Non-Standard Approach to J.F. Colombeau's Non-Linear Theory of Generalized Functions and a Soliton-Like Solution of Hopf's Equation

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Abstract

Let $\mathcal{T}$ stand for the usual topology on $\mathbb{R}^d$. J.F. Colombeau's non-linear theory of generalized functions is based on varieties of families of differential commutative rings $\mathcal{G} \stackrel{\text{def}}{=} \{\mathcal{G}(\Omega)\}_{\Omega\in\mathcal{T}}$ such that: 1) Each $\mathcal{G}$ is a **sheaf** of differential rings (consequently, each $f \in \mathcal{G}(\Omega)$ has a **support** which is a closed set of Ω). 2) Each $\mathcal{G}(\Omega)$ is supplied with a chain of sheaf-preserving embeddings $\mathcal{C}^\infty(\Omega) \subset \mathcal{D}'(\Omega) \subset \mathcal{G}(\Omega)$, where $\mathcal{C}^\infty(\Omega)$ is a **differential subring** of $\mathcal{G}(\Omega)$ and the space of L. Schwartz's distributions $\mathcal{D}'(\Omega)$ is a **differential linear subspace** of $\mathcal{G}(\Omega)$. 3) The ring of the scalars $\widetilde{\mathbb{C}}$ of the family $\mathcal{G}$ (defined as the set of the functions in $\mathcal{G}(\mathbb{R}^d)$ with zero gradient) is a non-Archimedean ring with zero devisors containing a copy of the complex numbers $\mathbb{C}$. Colombeau theory has numerous applications to ordinary and partial differential equations, fluid mechanics, elasticity theory, quantum field theory and more recently to general relativity. The main purpose of our **non-standard version of Colombeau' theory** is the improvement of the scalars: in our approach the set of scalars is always an algebraically closed non-Archimedean Cantor complete field. This leads to other improvements and simplifications such as reducing the number of quantifiers and possibilities for an axiomatization of the theory. As an application we shall prove the existence of a weak soliton-like solution of Hopf's equation improving a similar result, due to M. Radyna, obtained in the framework of V. Maslov's theory.

MSC: Functional Analysis (46F30); Generalized Solutions of PDE (35D05).

1 J. F. Colombeau's Non-Linear Theory of Generalized Functions

Let $\mathcal{T}$ stand for the usual topology on $\mathbb{R}^d$. **J.F. Colombeau's non-linear theory of generalized functions is based on varieties of families of differential commutative rings:**

$$\mathcal{G} \stackrel{\text{def}}{=} \{\mathcal{G}(\Omega)\}_{\Omega \in \mathcal{T}},$$

such that:

1. Each $\mathcal{G}$ is a **sheaf of differential rings** (consequently, each $f \in \mathcal{G}(\Omega)$ has a **support** which is a closed set of Ω).

2. The **ring of the scalars** of the family $\mathcal{G}$

   $$\widetilde{\mathbb{C}} = \{f \in \mathcal{G}(\mathbb{R}^d) \mid \nabla f = 0 \text{ on } \mathbb{R}^d\},$$

   is a non-Archimedean **ring with zero devisors** containing a copy of the complex numbers $\mathbb{C}$.

3. Each $\mathcal{G}(\Omega)$ is supplied with a chain of **sheaf-preserving embeddings**

   $$\mathcal{E}(\Omega) \hookrightarrow \mathcal{D}'(\Omega) \hookrightarrow \mathcal{G}(\Omega),$$

   where $\mathcal{E}(\Omega) \stackrel{\text{def}}{=} \mathcal{C}^\infty(\Omega)$ is a **differential subring** of $\mathcal{G}(\Omega)$ and the space of L. Schwartz's distributions $\mathcal{D}'(\Omega)$ is a **differential linear subspace** of $\mathcal{G}(\Omega)$.

4. Colombeau's theory has numerous **applications to PDE, elasticity theory, quantum field theory and more recently to general relativity**.

5. **The main purpose** of our non-standard version of Colombeau' theory is the **improvement of the scalars:** in our approach the set of scalars is always an **algebraically closed non-archimedean Cantor complete fields**.

6. The improvement of the properties of the scalars leads to **other simplifications** improvements such as reducing the number of quantifiers and possibilities for an axiomatization of the theory.

Remark 1.1 (A Non-Standard Sheaf) *The collection*

$$\{{}^*\mathcal{E}(\Omega)\}_{\Omega\in{}^*\mathcal{T}},$$

is a sheaf of differential rings on ${}^*\mathbb{R}^d$*, but*

$$\{{}^*\mathcal{E}(\Omega)\}_{\Omega\in\mathcal{T}},$$

is **not a sheaf on** $\mathbb{R}^d$ *!!!!!!!!*

Example 1.1 (A Counter Example) *Let* $\varphi \neq 0$ *and* $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$*.*

$$f(x) = {}^*\varphi(x-\nu).$$

However,

$$\bigcup_{n\in\mathbb{N}}(0,n) = \mathbb{R}_+,$$

and $f \upharpoonright (0,n) = f|{}^*(0,n) = 0$ *for all* n*. Yet,*

$$f \upharpoonright \mathbb{R}_+ = f|{}^*\mathbb{R}_+ = f \neq 0.$$

2 Non-Archimedean Hulls

In what follows ${}^*\mathbb{C}$ stands for a **non-standard extension of the field of the complex numbers** $\mathbb{C}$. Here is the **summary** of our non-archimedean hull theory:

1. Let $\mathcal{F}$ be a **convex subring** in ${}^*\mathbb{C}$, i.e. $\mathcal{F}$ is a subring of ${}^*\mathbb{C}$ such that

   $$(\forall x \in {}^*\mathbb{C})(\forall y \in \mathcal{F})(|x| \leq |y| \Rightarrow x \in \mathcal{F}).$$

   We denote by $\mathcal{F}_0$ the set of all **non-invertible elements** of $\mathcal{F}$, i.e.

   $$\mathcal{F}_0 = \{x \in \mathcal{F} \mid x = 0 \vee 1/x \notin \mathcal{F}\}.$$

2. We denote by

   $$\widehat{\mathcal{F}} = \mathcal{F}/\mathcal{F}_0,$$

   the corresponding **factor ring** and by $q : \mathcal{F} \to \widehat{\mathcal{F}}$ the corresponding **quotient mapping**.

   If $x \in \mathcal{F}$, we write $\widehat{x} \in \widehat{\mathcal{F}}$ instead of $q(x)$.

   We say that $\widehat{\mathcal{F}}$ is a **non-Archimedean hull** whenever $\widehat{\mathcal{F}}$ is a non-Archimedean field.

3. We $\mathbb{C} \subseteq \widehat{\mathcal{F}}$ by letting $c = \widehat{c}$ for all $c \in \mathbb{C}$.

4. Let $\mathcal{F}^d = \mathcal{F} \times \mathcal{F} \times \cdots \mathcal{F}$ and $\widehat{\mathcal{F}}^d = \widehat{\mathcal{F}} \times \widehat{\mathcal{F}} \times \cdots \widehat{\mathcal{F}}$ (d times). If $x = (x_1, x_2, \cdots, x_d) \in \mathcal{F}$, we shall write $\widehat{x} = (\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_d) \in \widehat{\mathcal{F}}$. We denote by $|| \cdot ||$ the usual Euclidean norm in either $\mathcal{F}^d$ or $\widehat{\mathcal{F}}^d$. If $X \subseteq \mathbb{R}^d$, the set

   $$\mu_{\mathcal{F}}(X) = \{x + dx \mid x \in X, dx \in \widehat{\mathcal{F}}^d,\ ||dx|| \approx 0\},$$

   is the **monad** of X in $\widehat{\mathcal{F}}^d$.

5. We define the ring of $\mathcal{F}$**-moderate functions** and the ideal of the $\mathcal{F}$**-negligible functions** in ${}^*\mathcal{E}(\Omega)$ by

$$\mathcal{M}_{\mathcal{F}}(\Omega) = \{f \in {}^*\mathcal{E}(\mathbb{R}^d) \mid (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega)(\partial^\alpha f(x) \in \mathcal{F})\},$$
$$\mathcal{N}_{\mathcal{F}}(\Omega) = \{f \in {}^*\mathcal{E}(\mathbb{R}^d) \mid (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega)(\partial^\alpha f(x) \in \mathcal{F}_0)\},$$

respectively, and we define also the **factor ring:**

$$\widehat{\mathcal{E}}_{\mathcal{F}}(\Omega) = \mathcal{M}_{\mathcal{F}}(\Omega)/\mathcal{N}_{\mathcal{F}}(\Omega).$$

We say that $\widehat{\mathcal{E}}_{\mathcal{F}}(\Omega)$ is a **differential ring generated by** $\mathcal{F}$.

If $f \in \mathcal{M}_{\mathcal{F}}(\Omega)$, then we denote by $\widehat{f} \in \widehat{\mathcal{E}}_{\mathcal{F}}(\Omega)$ the corresponding equivalence class.

Summarizing: For every convex subring $\mathcal{F}$ of ${}^*\mathbb{C}$ there is a unique differential ring of generalized functions:

$$\mathcal{F} \to \widehat{\mathcal{E}}_{\mathcal{F}}(\Omega).$$

6. We define the embedding

$$\mathcal{E}(\Omega) \hookrightarrow \widehat{\mathcal{E}_{\mathcal{F}}}(\Omega),$$

by $f \to {}^*f$, where ${}^*f$ is the **non-standard extension** of f.

7. Let $\Omega, \mathcal{O} \in \mathcal{T}$ be two open sets of $\mathbb{R}^d$ such that $\mathcal{O} \subseteq \Omega$. Let $\widehat{f} \in \widehat{\mathcal{E}}_{\mathcal{F}}(\Omega)$. We define a **restriction** of $\widehat{f}$ on $\mathcal{O}$ by the formula

$$\widehat{f} \restriction \mathcal{O} = \widehat{f|{}^*\mathcal{O}},$$

where ${}^*\mathcal{O}$ is the non-standard extension of $\mathcal{O}$ and $f|{}^*\mathcal{O}$ is the pointwise restriction of f on ${}^*\mathcal{O}$.

8. Let $\widehat{f} \in \widehat{\mathcal{E}_{\mathcal{F}}}(\Omega)$ and $\widehat{x} \in \mu_{\mathcal{F}}(\Omega)$. We define the **value of $\widehat{f}$ at $\widehat{x}$** by the formula

$$\widehat{f}(\widehat{x}) = \widehat{f(x)}.$$

We shall use the same notation, $\widehat{f}$, for the corresponding value-mapping $\widehat{f} : \mu_{\mathcal{F}}(\Omega) \to \widehat{\mathcal{F}}$.

9. **Simplified Notation:** We shall sometimes **drop $\mathcal{F}$, as a lower-index**, in $\mathcal{M}_{\mathcal{F}}(\Omega)$, $\mathcal{N}_{\mathcal{F}}(\Omega)$, $\widehat{\mathcal{E}}_{\mathcal{F}}(\Omega)$, $\mu_{\mathcal{F}}(\Omega)$, etc. and write simply

$$\mathcal{M}(\Omega),\ \mathcal{N}(\Omega),\ \widehat{\mathcal{E}}(\Omega),\ \mu(\Omega), \ldots,$$

when no confusion could arise.

Theorem 2.1 (Some Basic Results) *Let $\mathcal{F} \subseteq {}^*\mathbb{C}$ be a convex subring of ${}^*\mathbb{C}$. Then:*

1. *$\mathcal{F}_0$ is a convex maximal ideal in $\mathcal{F}$.*
2. *$\widehat{\mathcal{F}}$ is an* **algebraically closed field**. *Consequently, $\{\pm|x| : x \in \widehat{\mathcal{F}}\}$ is a* **real closed field**.
3. *$\mathcal{M}_{\mathcal{F}}(\Omega)$ is a differential subring of ${}^*\mathcal{E}(\Omega)$ and $\mathcal{N}_{\mathcal{F}}(\Omega)$ is a differential ideal in $\mathcal{M}_{\mathcal{F}}(\Omega)$.*
4. *Let $\mathcal{T}$ stands for the usual topology on $\mathbb{R}^d$. Then the collection*
$$\widehat{\mathcal{E}}_{\mathcal{F}} \stackrel{def}{=} \{\widehat{\mathcal{E}}_{\mathcal{F}}(\Omega)\}_{\Omega\in\mathcal{T}},$$
is a **sheaf of differential rings** *in the sense that:*
$$(\forall \Omega, \mathcal{O} \in \mathcal{T}) \left(F \in \widehat{\mathcal{E}}_{\mathcal{F}}(\Omega) \text{ and } \mathcal{O} \subseteq \Omega \text{ implies } F \restriction \mathcal{O} \in \widehat{\mathcal{E}}_{\mathcal{F}}(\mathcal{O})\right).$$
Consequently, **every** $F \in \widehat{\mathcal{E}}_{\mathcal{F}}(\Omega)$ **has a support** $\operatorname{supp}(F)$ *which is* **closed set of** Ω **(not of** ${}^*\Omega$ **!!!!!).**

5. *Each $\widehat{\mathcal{E}}_{\mathcal{F}}(\Omega)$ is a* **differential ring of generalized functions with values in** $\widehat{\mathcal{F}}$, *i.e.*
$$(\forall F \in \widehat{\mathcal{E}}_{\mathcal{F}}(\Omega))(\forall x \in \mu_{\mathcal{F}}(\Omega)) \left[F(x) \in \widehat{\mathcal{F}}\right].$$

6. *The* **ring of scalars** *of the sheaf* $\widehat{\mathcal{E}}_{\mathcal{F}} \overset{def}{=} \{\widehat{\mathcal{E}}_{\mathcal{F}}(\Omega)\}_{\Omega \in \mathcal{T}}$ **coincides with the field** $\widehat{\mathcal{F}}$, *i.e.*

$$\{F \in \widehat{\mathcal{E}}_{\mathcal{F}}(\mathbb{R}^d) \mid \nabla F = 0 \textit{ on } \mathbb{R}^d\} = \widehat{\mathcal{F}}.$$

Consequently, each $\widehat{\mathcal{E}}_{\mathcal{F}}(\Omega)$ *is a* **differential algebra over the field** $\widehat{\mathcal{F}}$.

7. $\mathcal{E}(\Omega)$ *is a* **differential subalgebra of** $\widehat{\mathcal{E}}_{\mathcal{F}}(\Omega)$ *over* $\mathbb{C}$ *under the embedding* $f \to {}^*f$. *We shall often write this as an inclusion*

$$\mathcal{E}(\Omega) \hookrightarrow \widehat{\mathcal{E}}_{\mathcal{F}}(\Omega).$$

3 How We Justify Our Hull Construction

We have to prove the following: Let $\mathcal{F}$ be a convex subring of ${}^*\mathbb{C}$. Then

1. $\mathbb{C} \subset \mathcal{F}({}^*\mathbb{C}) \subseteq \mathcal{F} \subseteq {}^*\mathbb{C}$.

2. *There exists* **maximal fields** $\mathbb{M} \subset \mathcal{F}$ (Zorn Lemma).

3. Every maximal field $\mathbb{M}$ is an **algebraically closed field**.

4. Let $\mathbb{M}$ be a maximal field. We have the **following characterization** of $\mathcal{F}$ and $\mathcal{F}_0$ (see the beginning of this section):

$$\mathcal{F} = \{x \in \mathcal{F} \mid (\exists \varepsilon \in \mathbb{M}_+)(|x| \leq \varepsilon\}, \tag{1}$$
$$\mathcal{F}_0 = \{x \in \mathcal{F} \mid (\forall \varepsilon \in \mathbb{M}_+)(|x| < \varepsilon\}. \tag{2}$$

   Consequently, $\mathcal{F}_0$ **is a convex maximal ideal in** $\mathcal{F}$ and the factor ring $\widehat{\mathcal{F}} = \mathcal{F}/\mathcal{F}_0$ **is a field**.

5. The fields $\mathbb{M}$, $\widehat{\mathbb{M}}$ and $\widehat{\mathcal{F}}$ are mutually isomorphic.

6. There exists an embedding $\widehat{\mathcal{F}} \subseteq {}^*\mathbb{C}$ and a **quasi-standard part mapping**

$$\widehat{\mathrm{st}} : \mathcal{F} \to {}^*\mathbb{C}$$

   with range $\widehat{\mathrm{st}}[\mathcal{F}] = \widehat{\mathcal{F}}$.

SEVERAL EXAMPLES:

Example 3.1 (Nothing New) *Let* $\mathcal{F} = \mathcal{F}({}^*\mathbb{C})$*. In this case*

$\mathcal{F}_0 = \mathcal{I}({}^*\mathbb{C})$,
$\widehat{\mathcal{F}} = \mathbb{C}$,
$\mathcal{M}_{\mathcal{F}}(\Omega) = \{f \in {}^*\mathcal{E}(\Omega) \mid (\forall\alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega)(\partial^\alpha f(x) \in \mathcal{F}({}^*\mathbb{C}))\}$,
$\mathcal{N}_{\mathcal{F}}(\Omega) = \{f \in {}^*\mathcal{E}(\Omega) \mid (\forall\alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega)(\partial^\alpha f(x) \in \mathcal{I}({}^*\mathbb{C}))\}$.

Consequently, the corresponding hull coincides with the **familiar algebra of smooth functions:**

$$\widehat{\mathcal{E}}_{\mathcal{F}}(\Omega) = \mathcal{E}(\Omega).$$

The **quasi-standard part mapping** $\widehat{\mathrm{st}}$ **coincides with the usual standard part mapping** st.

Example 3.2 (Asymptotic Functions) *Let ρ be a positive infinitesimal in ${}^*\mathbb{R}$ and let*

$$\mathcal{F} = \mathcal{M}_\rho({}^*\mathbb{C}) = \{x \in {}^*\mathbb{C} : \ |x| \leq \rho^{-n} \ \textit{ for some } n \in \mathbb{N}\},$$

is the ring of the ρ**-moderate numbers** *in ${}^*\mathbb{C}$. In this case we have:*

$\mathcal{F}_0 = \mathcal{N}_\rho({}^*\mathbb{C}) = \{x \in {}^*\mathbb{C} : \ |x| \leq \rho^{n} \ \textit{ for all } n \in \mathbb{N}\},$
${}^\rho\mathbb{C} = \mathcal{M}_\rho({}^*\mathbb{C})/\mathcal{N}_\rho({}^*\mathbb{C}),$ **(A. Robinson's asymptotic numbers)**
$\mathcal{M}_\mathcal{F}(\Omega) = \mathcal{M}_\rho({}^*\mathcal{E}(\Omega)),$
$\mathcal{N}_\mathcal{F}(\Omega) = \mathcal{N}_\rho({}^*\mathcal{E}(\Omega)),$

where

$$\mathcal{M}_\rho({}^*\mathcal{E}(\Omega)) = \left\{f \in {}^*\mathcal{E}(\Omega) \mid (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega))\, [\partial^\alpha f(x) \in \mathcal{M}_\rho({}^*\mathbb{C})]\right\},$$
$$\mathcal{N}_\rho({}^*\mathcal{E}(\Omega)) = \left\{f \in {}^*\mathcal{E}(\Omega) \mid (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega))\, [\partial^\alpha f(x) \in \mathcal{N}_\rho({}^*\mathbb{C})]\right\}.$$

The corresponding factor ring

$${}^\rho\mathcal{E}(\Omega) = \mathcal{M}_\mathcal{F}(\Omega)/\mathcal{N}_\mathcal{F}(\Omega),$$

is **a differential algebra over the field of asymptotic numbers** ${}^\rho\mathbb{C}$*.*

1. *The field of* **real asymptotic numbers** ${}^\rho\mathbb{R}$ *was introduce by A. Robinson [74] (see also A. Robonson and A.H. Lightstone [56])*

2. *The functions in ${}^\rho\mathcal{E}(\Omega)$ are called* **asymptotic functions** *(M. Oberguggenberger and T. Todorov [66]). Here you will find the following result:*

Theorem 3.1 (Embedding of Schwartz Distributions in ${}^{\rho}\mathcal{E}(\Omega)$) *There exists an embedding $\Sigma_\Omega : \mathcal{D}'(\Omega) \to {}^{\rho}\mathcal{E}(\Omega)$ which preserves all linear operations in $\mathcal{D}'(\Omega)$ and the multiplication in $\mathcal{E}(\Omega) = \mathcal{C}^\infty(\Omega)$, in symbol,*

$$\mathcal{E}(\Omega) \hookrightarrow \mathcal{D}'(\Omega) \hookrightarrow {}^{\rho}\mathcal{E}(\Omega).$$

Proof: *(M. Oberguggenberger and T. Todorov [66])*

3. *The algebra ${}^{\rho}\mathcal{E}(\Omega)$ is, in a sense, a non-standard counterpart of a* **special Colombeau's algebra** $\mathcal{G}^s(\Omega)$ *(J. F. Colombeau [12]) with the important* **improvement of the properties of the scalars**.

Example 3.3 (Logarithmic Hull) *Let ρ be (as before) a positive infinitesimal in ${}^*\mathbb{R}$ and let*

$$\mathcal{F} = \mathcal{F}_\rho({}^*\mathbb{C}) = \{x \in {}^*\mathbb{C} : \ |x| < 1/\sqrt[n]{\rho} \ \textit{ for all } \ n \in \mathbb{N}\},$$

is the set of the ρ**-finite numbers** *in ${}^*\mathbb{C}$. In this case we have:*

$$\begin{aligned}
&\mathcal{F}_0 = \mathcal{I}_\rho({}^*\mathbb{C}) = \{x \in {}^*\mathbb{C} : \ |x| \le \sqrt[n]{\rho} \ \textit{ for some } \ n \in \mathbb{N}\},\\
&\widehat{{}^\rho\mathbb{C}} = \mathcal{F}_\rho({}^*\mathbb{C})/\mathcal{I}_\rho({}^*\mathbb{C}) \ \ \textbf{\textit{logarithmic field}},\\
&\mathcal{M}_\mathcal{F}(\Omega) = \mathcal{F}_\rho({}^*\mathcal{E}(\Omega)),\\
&\mathcal{N}_\mathcal{F}(\Omega) = \mathcal{I}_\rho({}^*\mathcal{E}(\Omega)),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{F}_\rho({}^*\mathcal{E}(\Omega)) &= \left\{ f \in {}^*\mathcal{E}(\Omega) \mid (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega)) \left[\partial^\alpha f(x) \in \mathcal{F}_\rho({}^*\mathbb{C})\right] \right\},\\
\mathcal{I}_\rho({}^*\mathcal{E}(\Omega)) &= \left\{ f \in {}^*\mathcal{E}(\Omega) \mid (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega)) \left[\partial^\alpha f(x) \in \mathcal{I}_\rho({}^*\mathbb{C})\right] \right\}.
\end{aligned}$$

For the corresponding **algebra of generalized functions**

$$\widehat{\mathcal{E}}_\mathcal{F}(\Omega) = \mathcal{F}_\rho({}^*\mathcal{E}(\Omega))/\mathcal{I}_\rho({}^*\mathcal{E}(\Omega)),$$

is a algebra over the logarithmic field $\widehat{{}^\rho\mathbb{C}}$. It seems that this algebra of generalized functions is **without counterpart in Colombeau's theory**.

Example 3.4 (The case $\mathcal{F} = {}^*\mathbb{C}$) *Let $\mathcal{F} = {}^*\mathbb{C}$. In this* *case*

$\mathcal{F}_0 = \{0\}$,
$\widehat{\mathcal{F}} = {}^*\mathbb{C}$,
$\mathcal{M}_{\mathcal{F}}(\Omega) = {}^*\mathcal{E}(\Omega)$,
$\mathcal{N}_{\mathcal{F}}(\Omega) = \{f \in {}^*\mathcal{E}(\mathbb{R}^d) \mid (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega)(\partial^\alpha f(x) = 0)\}$,
$\widehat{\mathcal{E}}(\Omega) = {}^*\mathcal{E}(\Omega)/\mathcal{N}_{\mathcal{F}}(\Omega)$.

The differential ring $\widehat{\mathcal{E}}(\Omega)$ is an algebra over the field ${}^*\mathbb{C}$. The algebra $\widehat{\mathcal{E}}(\Omega)$ is, in a sense, a **non-standard counterpart of Egorov algebra** *(Yu. V. Egorov [20]-[21]) with (at least) two important improvements:*

(a) *The ring of the scalars ${}^*\mathbb{C}$ of $\widehat{\mathcal{E}}(\Omega)$ constitutes an algebraically closed saturated field. In contrast, the the scalars of Egorov's algebra are a ring with zero divisors.*

(b) *We ave the following result:*

Theorem 3.2 (Embedding of Schwartz Distributions in $\widehat{\mathcal{E}}(\Omega)$) *There exists an embedding $\sigma_\Omega : \mathcal{D}'(\Omega) \to \widehat{\mathcal{E}}(\Omega)$ which preserves all linear operations in $\mathcal{D}'(\Omega)$ and the multiplication in the* **ring of polynomials** $\mathbb{C}[\Omega]$*, in symbol,*

$$\mathbb{C}[\Omega] \hookrightarrow \mathcal{D}'(\Omega) \hookrightarrow \widehat{\mathcal{E}}(\Omega).$$

where $\mathbb{C}[\Omega] \overset{def}{=} \mathbb{C}[x_1, x_2, ..., x_d] \,|\, \Omega$.

Proof: :

1. We construct ${}^*\mathbb{C} = \mathbb{C}^{\mathcal{D}(\mathbb{R}^d)}/\mathcal{U}$ and ${}^*\mathcal{E}(\Omega) = \mathcal{E}(\Omega)^{\mathcal{D}(\mathbb{R}^d)}/\mathcal{U}$, where $\mathcal{U}$ is a c^+-good ultrafilter on the index set $\mathcal{I} = \mathcal{D}(\mathbb{R}^d)$. Here $\mathcal{D}(\mathbb{R}^d) = \mathcal{C}_0^\infty(\mathbb{R}^d)$ and $c = \operatorname{card}(\mathbb{R})$).

2. The choice of the ultrafilter $\mathcal{U}$ is closely connected with Colombeau's theory: Let

   $$\mathcal{D}(\mathbb{R}^d) \overset{\text{def}}{=} \mathcal{B}_0 \supset \mathcal{B}_1 \supset \mathcal{B}_2 \supset \dots,$$

   where

   (3)

   $$\begin{aligned}\mathcal{B}_n = \{\varphi \in \mathcal{D}(\mathbb{R}^d) \; : \; & \\ & \int_{\mathbb{R}^d} \varphi(x)\, dx = 1, \\ & \int_{\mathbb{R}^d} x^\alpha \varphi(x)\, dx = 0 \;\text{ for all }\; \alpha \in \mathbb{N}_0^d,\; 1 \le |\alpha| \le n, \\ & ||x|| \ge 1/n \Rightarrow \varphi(x) = 0, \\ & 1 \le \int_{\mathbb{R}^d} |\varphi(x)|\, dx < 1 + \frac{1}{n}\, \}.\end{aligned}$$

Lemma 3.1 (Properties of $\mathcal{B}_n$) *$\mathcal{B}_n \neq \varnothing$ for all n.*

Proof: (M. Oberguggenberger and T. Todorov [66]).

Lemma 3.2 *There exists a c^+-good ultrafilter $\mathcal{U}$ on $\mathcal{D}(\mathbb{R}^d)$, where $c = \mathrm{card}(\mathbb{R})$, such that $(\forall n \in \mathbb{N})\,(\mathcal{B}_n \subset \mathcal{U})$.*

Proof: (C. C. Chang and H. Jerome Keisler [8]).

Notation:

(a) If $(A_\varphi) \in \mathbb{C}^{\mathcal{D}(\mathbb{R}^d)}$, we denote by $\langle A_\varphi \rangle \in {}^*\mathbb{C}$ the corresponding non-standard number.

(b) Similarly, if $(f_\varphi) \in \mathcal{E}(\Omega)^{\mathcal{D}(\mathbb{R}^d)}$, we denote $\langle f_\varphi \rangle \in {}^*\mathcal{E}(\Omega)$.

(c) If $\langle f_\varphi \rangle \in {}^*\mathcal{E}(\Omega)$, we shall often write $\widehat{f_\varphi} \in \widehat{\mathcal{E}}(\Omega)$ instead of the more precise $\widehat{\langle f_\varphi \rangle} \in \widehat{\mathcal{E}}(\Omega)$.

Example 3.5 (Canonical Infinitesimal) *Define* $(R_\varphi) \in \mathbb{C}^{\mathcal{D}(\mathbb{R}^d)}$ *by*

$$R_\varphi = \begin{cases} \sup\{||x|| \mid x \in \mathbb{R}^d,\ \varphi(x) \neq 0\}, & \varphi \neq 0, \\ 1, & \varphi = 0. \end{cases}$$

The nonstandard number $\rho = \langle R_\varphi \rangle$ *is a* **positive infinitesimal** *in* ${}^*\mathbb{R}$.

Example 3.6 (Non-Standard Delta Function) *Let* $id : \mathcal{D}(\mathbb{R}^d) \to \mathcal{D}(\mathbb{R}^d)$ *be the identity function on* $\mathcal{D}(\mathbb{R}^d)$, *given by* $id(\varphi) = \varphi$. *Notice that* $id \in \mathcal{E}(\Omega)^{\mathcal{D}(\mathbb{R}^d)}$. *let* $\delta \overset{def}{=} \widehat{\varphi} \in \widehat{\mathcal{E}}(\Omega)$. *We shall call the corresponding Here are some of the properties of this function:*

$$\int_{\mathbb{R}^d} \delta(x)\, dx = 1,$$

$$\int_{\mathbb{R}^d} |\delta(x)|\, dx \approx 1,$$

$$\int_{\mathbb{R}^d} \delta(x)\, x^\alpha\, dx = 0,\ |\alpha| \neq 0.$$

Example 3.7 (The Square of δ) *For* $\delta^2 = \widehat{\varphi^2}$ *we have*

(a) $\delta^2(x) = 0$ *for* $x \in {}^*\mathbb{R}^d,\ ||x|| \geq \rho$.

(b) $\int_{\mathbb{R}} \delta^2(x)\, dx = \int_{-\infty}^{\infty} \widehat{\varphi^2(x)}\, dx$ *is infitely large number in* ${}^*\mathbb{R}$.

3. The mapping $T \to \widehat{T \star \varphi}$ from $\mathcal{D}'(\mathbb{R}^d)$ to $\widehat{\mathcal{E}}(\mathbb{R}^d)$ satisfies the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C}[\mathbb{R}^d] & \xrightarrow{P \to T_P} & \mathcal{D}'(\mathbb{R}^d) \\ {\scriptstyle P \to {}^*\widehat{P}} \downarrow & & \downarrow {\scriptstyle T_P \to \widehat{T_P \star \varphi}} \\ \widehat{\mathcal{E}}(\mathbb{R}^d) & \xrightarrow[id]{} & \widehat{\mathcal{E}}(\mathbb{R}^d), \end{array}$$

where id is the identity mapping and

$$\langle T_P, \tau \rangle = \int_{\mathbb{R}^d} P(x)\tau(x)\, dx, \quad \tau \in \mathcal{D}(\mathbb{R}^d),$$
$$T_P \star \varphi = P \star \varphi.$$

We have to show that ${}^*\widehat{P} = \widehat{P \star \varphi}$ in $\widehat{\mathcal{E}}(\mathbb{R}^d)$. The Taylor formula gives $P(x-t) = P(x) + \sum_{|\alpha|=1}^{p} \frac{(-1)^{|\alpha|}\partial^\alpha P(x)}{\alpha!}\, t^\alpha$, where p is the degree of P. It follows

$$(P \star \varphi)(x) = \int_{\mathbb{R}^d} P(x-t)\varphi(t)dt =$$
$$= P(x)\int_{\mathbb{R}^d} \varphi(t)dt + \sum_{|\alpha|=1}^{p} \frac{(-1)^{|\alpha|}\partial^\alpha P(x)}{\alpha!} \int_{\mathbb{R}^d} t^\alpha \varphi(t)dt = P(x),$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and all $x \in \mathbb{R}^d$. Notice that if $\varphi \in \mathcal{B}_n$ for some $n \geq p$, then $\int_{\mathbb{R}^d} \varphi(t)dt = 1$ and $\int_{\mathbb{R}^d} t^\alpha \varphi(t)dt = 0,\ |\alpha| = 1, 2, \ldots, p$. Thus we have

$$\mathcal{B}_n \subseteq \{\varphi \mid P \star \varphi = P\},$$

implying $\{\varphi \mid P \star \varphi = P\} \in \mathcal{U}$, as required.

4. If $\Omega \neq \mathbb{R}^d$, we extend the embedding by

$$T \to (\widehat{{}^*T\Pi_{\Omega,\varphi})} \star \varphi,$$

from $\mathcal{D}'(\Omega)$ to $\widehat{\mathcal{E}}(\Omega)$, where $\Pi_{\Omega,\varphi} \in {}^*\mathcal{D}(\mathbb{R}^d)$ is a **cut-off-function**, i.e.

$$\widehat{\Pi_{\Omega,\varphi}}(x) = 0, \quad \text{for all } x \in \mu(\Omega).$$

4 Example from Masslov Theory: Weak Solution of Hopf's Equation

Theorem 4.1 (M. Radyna [70]) M. Radyna proves the following result: *For every $n \in \mathbb{N}$ there exists a function $\Theta_n \in \mathcal{S}(\mathbb{R})$ such that the function*

$$u(x, t, \varepsilon, n) = u_0 + \frac{A}{\epsilon}\, \Theta_n \left(\frac{x - vt}{\epsilon} \right),$$

satisfies:

$$\left| \int_{\mathbb{R}} [u_t(x, t, \varepsilon, n) + u(x, t, \varepsilon, n) u_x(x, t, \varepsilon, n)] \tau(x)\, dx \right| < \epsilon^n,$$

for **every test function** $\tau \in \mathcal{D}(\mathbb{R})$**, every** $t \in \mathbb{R}$ **and all sufficiently small** $\epsilon \in \mathbb{R}$.

We say that the family $u(x, t, \varepsilon, n)$ is a **weak solution of order** n **to Hopf's equation:**

$$u_t(x, t) + u(x, t) u_x(x, t) = 0. \tag{4}$$

in the sense of Masslov approach.

In contrast to M. Radyna's result we have the following result:

Theorem 4.2 *Let ρ be a positive infinitesimal in ${}^*\mathbb{R}$ and $A, v, u_0 \in \mathcal{M}_\rho({}^*\mathbb{R})$, $A > 0$. There exists a function $\Theta \in {}^*\mathcal{S}(\mathbb{R})$ (***not depending on** *n), with $\int_{{}^*\mathbb{R}} \Theta(x)dx = 1$, such that the function:*

$$u(x,t) = u_0 + \frac{A}{\rho}\,\Theta\left(\frac{x - vt}{\rho}\right),$$

satisfies:

$$\left|\int_{\mathbb{R}} [u_t(x,t) + u(x,t)u_x(x,t))\tau(x)\,dx\right| < \rho^n,$$

for every test function $\tau \in \mathcal{D}(\mathbb{R})$ and every finite $t \in {}^*\mathbb{R}_+$ and **for all** *$n \in \mathbb{N}$.*

Corollary 4.1 *Let ρ be a positive infinitesimal in ${}^*\mathbb{R}$. Then:*

1. *There exists and an* **asymptotic function**

$$U(x,t) = \widehat{u(x,t)} \in {}^\rho\mathcal{E}(\mathbb{R}^2),$$

which is a **weak solution of Hopf's equation**

$$u_t(x,t) + u(x,t)u_x(x,t) = 0, \tag{5}$$

in the sense that

$$\int_{\mathbb{R}} [U_t(x,t) + U(x,t)U_x(x,t)]\, \tau(x)\, dx = 0,$$

for every test function $\tau \in \mathcal{D}(\mathbb{R})$ and every **finite** $t \in {}^*\mathbb{R}_+$.

2. *We have the* **formula for the amplitude***:*

$$A = \frac{2(\widehat{v} - \widehat{u_0})\widehat{\rho}}{\int_{\mathbb{R}} \widehat{\Theta}^2(y)dy}.$$

   (a) *Infinitesimal amplitude A and finite velocities v, u_0* **(small signals)***;*

   (b) *Finite (or even infinitely large) amplitude A and infinitetely large velocities v, u_0* **(explosion)***.*

3. $U(x,t)$ **obeys the conservation law** *in the sense that for* **all real** $a, b \in \mathbb{R}$*, and* **every finite** $t \in {}^*\mathbb{R}_+$,

$$\frac{d}{dt}\int_a^b U(x,t)dx = \frac{1}{2}[U^2(a,t) - U^2(b,t)], \tag{6}$$

where the equality in (6) is in ${}^\rho\mathbb{C}$.

Proof of the Theorem:

Step 1 (Standard Part of the Proof): Let $f \in \mathcal{S}(\mathbb{R})$ with $\int_{\mathbb{R}} f(x)\, dx = 1$. We replace $u(x,t) = u_0 + \frac{A}{\rho}\, {}^*f\left(\frac{x-vt}{\rho}\right)$ in the integral:

$$\int_{^*\mathbb{R}} [u_t + uu_x]\, {}^*\tau(x)dx$$

$$= \frac{(u_0 - v)A}{\rho^2} \int_{^*\mathbb{R}} {}^*f'\left(\frac{x-vt}{\rho}\right)\, {}^*\tau(x)dx + \frac{A^2}{2\rho^2} \int_{^*\mathbb{R}} \left({}^*f^2\left(\frac{x-vt}{\rho}\right)\right)_x {}^*\tau(x)dx$$

Integrating by parts and making the substitution $y = \frac{x-vt}{\rho}$ gives

$$= \int_{^*\mathbb{R}} \left[(v - u_0)\, A\, {}^*f(y) - \frac{A^2}{2\rho}\, {}^*f^2(y)\right] {}^*\tau'(vt + \rho y)dy =$$

Step 2: By Taylor expansion for $\tau'(vt + \rho y)$, we obtain the **asymptotic expansion in the powers of** ρ:

$$= \sum_{n=0}^{m} \int_{^*\mathbb{R}} \left[(v - u_0)\, A\, {}^*f(y) - \frac{A^2}{2\rho} {}^*f^2(y)\right] y^n\, \frac{{}^*\tau^{(n+1)}(vt)}{n!}\, \rho^n\, dy + R_m(\tau)$$

where the **remainder term** is

$$R_m(\tau) = \rho^{m+1} \int_{^*\mathbb{R}} \left[(v - u_0)\, A\, {}^*f(y) - \frac{A^2}{2\rho}\, {}^*f^2(y)\right] \frac{{}^*\tau^{(m+2)}(\eta(y,t))}{(m+1)!} y^{m+1} dy$$

We **impose the condition on the coefficients to be zero**:

$$\int_{\mathbb{R}} \left[(v - u_0)\, A\, f(y) - \frac{A^2}{2\rho} f^2(y)\right] y^n dy = 0, \qquad 0 \le n \le m,$$

and $|R_m(\tau)| < \rho^{m+k}$ (for some fixed k).

Step 3: When $m = 0$, we have that:

$$A = \frac{2(v - u_0)\rho}{\int_{\mathbb{R}} f^2(y)dy}$$

Replacing this value of A, we have that for every m,

$$\left(\int_{\mathbb{R}} f^2(y)dy\right)\left(\int_{\mathbb{R}} f(y)y^n dy\right) = \int_{\mathbb{R}} f^2(y)y^n dy,\ 0 \leq n \leq m$$

Define

$$S_m = \left\{f \in \mathcal{S}(\mathbb{R})\ :\ \int_{\mathbb{R}} f(x)x^n dx = \frac{\int_{\mathbb{R}} f^2(x)x^n dx}{\int_{\mathbb{R}} f^2(x)dx},\ 0 \leq n \leq m\right\}$$

We have $S_m \neq \varnothing$ for all $m \in \mathbb{N}$, by (M. Radyna [70] p. 275).

This is the end of the "standard part of the proof".

Step 4 (Non-Standard Part of the Proof): We define the **internal sets:**

$$\begin{aligned}\mathcal{A}_m = \{f \in {}^*S_m\ :&|\ln\rho|^{-1}\int_{^*\mathbb{R}} |f(x)x^n| < 1/m\\ &|\ln\rho|^{-1}\int_{^*\mathbb{R}} |f^2(x)x^n|dx < 1/m,\\ &|\ln\rho|^{-1}\,|f^{(n)}(x)| < 1/m,\ x \in {}^*\mathbb{R},\ |x| \leq m,\ 0 \leq n \leq m\}\end{aligned}$$

and observe that $\mathcal{A}_1 \supset \mathcal{A}_2 \supset \mathcal{A}_3 \supset \ldots$. Also $\mathcal{A}_m \neq \varnothing$ for all $m \in \mathbb{N}$. Indeed, $f \in S_m$ **implies** ${}^*f \in \mathcal{A}_m$ **because the integrals of** ${}^*f$ **is a standard (real) number and**

$|\ln\rho|^{-1}$ **is infinitesimal**. Thus there exists

$$\Theta \in \bigcap_{m=1}^{\infty} \mathcal{A}_m,$$

by the **Saruration Principle**. Notice that (due to the logarithmic term $|\ln\rho|^{-1}$)

$$\int_{^*\mathbb{R}} |\Theta(x)x^n| \leq |\ln\rho|,$$
$$\int_{^*\mathbb{R}} |\Theta^2(x)x^n|dx \leq |\ln\rho|,$$

which guarantees the estimation of the residual term $R_m(\tau)$. We have the **formula for the amplitude**:

$$A = \frac{2(v-u_0)\rho}{\int_{\mathbb{R}} \Theta^2(y)dy}.$$

▲

Remark 4.1 *Notice that (due to the logarithmic term $|\ln\rho|^{-1}$)*

$$\Theta \in \mathcal{M}_\rho(^*\mathcal{S}(\mathbb{R})) \subset \mathcal{M}_\rho(^*\mathcal{E}(\mathbb{R})),$$

which is important in the factorization in Corollary !!!

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