$X' = \big\{\alpha \in \mathbb{T}^q;\ (\forall\, x \in X)(\alpha^x = 1)\big\}$$

the annihilator of X in $\mathbb{T}^q$; it is always a subgroup of $\mathbb{T}^q$.

If $H \subseteq \mathbb{Z}^q$ is an integral point lattice in $\mathbb{R}^q$, then, by the PvK duality, there are canonic group isomorphisms

$$\widehat{H} \cong \mathbb{T}^q/H' \qquad \text{and} \qquad \widehat{\mathbb{Z}^q/H} \cong H'$$

We also denote

$$B_1 = \{x \in \mathbb{R}^q;\ \|x\|_1 \leq 1\}, \qquad \text{and} \qquad B_\infty = \{x \in \mathbb{R}^q;\ \|x\|_\infty \leq 1\},$$

the closed unit balls with respect to the ℓ_1-norm $\|x\|_1 = \sum_{i=1}^q |x_i|$, and with respect to the ℓ_∞-norm $\|x\|_\infty = \max_{i \leq q} |x_i|$ on $\mathbb{R}^q$, respectively. The interior of any set $X \subseteq \mathbb{R}^q$ or $X \subseteq \mathbb{T}$ is denoted by X°. Additionally,

$$\Lambda_\delta = \{c \in \mathbb{T};\ |\arg c| \leq \delta\},$$

and

$$\mathbb{T}^q(X, A) = \{\alpha \in \mathbb{T}^q;\ (\forall\, x \in X)(\alpha^x \in A)\},$$

for any $X \subseteq \mathbb{Z}^q$, $A \subseteq \mathbb{T}$.

Theorem 5. *Let $0 < \delta < \frac{2\pi}{3}$, $\varepsilon > 0$ and $1 \leq q \in \mathbb{N}$. Then there exists a positive integer $n \in \mathbb{N}$ (depending just on δ, ε and q) such that for every integral point lattice $H \subseteq \mathbb{Z}^q$ we have*

$$\mathbb{T}^q(H \cap nB_1, \Lambda_\delta) \subseteq H' \cdot (\Lambda_\varepsilon^\circ)^q,$$

i.e., for each $\alpha \in \mathbb{T}^q$, satisfying $\alpha^x \in \Lambda_\delta$ for any $x \in H$ such that $\|x\|_1 \leq n$, there is a $\beta \in H'$, such that $|\arg(\alpha_j/\beta_j)| < \varepsilon$ whenever $1 \leq j \leq q$.

Sketch of proof. The result can be derived from Theorem 4 by rather elementary, though not quite straightforward accounts, using the fact that the abelian group $G = \mathbb{Z}^q/H$ is generated by the at most q-element set $S = \{e_1 + H, \dots, e_q + H\}$, where $e_i \in \mathbb{Z}^q$ has 1 in the ith place and 0's elwewhere.

The first integer $n \geq 1$ satisfying the condition of Theorem 5 for δ, ε and q will be denoted by $N(\delta, \varepsilon, q)$.

Remark. Gordon's theorem seems to add certain uniformity to the discrete-compact case of PvK duality. The latter implies the existence of such an n depending not just on δ, ε and q but also on H.

On the other hand, using the PvK-duality in full generality, one can prove the following more general but less uniform result (the weakened version of the last theorem being a special case for the discrete group $G = \mathbb{Z}^q$ and $S_n = \mathbb{Z}^q \cap nB_1$):

Let G be a σ-compact LCA group and $\widehat{G}$ be its dual group. Assume that $S_n \subseteq S_{n+1}$ is a sequence of symmetric compact neighborhoods of the unit element $1 \in G$, such that $G = \bigcup_{n\in\mathbb{N}} S_n$, and H is a closed subgroup of G. Then for any $\delta \in (0, 2\pi/3)$, $\varepsilon > 0$, there is a positive integer $n \in \mathbb{N}$ such that

$$\widehat{G}(H \cap S_n, \Lambda_\delta) \subseteq H' \cdot \widehat{G}(S_1, \Lambda_\varepsilon^\circ),$$

where

$$\widehat{G}(S, A) = \{\varphi \in \widehat{G};\ \varphi[S] \subseteq A\}$$

for $S \subseteq G$, $A \subseteq \mathbb{T}$, and $H' = \widehat{G}(H, 1)$ is the annihilator of H in $\widehat{G}$.

Probably, the proof of Gordon's conjecture would require to add similar uniformity to the general PvK duality, as well.

For any set $X \subseteq \mathbb{R}^q$, let us denote

$$X^+ = \{y \in \mathbb{R}^q;\ (\forall\, x \in X)(xy \in \mathbb{Z})\}$$

its integral annihilator, where $xy = x_1y_1 + \ldots + x_qy_q$ is the usual scalar product in $\mathbb{R}^q$.

If $H \subseteq \mathbb{Z}^q$ is an integral point lattice in $\mathbb{R}^q$, then $\mathbb{Z}^q = (\mathbb{Z}^q)^+ \subseteq H^+$, and the dual group of the quotient $\mathbb{Z}^q/H$ is isomorphic to $H' \cong H^+/\mathbb{Z}^q$.

The *dual* (also called *polar* or *reciprocal*) lattice of a point lattice $H \subseteq \mathbb{R}^q$ is defined by

$$H^\star = H^+ \cap [H].$$

Thus, for a full rank lattice, we have $H^\star = H^+$.

We also denote

$$X^{(\delta)} = \{y \in \mathbb{R}^q;\ (\forall\, x \in X)(\exists\, c \in \mathbb{Z})(|xy - c| \leq \delta)\},$$

for $X \subseteq \mathbb{R}^q$, $\delta > 0$.

With all this in mind, changing the scale from 2π to 1 one can readily translate Theorem 5 into the language of duals of full rank integral point lattices.

Theorem 6. *Let $0 < \delta < \frac{1}{3}$, $\varepsilon > 0$ and $1 \leq q \in \mathbb{N}$. Then for each $n \geq N(\delta, \varepsilon, q)$ and every full rank integral point lattice $H \subseteq \mathbb{Z}^q$ we have*

$$(H \cap nB_1)^{(\delta)} \subseteq H^\star + \varepsilon B_\infty^\circ,$$

i.e., for each $u \in (H \cap nB_1)^{(\delta)}$ there is a $v \in H^\star$ such that $\|u - v\|_\infty < \varepsilon$.

Final remark. It would be interesting if somebody could prove any of the Theorems 4, 5 or 6 in a more constructive way, avoiding the assumption of existence of nontrivial ultrafilters on $\mathbb{N}$, as well as any higher choice related axioms of set theory. E.g., one could try to prove any of them by induction on q.

Furthermore, restricting the values of δ and ε to some sequences of the form $\delta_k = 1/a_k$, $\varepsilon_k = 1/b_k$, where $a_k > 3$, $b_k > 0$ are some fixed strictly increasing primitive recursive sequences of integers, Theorem 6 can relatively easily be re-formulated within the language of Peano arithmetic (PA). Thus it is natural to ask the following question:

Question. *Is (the above modification of) Theorem 6 provable in PA?*

A closely related question can be stated as follows:

Question. *Let $W(k)$ denote the first $n \geq 1$ satisfying the conclusion of Theorem 6 for, say, $\delta = 1/(k+4)$, $\varepsilon = 1/(k+1)$, $q = k+1$. Is the function $W\colon \mathbb{N} \to \mathbb{N}$ (primitive) recursive?*