A combinatorial infinitesimal representation of Lévy processes

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Lévy processes: definition

Definition* 1.** *Consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let $d \in \mathbb{N}$. A stochastic process $x : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ is called a* Lévy process *if and only if it is pinned to zero and has* ***stationary and independent increments*, i.e.*

1. *$x_0 = 0$ on Ω,*
2. *$x_t - x_s$ is independent of $\mathcal{F}_s = \sigma\left(x_u \ : \ u \leq s\right)$ for all $t > s$,*
3. *the law of $x_t - x_s$ equals the law of x_{t-s} for all $t \geq s$, and*
4. *$\mathbb{P}$-almost all paths of $(x_t)_{t \in \mathbb{R}_+}$ are right-continuous with left limits (càdlàg).*

Translation-invariant Markovian semigroups

There is a one-to-one correspondence between

- Lévy processes on the space $D[0,+\infty)$ of **càdlàg paths** in $\mathbb{R}^d$
- Markovian semigroups $(p_t)_{t\in\mathbb{R}_+}$ on $\mathbb{R}^d$ that are
  1. **continuous** (i.e. $t \mapsto p_t$ is continuous with respect to the vague topology) and
  2. **(space-)translation invariant** (in the sense that $p_t f(\cdot + z) = p_t f$ for all $z \in \mathbb{R}^d$, $t \geq 0$ and any nonnegative Lebesgue-Borel measurable $f : \mathbb{R}^d \to \mathbb{R}$).

This bijection is given by
$(x_\cdot, (\Omega, \mathcal{A}, \mathbb{P})) \rightarrowtail (\mathbb{P}_{x_t})_{t\geq 0}$,
$\left((p_J)_{J\in\mathbb{R}_+{}^{<\aleph_0}}\right) \leftarrowtail (p_t)_{t\geq 0}$
(Ionescu-Tulcea-Kolmogorov).

Lévy-Khintchine formula

Theorem* 1 *(Lévy-Khintchine formula, cf. e.g. Revuz and Yor [5], Sato [6]). *Consider a Markovian semigroup* $(p_t)_{t\in\mathbb{R}_+}$ *on* $\mathbb{R}^d$, $d \in \mathbb{N}$. $(p_t)_{t\in\mathbb{R}_+}$ *is continuous and translation invariant if and only if the infinitesimal generator* ℓ *of* $(p_t)_{t\in\mathbb{R}_+}$ *can be written as*

$$\ell : f \mapsto \frac{1}{2}\sum_{i,j=1}^{d} {\sigma_{i,j}}^2 \partial_i \partial_j f + \sum_{i=1}^{d} \gamma_i \partial_i f + \int_{\mathbb{R}^d} \left(f(\cdot + y) - f\right) \nu(dy)$$

where $\sigma \in {\mathbb{R}_+}^{d\times d}$ *is a* ***symmetric*** $d \times d$*-**matrix with nonnegative entries***, $\gamma \in \mathbb{R}^d$, *and* ν *is a* ***Radon measure*** *on* $\mathbb{R}^d$ *satisfying*

1. $\nu\{0\} = 0$,
2. $\int_{B_1(0)} |y|^2 \nu(dy) < +\infty$,
3. $\nu\left[\complement B_1(0)\right] < +\infty$,

$\complement B_1(0)$ *denoting the complement of the unit ball in* $\mathbb{R}^d$ *centered at* 0.

Overview of this paper: theory

In this paper we shall, using results of Lindstrøm's [4], construct a **particularly simple internal analogue of** ℓ for a given positive infinitesimal $h > 0$.

This will be an internal operator L such that for all test functions $f \in C_0^\infty(\mathbb{R}^d)$ one has

$$\forall \underline{x} \in {}^*\mathbb{R}^d \forall x \in \mathbb{R}^d \left(\underline{x} \approx x \Rightarrow L^* f\left(\underline{x}\right) \approx \ell f(x)\right)$$

and in addition, the internal translation-invariant Markovian semigroup $P = (P_t)_{t \in h \cdot {}^*\mathbb{N}_0}$ generated by L shall be proven to be the **internal convolution** of

- a weighted multiple of **Anderson's random walk** as well as
- the superposition of hyperfinitely many **independent stochastic jumps**,

corresponding to the diffusion and jump (or: Lévy measure) parts of the initial Lévy process, respectively.

L is said to generate the ***reduced lifting***.

Application: Towards a weak notion of completeness for Lévy markets (1)

Let $d = 1$ and let Λ denote the internal Lévy measure of the reduced lifting of a given Lévy process $x.$, that is: the set of pairs of possible jump sizes and intensities. Suppose, the **Lévy measure** ν **of** $x.$ **is concentrated on** $\mathbb{R}_{>0}$ ("all risks are insured against").

One can show that there is a reduced lifting of $x.$ that, at each time, is the **independent sum** of

- a weighted multiple of **Anderson's random walk** (the lifting of the diffusion part), and
- the superposition of $m \in {}^*\mathbb{N} \setminus \mathbb{N}$ **independent stochastic jumps**, each of which is **greater than** rh, occurring at a probability given by the internal Lévy measure Λ which is derived from ${}^*\nu$.

Application: Towards a weak notion of completeness for Lévy markets (2)

Thus, the Markov kernel P_h generating the internal Markov chain of the reduced lifting of x. can be **decomposed according to**

$$P_h = Q^{(0)} \cdots Q^{(m)}$$

wherein

- $Q^{(0)} : f \mapsto$ $p_0 f\left(\cdot - \sigma h^{\frac{1}{2}}\right) + (1-p_0) f\left(\cdot + \sigma h^{\frac{1}{2}}\right)$ for some $\sigma > 0$, $p_0 \in (0,1)$, and
- for all $i \in \{1, \ldots, m\}$,

  $$Q^{(i)} : f \mapsto (1-p_i) f(\cdot) + p_i f(\cdot + \underline{\alpha}_i),$$

  $\underline{\alpha}_i > rh$, $i \in \{1, \ldots, m\}$, being the **jumps of the reduced lifting**, counted with multiplicity.

This **reduces the 2^{m+1}-nomial market model P_h to a binomial one** (uniquely up to permutations of $\{\underline{\alpha}_i\}_i$).

Notation (1)

Fix an infinite hyperfinite number H, and let $\bar{\mathbb{L}} := \frac{1}{H} \cdot {}^*\mathbb{Z}^d$ be the **lattice of mesh size** $\eta := \frac{1}{H}$. For an arbitrary $N \in {}^*\mathbb{N} \setminus \mathbb{N}$, we set $[-N, N]^d \cap \bar{\mathbb{L}} =: \mathbb{L}$, thereby ensuring that $\mathbb{L}$ is **hyperfinite**.

By $\rho := \rho_{\mathbb{L}}$ we shall denote the $\mathbb{L}$-**rounding operation** $\rho_{\mathbb{L}}$, defined by

$$\rho_{\mathbb{L}} : x \mapsto \left(\sup\left\{y_i \leq x_i \;\; : \;\; y \in \mathbb{L}\right\} \vee -N\right)_{i=1}^{d},$$

Owing to the particular shape of $\mathbb{L}$ as a discrete set, the supremum in each component is even a maximum and $\rho : {}^*\mathbb{R}^d \to \mathbb{L}$.

Notation (2)

Let $0 < h \approx 0$ and define, for $\alpha \in \mathbb{L}$ and $\lambda \in {}^*\mathbb{R}_{>0}$, firstly the **internal infinitesimal generator** for a **hyperfinite Poisson process**

$$L_h^{(\alpha)} : f \mapsto f(\cdot + \alpha) - f$$

and the corresponding **internal Markov kernel**

$$P_h^{(\alpha)} := f + hL_h^{(\alpha)} = f + h\left(f(\cdot + \alpha) - f\right).$$

Varying the intensity, we also set

$$P_h^{(\alpha,\lambda)} := f + h\lambda L_h^{(\alpha)}.$$

These kernels generate **internal Markov chains** via

$$\forall t \in h^*\mathbb{N}_0 \quad P_t^{(\alpha,\lambda)} := \left(P_h^{(\alpha,\lambda)}\right)^{\circ(t/h)}, P_t^{(\alpha)} := P_t^{(\alpha,1)}.$$

The modified **internal infinitesimal generator** for infinitesimal $t \in h^*\mathbb{N}_0$ is defined by

$$L_t^{(\alpha)} : f \mapsto \frac{P_t^{(\alpha)} f - f}{t}.$$

Existence of a twice S-continuous S-infinitesimal generator for superpositions

The **(modified) internal infinitesimal generator has a standard part**:

***Lemma* 1.** *Let $f:\mathbb{R}^d \to \mathbb{R}$ continuous with compact support, and consider a Radon measure ν, whether finite or infinite, on $\mathbb{R}^d$. Then for any two hyperfinite real numbers $y \approx x$ and all $0 \approx t \in h^*\mathbb{N}$, one has:*

$$\underbrace{\int_{\mathbb{L}} L_t^{(\underline{\alpha})} f(y) \left({}^*\nu \circ \rho^{-1}\right)(d\underline{\alpha})}_{=\sum_{\underline{\alpha}\in\mathbb{L}} L_t^{(\underline{\alpha})} f(y) \cdot {}^*\nu[\rho^{-1}\{\underline{\alpha}\}]}$$

$$\approx \underbrace{\int_{\mathbb{L}} L_h^{(\underline{\alpha})} f(x) \left({}^*\nu \circ \rho^{-1}\right)(d\underline{\alpha})}_{=\sum_{\underline{\alpha}\in\mathbb{L}} L_h^{(\underline{\alpha})} f(x) \cdot {}^*\nu[\rho^{-1}\{\underline{\alpha}\}]}.$$

Proof idea. A combination of elementary estimates yields the result for finite ν; the general result will follow by truncation and monotone convergence. $\square$

Notation: composition of hyperfinite kernels

Let AB for two hyperfinite translation-invariant kernels A, B on ${}^*\mathbb{R}^d$ denote the **translation-invariant kernel** obtained by **convolving the two associated measures**: If

$$A : f \mapsto \sum_i p_i f\left(\cdot - \underline{\alpha}_i\right), \quad B : f \mapsto \sum_j q_j f\left(\cdot - \underline{\beta}_j\right),$$

then we define the product of A and B as

$$AB : f \mapsto \sum_{\underline{\gamma} \in \left\{\underline{\alpha}_i\right\}_i \cup \left\{\underline{\beta}_j\right\}_j} \left(\sum_{\substack{i,j \\ \underline{\alpha}_i + \underline{\beta}_j = \gamma}} p_i q_j \right) f\left(\cdot - \underline{\gamma}\right).$$

Then $AB = BA$ is again a hyperfinite translation-invariant kernel and we can **define the product** $\prod_{A \in \mathbb{A}} A$ for a hyperfinite set $\mathbb{A}$ of hyperfinite translation-invariant kernels **recursively** in the internal cardinality of $\mathbb{A}$. Analogously, powers of hyperfinite translation-invariant kernels can be defined.

Superposition of hyperfinite Poisson processes (1)

***Lemma* 2.** *Consider an internal hyperfinite family* $\{x_i\}_{i<M} \subseteq \mathbb{L}$ *of vectors in* ${}^*\mathbb{R}^d$ *and an internal hyperfinite family of positive hyperreal numbers* $(\lambda_i)_{i<M}$ *such that:*

1. $C_0 := \sum_{1\leq|x_i|} \lambda_i$ *as well as* $C_1 := \sum_{|x_j|\leq 1} \lambda_j |x_j|^2$ *are finite;*
2. *Setting* $C_2 := \sum_{|x_i|\leq 1} \lambda_i$ *and* $C_3 := \sum_{|x_j|<1} \lambda_j |x_j|$*, one has*

$$N\sqrt{h} \approx C_3\sqrt{h} \approx C_2 \cdot h \approx 0.$$

(These requirements may be read as ***regularity conditions on the measure*** $A \mapsto \sum_i \lambda_i \chi_A(x_i)$*)*

Define, for $t \in h \cdot {}^*\mathbb{N}$,

$$\forall i < M \quad Q_t^{(i)} := P_t^{(x_i,\lambda_i)}.$$

*Then for all $f \in C^2\left(\mathbb{R}^d, \mathbb{R}\right)$ with a finite $C^2(\mathbb{R}^d)$-norm, **there exists an** $R = R(f) \in {}^*\mathbb{R}$ with $hR \approx 0$ such that for all $k < M$,*

$$\sum_{i \geq k} \left| \frac{\sum_{j=i+1}^{M-1} \left(Q_h^{(j)} - \mathrm{id}\right)}{h} \left(\frac{Q_h^{(i)} - \mathrm{id}}{h}\right) f \right| \leq R.$$

Moreover, this $R(f)$ can be chosen to be a 1-homogeneous function in f by setting

$$\begin{aligned} R(f) := \left(4{C_0}^2 + 4C_0C_2\right) \sup_{\mathbb{R}^d} |f| \\ + \left(NC_3 + C_0C_1 + 4C_2C_1\right) \sup_{\mathbb{R}^d} |f''|. \end{aligned}$$

Proof idea for Lemma 2. Apply the **transfer principle to Taylor's Theorem** for functions in $C^2\left(\mathbb{R}^d, \mathbb{R}\right)$. $\square$

Superposition of hyperfinite Poisson processes (2)

Remark* 1.** *The conditions imposed in assumption (2) of Lemma 2 can be viewed as conditions on the internal measure* Λ *on* $\mathbb{L} = \bar{\mathbb{L}} \cap [-N, N]^d$, *induced by* $(\lambda_i)_i$ *and* $(x_i)_i$ *via* $A \mapsto \sum_{i<M} \lambda_i \chi_A(x_i)$. *They are exactly the* ***regularity properties of Lindstrøm's hyperfinite Lévy measure *(as constructed in the proof in his hyperfinite representation theorem for standard Lévy processes [4, Theorem 9.1]).*

Superposition of hyperfinite Poisson processes and internal Lévy measures

Theorem* 2.** *Under the assumptions and with the notation of the previous Lemma 2: For any* $f \in C^2\left(\mathbb{R}^d, \mathbb{R}\right)$ *with* $\|f\|_{C^2(\mathbb{R}^d)} < +\infty$ *and for all* $y \in {}^*\mathbb{R}$, *the **central approximate identity

$$\frac{\prod_{i<M} Q_h^{(i)} f - f}{h}(y) \approx \sum_{i<M} \frac{Q_h^{(i)} f - f}{h}(y)$$
$$= \int_{\mathbb{L}} \left(f\left(y + \underline{\alpha}\right) - f(y)\right) \Lambda(d\underline{\alpha}),$$

holds, Λ *being the internal measure on* $\mathbb{L}$ *defined by* $\Lambda : P(\mathbb{L}) \to {}^*\mathbb{R}_{\geq 0}$, $A \mapsto \sum_{i<M} \lambda_i \chi_A(x_i)$.

Proof idea. Prove **inductively in** $M \in {}^*\mathbb{N}$ that **there is an** $R > 0$ (the same as in Lemma 2) such that

$$\left| \frac{\prod_{i<M} Q_h^{(i)} f - f}{h} - \sum_{i<M} \frac{Q_h^{(i)} f - f}{h} \right| \leq 3Rh \approx 0.$$

$\square$

Hyperfinite random walks

For the following, let $\Omega := (\Omega, L(\mathcal{A}), L(\mu))$ be a **hyperfinite Loeb probability space** such that $(\Omega, \mathcal{A}, \mu)$ is an internal probability space, let $\mathbb{T} := h^*\mathbb{N} \cap [0,1]$ with $h = \frac{1}{N}$ for some $N \in {}^*\mathbb{N} \setminus \mathbb{N}$, and fix $d \in \mathbb{N}$.

Adopting the terminology of Lindstrøm's [4]:

***Definition* 2.** *[4, Definitions 1.1] Consider an internal stochastic process $X : \Omega \times \mathbb{T} \to {}^*\mathbb{R}^d$. X is called a* hyperfinite random walk with increments from A and transition probabilities $\{p_a\}_{a \in A}$ *if and only if*

1. *$X_0 = 0$ on Ω,*
2. *The **increments** ΔX_t, $t \in \mathbb{T} \cap [0,1)$, defined by $\Delta X_t := X_{t+\frac{1}{N}} - X_t$, form a hyperfinite set of *-**independent** internal random variables, and*
3. *For all $t \in \mathbb{T}$ with $t < 1$ and for all $a \in A$,*

$$\mu \{\Delta X_t = a\} = p_a.$$

Hyperfinite Lévy processes

Let $\mathrm{Fin}\left({}^*\mathbb{R}^d\right)$ denote, as usual, the subset of finite elements of ${}^*\mathbb{R}^d$.

***Definition* 3.** *[4, Definition 1.3]*
$X : \Omega \times \mathbb{T} \to {}^*\mathbb{R}^d$ *is called a* **hyperfinite Lévy** *process if and only if*

1. *X is a hyperfinite random walk and*
2. $L(\mu)\left[\bigcap_{t\in\mathbb{T}\cap[0,1)}\left\{X_t \in \mathrm{Fin}\left({}^*\mathbb{R}^d\right)\right\}\right] = 1$ *(**almost every path remains finite**).*

Hyperfinite Lévy processes: Lindstrøm's criteria

The proof of the subsequently stated Theorem 4 will be based on the following result by Lindstrøm which characterises **hyperfinite Lévy processes** via the **regularity of the associated infinitesimal Markov kernels**:

***Theorem* 3.** *[4, Theorem 4.3] Let X be a hyperfinite random walk with increments from A and transition probabilities $\{p_a\}_{a\in A}$. X is a hyperfinite Lévy process if and only if all of the following conditions are satisfied:*

1. *For all $k \in \mathrm{Fin}\left({}^*\mathbb{R}\right) \setminus \mathrm{st}^{-1}\{0\}$,* $\frac{1}{h}\sum_{|a|\leq k} a p_a \in \mathrm{Fin}\left({}^*\mathbb{R}^d\right)$.
2. *For all $k \in \mathrm{Fin}\left({}^*\mathbb{R}\right)$,* $\frac{1}{h}\sum_{|a|\leq k} |a|^2 p_a \in \mathrm{Fin}\left({}^*\mathbb{R}^d\right)$.
3. $\lim_{k\to\infty} {}^\circ\left(\frac{1}{h}\sum_{|a|>k} p_a\right) = 0$, *i.e. for all $\varepsilon \in \mathbb{R}_{>0}$ there exists some $n_\varepsilon \in \mathbb{N}$ such that for all $k \geq n_\varepsilon$, one has $\frac{1}{h}\sum_{|a|>k} p_a \leq \varepsilon$.*

Hyperfinite Lévy processes as convolutions of discrete jumps (1)

Whether the internal infinitesimal generator comes from a superposition or a Lévy measure: The property of **generating a hyperfinite Lévy process is unaffected**:

***Theorem* 4.** *Under the hypotheses and with the notation of the previous Lemma 2: The internal infinitesimal generator* $f \mapsto \sum_{i<M} \frac{Q_h^{(i)} f - f}{h}$ *generates a hyperfinite Lévy process (rather than a mere hyperfinite random walk) if and only if so does the internal infinitesimal generator* $f \mapsto \frac{\prod_{i<M} Q_h^{(i)} f - f}{h}$.

Let A, B denote the sets of increments and $\{p_a \ : \ a \in A\}$, $\{p'_b \ : \ b \in B\}$ the sets of transition probabilities corresponding to the internal infinitesimal generators $f \mapsto \sum_{i<M} \frac{Q_h^{(i)} f - f}{h}$, $f \mapsto \frac{\prod_{i<M} Q_h^{(i)} f - f}{h}$, respectively.

Hyperfinite Lévy processes as convolutions of discrete jumps (2)

Proof sketch for Theorem 4.

By Theorem 2, for all $f \in C^2(\mathbb{R})$ with finite C^2-norm:

$$\frac{1}{h}\sum_{a\in A} f(a)p_a \approx \frac{1}{h}\sum_{b\in B} f(b)p'_b. \qquad (1)$$

Approximate the "integrands" occurring in Lindstrøm's criteria [4, Theorem 4.3] by test functions f, and apply (1). Then one can first verify

$$(\text{(3) for } A) \Leftrightarrow (\text{(3) for } B)\,.$$

Also, one can prove

$$(\text{(1) for } A, \text{(2) for } A) \Leftrightarrow (\text{(1) for } B, \text{(2) for } B)$$

for $d = 1$, and by considering each half-axis separately also for arbitrary d.

Thanks to Lindstrøm's [4, Theorem 4.3] criteria, the equivalence assertion follows. $\square$

Main result: existence of the reduced lifting

***Theorem* 5.** *Given a positive infinitesimal $h = \frac{1}{M}$ for $M \in {}^*\mathbb{N} \setminus \mathbb{N}_0$ and a hyperfinite lattice $\mathbb{L} = \eta^*\mathbb{Z}^d \cap [-N, N]^d$ of infinitesimal mesh η, any Lévy process $x.$ with infinitesimal generator ℓ is adapted-equivalent to the standard part of a unique hyperfinite Lévy process $X.$ for time mesh h with the following property: The jump part of the internal translation-invariant **Markov chain corresponding to $X.$ is generated by the measure** $\bigstar_{\underline{\alpha} \in \mathbb{L} \setminus \{0\}} \Lambda_{\underline{\alpha}}$ (when viewed as a kernel), which is an internal convolution of measures of the shape*

$$\Lambda_{\underline{\alpha}} = \lambda_{\underline{\alpha}} \delta_{\underline{\alpha}} + \left(1 - \lambda_{\underline{\alpha}}\right) \delta_0$$

for $\underline{\alpha} \in \mathbb{L} \setminus \{0\}$, wherein

$$\forall \underline{\alpha} \in \mathbb{L} \setminus \{0\} \quad \lambda_{\underline{\alpha}} = \left({}^*\nu \circ (\rho_{\mathbb{L}})^{-1}\right) \{\underline{\alpha}\}.$$

The diffusion part will be a weighted multiple of Anderson's random walk, *-independent from the jump part at any time $t \in \mathbb{T}$.

Comments on the main result

***Definition* 4.** *The lifting $X_{\cdot}$ of Theorem 5 shall be called a* reduced lifting *of its standard part.*

The main idea of this paper is to combine Lindstrøm's hyperfinite representation theorem for standard Lévy processes [4, Theorem 9.1] with the previous results to obtain a particularly simple lifting of a given Lévy process.

Remark* 2.** *The hyperfinite measure Λ referred to in Theorem 5* ***corresponds to an internal sum of independent hyperfinite Poisson processes *with jump directions in the hyperfinite lattice $\mathbb{L} \setminus \{0\}$.*

***Remark* 3.** *As one might already see from the statement of the Theorem, the proof of this Theorem exploits the universality of hyperfinite adapted spaces as discovered in the model theory of stochastic processes by Keisler et al. ([2],[3],[1]).*

Proof of the main theorem (1)

The proof of Theorem 5 will make use of Theorems 2 and 4. In addition, the following Lemma which may be interesting in its own right and is an application of Lemma 1:

***Lemma* 3.** *Consider a pure-jump hyperfinite Lévy process $Y.$ with internal infinitesimal generator L_h, and let $y.$ be its standard part. Then $Y.$ right-lifts $y.$ (due to Lindstrøm [4]), and if ℓ denotes the infinitesimal generator of $y.$, then the **pointwise standard part of any L_u for** $h^*\mathbb{N} \ni u \approx 0$ is ℓ: For all smooth functions with compact support f,*

$$\forall \underline{x} \approx x \in \mathbb{R} \forall u \in h^*\mathbb{N} \cap \mathrm{st}^{-1}\{0\}$$
$$L_u f\left(\underline{x}\right) \approx L_h f\left(\underline{x}\right) \approx \ell f(x).$$

The result referred to is

***Theorem* 6.** *[4, Proposition 6.3, Theorem 6.6] The **right standard part** $x.$ of any hyperfinite Lévy process $X.$ exists and is a **Lévy process**.*

Proof of the main theorem (2)

Furthermore, the proof of Theorem 5 employs the following representation of standard Lévy processes that was proven by Lindstrøm (although in the language of Fourier transforms rather than semigroups):

Theorem 7. *[4, Theorem 9.1] Given a drift vector $\gamma \in \mathbb{R}^d$, a covariance matrix $\sigma = (\sigma_{i,j})_{i,j\in\{1,\dots,d\}}$ (that is, a symmetric $d \times d$-matrix with nonnegative entries) and a Borel measure on $\mathbb{R}^d \setminus \{0\}$ such that $\int_{B_1(0)} |y^2|\nu(dy) < +\infty$ and $\nu\left[\complement B_1(0)\right] < +\infty$),* ***there is a hyperfinite Lévy process*** *$X.$* ***with standard part*** *$x.$ such that the infinitesimal generator of the Markovian semigroup corresponding to $x.$ is*

$$\ell : f \mapsto \frac{1}{2}\sum_{i,j=1}^{d} {\sigma_{i,j}}^2 \partial_i \partial_j f + \sum_{i=1}^{d} \gamma_i \partial_i f + \int_{\mathbb{R}^d} \left(f(\cdot + y) - f)\right) \nu(dy).$$

Proof of the main theorem (3)

Proof sketch for Theorem 5. 1. Without loss of generality, consider only pure jump processes $x.$ (cf. the Lévy-Khintchine formula).

2. By **adapted universality**, there is a process $y.$ equivalent to $x.$ on any hyperfinite adapted probability space Ω.

3. Apply **Lindstrøm's representation theorem** for standard Lévy processes to $y.$ to find a hyperfinite (and pure-jump) Lévy process Y with infinitesimal generator $L_h : f \mapsto \sum_{\underline{\alpha} \in \mathbb{L}} \lambda_{\underline{\alpha}} \cdot (f(\cdot + \underline{\alpha}) - f)$.

4. By Theorems 2 and 4, there is a **hyperfinite Lévy process** Z with internal infinitesimal generator

$$K_h : f \mapsto \frac{\prod_{i<M} Q_h^{(i)} f - f}{h} = \frac{\left(\bigstar_{\underline{\alpha} \in \mathbb{L}} \Lambda_{\underline{\alpha}}\right) \star f - f}{h}$$

which has the same standard part (in a pointwise sense) as the internal infinitesimal generator L_h.

5. Therefore, due to Lemma 3, the standard parts $y.$ of $Y.$ and $z.$ of $Z.$ have the **same infinitesimal generator** and thus the same finite-dimensional distributions.

6. Since $y.$ and $z.$ are Markov processes, their adapted equivalence follows from the equality of the finite-dimensional distributions.

$\square$

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