A Game on the Universe of Sets

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Working in ZF minus Regularity, we consider a two persons game on the universe of sets. In this game, the players choose in turn an element of a given set, an element of this element, etc.; a player *wins* if its adversary cannot make any following move, i.e. if he could choose the empty set. (The game, but not any our result, can be found in [1], where it is considered in NF. A close game is mentioned in [2].) A set is said to be *winning* if it has a winning strategy for some player. The class W of winning sets admits a natural hierarchy: Let a set be 2γ-*winning* if every its element is $2\delta+1$-winning for some $\delta < \gamma$, and $2\gamma+1$-*winning* if some of its elements is 2γ-winning. Let W_ν be the class of ν-winning sets. Then $W = \bigcup_\nu W_\nu$ and each level $S_\nu = W_\nu - \bigcup_{\mu<\nu} W_\mu$ is nonempty. Let HW be the class of hereditarily winning sets and V_∞ the class of well-founded sets.

Theorem 1. *HW is an inner model and $HW \supseteq V_\infty$. Moreover, each of four possible cases: $V = HW = V_\infty$, $V \neq HW = V_\infty$, $V = HW \neq V_\infty$, and $V \neq HW \neq V_\infty$ is consistent.*

A winning set can be not only non-well-founded but slightly surprisingly without $\in$-minimal elements; the next theorem completely describes such cases.

Theorem 2. *Let A be a class of ordinals. The assertion "S_ν contains sets without $\in$-minimal elements iff $\nu \in A$" is consistent iff either A is empty, or $A = \{\nu > 1 : \nu \text{ is odd}\}$, or else $A = \{\nu > 1 : \nu \text{ is odd or } \nu \geq \mu\}$ for some μ of cofinality $\leq \omega$.*

For consistency results, we propose a new method for getting models with non-well-founded sets (different from the customary method of [2], [3], and [4]; cf. [5]).

In conclusion, we consider the question how long can this game be in general case. Let Pr be a certain natural probability over the class V_ω of hereditarily finite well-founded sets.

Theorem 3.

$$\Pr(S_n \cap V_\omega) = \begin{cases} 1/2 & \text{if } n \in \{1,3\}, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Thus for almost all elements of V_ω the game ends either at 1 or at 3 moves, and so the first player wins almost always.

Both last theorems display a difference between odd- and even-winning sets by showing that the latter are more complicated and more rare objects.

References

[1] Thomas E. Forster. *Set theory with a universal set, exploring an untyped universe*. Oxford Univ. Press, NY, 1995 (2nd ed.). [2] Jon Barwise and Lawrence Moss. *Vicious circles*. CSLI Lecture Notes, 60, Stanford, Calif., 1996. [3] Peter Aczel. *Non-well-founded sets*. CSLI Lecture Notes, 14, Stanford, Calif., 1988. [4] Giovanna d'Agostino. *Modal logic and non-well-founded set theory: translation, bisimulation, interpolation*. ILLC, Diss. Ser., 4 (1998). [5] Denis I. Saveliev. *Representations of classes by sets, reflection principles, and other consequences of axioms concerning well-founded relations*. To appear.