Nonhomogeneous stochastic Navier-Stokes equations

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The stochastic non-homogeneous (i.e. non-constant density) incompressible Navier-Stokes equations with *multiplicative* noise are:

(Velocity) $$\rho du = [\nu \Delta u - < \rho u, \nabla > u - \nabla p + \rho f(t,u)]\, dt + \rho g(t,u) dw_t \tag{1}$$

(Density) $$\frac{\partial \rho}{\partial t} + < u, \nabla > \rho = 0 \tag{2}$$

(Incompressibility) $\mathrm{div}\, u = 0$

(Boundary condition) $u|_{\partial D} = 0$

(Initial conditions) $u|_{t=0} = u_0$ and $\rho|_{t=0} = \rho_0$

These model the velocity u and density ρ of a mixture of viscous incompressible fluids of varying density in a bounded domain $D \subset \mathbb{R}^d$ ($d = 2, 3$). As usual p is the pressure; f represents external forces and the term gdw (where w is a Wiener process) represents additional random forces.

(1) $g = 0$ gives the *deterministic* nonhomogenous equations. Kazhikhov (1974) - assuming $\rho_0 \geq m > 0$ and Simon (1978,1990), Kim (1987) assuming only $\rho_0 \geq 0$. More recently: local existence of strong solutions have been obtained (Boldrini–Medar (2003), Choe, Cho & Kim (2003,2004)

(2) The *stochastic* equations with *additive* noise ($dG = gdw$ does not depend on u) - Yashima (1992) assuming $\rho_0 \geq m > 0$. Solved essentially pathwise.

(3) Here: the *stochastic* equations with general *multiplicative* noise are solved for $d = 2, 3$ assuming $\rho_0 \geq m > 0$.

Techniques: Loeb spaces, hyperfinite dimensional approximation and standardization. (This gives possibly simpler proof for the deterministic equations).

Hilbert space formulation for Navier-Stokes equations

(a) the velocity field

$$u(t, \omega) \in \mathbf{H} \subseteq L^2(D; \mathbb{R}^d)$$

$\mathbf{H}$ is the Hilbert subspace of divergence free vector fields on the physical domain $D \subset \mathbb{R}^d$ ($d = 2$ or 3). D is bounded, open with a sufficiently smooth boundary.

$\mathbf{V} \subset \mathbf{H}$ is the subspace of "differentiable" velocity fields on D.

The self-adjoint extension of the projection of $-\Delta$, denoted by A has an orthonormal basis of eigenfunctions $\{e_k\}_{k\in\mathbb{N}} \subset \mathbf{H}$ with eigenvalues $0 < \lambda_k \nearrow \infty$. For $u \in \mathbf{H}$ write $u = \sum u_k e_k$. Write $\mathbf{H}_n = \text{span}\{e_1, e_2, \ldots, e_n\}$ and Pr_n for the projection onto $\mathbf{H}_n$. Each $u \in \mathbf{H}_n$ is still a velocity field on the whole of D.

(b) the density $\rho(t, \omega)$ is assumed to belong to $L^\infty(D)$.

(c) the "noise" is taken here to mean a Wiener process with values in the space $\mathbf{H}$ (i.e. each value is an entire velocity field).

The chief difficulties with the Navier-Stokes equations stem from the unbounded quadratic term $< \rho u, \nabla > u$ and usually (in physical dimension 3) they can only be solved in a weak sense (one of the Millennium problems: strong existence in dim $d = 3$) even for constant density. For non-constant density (as here) there are additional problems to do with the feedback from the density equation. For this reason an even weaker type of solution is generally sought. **Weak** means in the same sense as for a weak topology: the equations are "tested" against suitable test functions (see below).

Definition of solution

The definition of a weak solution to the stochastic equations is the natural generalization of that used by Kazhikov for the case $g = 0$. Both the velocity and the density will be stochastic processes living on an adapted probability space $\mathbf{\Omega} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$

Definition 1 *Given* $u_0 \in \mathbf{H}$, $\rho_0 \in L^\infty(D)$, $f : [0,T] \times \mathbf{H} \to \mathbf{H}$ *and* $g : [0,T] \times \mathbf{H} \to L(\mathbf{H},\mathbf{H})$ *a pair of stochastic processes* (ρ, u) *is a* weak solution *to the stochastic nonhomogeneous Navier-Stokes equations if*

(i) $u \in L^2([0,T] \times \Omega, \mathbf{V})$ *and for a.a.* ω

$$u(\cdot,\omega) \in L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V})$$

(ii) $\rho \in L^\infty([0,T] \times D \times \Omega)$

*(iii) (***Velocity***) for almost all* $T_0 \le T$, *for all* $\Phi \in C^1(0,T;\mathbf{V})$

$$\begin{aligned}(\rho(T_0)u(T_0), \Phi(T_0)) &- (\rho_0 u_0, \Phi(0)) \\ &= \int_0^{T_0} \Big[(\rho u, \Phi' + \langle u, \nabla\rangle\Phi) - \nu((u,\Phi)) + (\rho f, \Phi)\Big]\,dt + \int_0^{T_0} (\Phi, \rho g)\,dw\end{aligned}$$

*(iv) (***Density***) for all* $\varphi \in C^1(0,T;H^1(D))$, *for all* $T_0 \le T$

$$(\rho(T_0), \varphi(T_0)) - (\rho_0, \varphi(0)) = \int_0^{T_0} (\rho, \varphi' + \langle u, \nabla\rangle\varphi)\,dt$$

(v) $\rho(0) = \rho_0$ *and* $u(0) = u_0$

When $g = 0$ this gives Kazhikhov's original definition of a weak solution for the deterministic equations.

Main Theorem *Suppose that $u_0 \in H$ and $\rho_0 \in L^\infty(D)$ with $0 < m \leq \rho_0(x) \leq M$, and f, g satisfy natural continuity and growth conditions. Then there is a weak solution (ρ, u) to the stochastic nonhomogeneous Navier-Stokes equations with*

$$\mathbb{E}\left(\sup_{t \leq T} |u(t)|^2 + \nu \int_0^T ||u(t)||^2 \, dt\right) < \infty$$

and for almost all ω, for all t

$$m \leq \rho(t,x) \leq M \quad \text{for almost all} \quad x$$

Main idea of the proof

1. Solve a modified hyperfinite dimensional approximation of the equations with velocity field $U(\tau,\omega)$ with values in $\mathbf{H}_N$, using the transfer of finite dimensional SDE theory. This will live on an internal adapted probability space $\mathbf{\Omega}_0 = (\Omega, \mathcal{A}, (\mathcal{A}_\tau)_{\tau\geq 0}, \mathit{\Pi})$ carrying an internal Wiener process $W(\tau,\omega)$ also with values in $\mathbf{H}_N$. The density will take the form $R(\tau,\omega)$ with values in ${}^*C^1(D) \subset {}^*L^\infty(D)$.

2. Prove an "energy estimate" showing that for almost all (τ,ω) the field $U(\tau,\omega)$ is nearstandard.

3. Show that for almost all (τ,ω) the density $R(\tau,\omega)$ is nearstandard

4. Establish appropriate S-continuity in the time variable τ

5. Take standard parts $u({}^\circ\tau,\omega) = {}^\circ U(\tau,\omega)$ and $\rho({}^\circ\tau,\omega) = {}^\circ R(\tau,\omega)$

6. Show that the pair (u,ρ) is a solution to the stochastic nonhomogeneous Navier-Stokes equations on the adapted Loeb space

$$\mathbf{\Omega} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$$

where $P = \mathit{\Pi}_L$, $\mathcal{F} = L(\mathcal{A})$ and $(\mathcal{F}_t)_{t\geq 0}$ is the usual filtration obtained from $(\mathcal{A}_\tau)_{\tau\geq 0}$ in the usual way.

Step 1(a) in the solution is to solve the density equation for a single path of the evolution of the velocity in any of the finite dimensional subspaces $\mathbf{H}_n$:

Lemma 1 *If* $y = (y_t)_{t\in[0,T]} \in C(0,T;\mathbf{H}_n)$ *and* $\rho_0 \in C^1(D)$ *with*

$$0 < m \leq \rho_0(x) \leq M$$

then the equation

$$\frac{\partial \rho}{\partial t}(t,x) + < y(t), \nabla > \rho(t,x) = 0 \qquad (3)$$
$$\rho(0,x)) = \rho_0(x)$$

has a unique solution $\rho(t,x) \in C^1([0,T]\times D)$. *The solution has*

$$0 < m \leq \rho(t,x) \leq M$$

for all (t,x). *The dependence of* ρ *on* y *is continuous; that is, if* $r(y)$ *denotes the solution to the density equation (3), so that*

$$r : C(0,T;\mathbf{H}_n) \to C^1([0,T]\times D)$$

then r *is continuous with respect to the uniform topologies on both sides.*

Hyperfinite approximation of dimension N (infinite).

This is for a pair of internal stochastic processes (R, U) with $R : {}^*[0,T] \times \Omega \to {}^*C^1(D)$ and $U : {}^*[0,T] \times \Omega \to \mathbf{H}_N$ where Ω carries the internal space $\mathbf{\Omega}_0$ with internal Wiener process W: for *a.a. ω

$$
\begin{aligned}
R(\tau)dU(\tau) = [-R(\tau)\langle U(\tau), \nabla\rangle U(\tau) - \nu AU(\tau) + R(\tau)^* f(\tau, U(\tau))]d\tau \\
+ R(\tau)^* g(\tau, U(\tau))dW_\tau \\
\frac{dR}{d\tau} + < U(\tau), \nabla > R(\tau) = 0
\end{aligned}
$$

with prescribed initial conditions $U(0) = U_0 \in \mathbf{H}_N$ and $R(0) = R_0 \in {}^*C^1(D)$. We need to modify these equations to avoid blow up caused by the quadratic term. Fix an infinite number κ and for $V \in \mathbf{H}_N$ define the truncation $\overline{V}$ by

$$
\overline{V} = \begin{cases} V & \text{if } |V| \leq \kappa \\ \kappa V/|V| & \text{if } |V| \geq \kappa \end{cases}
$$

The modified equations are then

$$
\begin{aligned}
R(t)dU(\tau) = [-R(\tau)\langle \overline{U}(\tau), \nabla\rangle U(\tau) - \nu AU(\tau) + R(\tau)^* f(\tau, U(\tau))]d\tau \quad (4) \\
+ R(\tau)^* g(\tau, U(\tau))dW_\tau \\
\frac{dR}{d\tau} + < \overline{U}(\tau), \nabla > R(\tau) = 0 \quad (5)
\end{aligned}
$$

For these we have:

Theorem 1 *If $U_0 \in \mathbf{H}_N$ is finite and $R_0 \in {}^*C^1(D)$ with $0 < m \leq R_0(\xi) \leq M$ then the internal modified equations (4,5) have an internal solution (R, U) with the following properties:*

(a) There is a finite constant E (independent of N) such that

$$\mathbb{E}\left(\sup_{\tau \leq T} |U(\tau)|^2 + \nu \int_0^T ||U(\sigma)||^2 \, d\sigma\right) < E \tag{6}$$

(b) For *a.a. ω, for all τ and ξ

$$m \leq R(\tau, \xi, \omega) \leq M$$

The internal modified hyperfinite dimensional equations are solved by using the function $r(y)$ giving the density for a single velocity path to continuously feedback into the velocity equation, giving a single hyperfinite dimensional past-dependent stochastic equation for the velocity. This can be solved by "standard" techniques.

A solution to the stochastic non-homogeneous Navier-Stokes equations will be obtained by taking standard parts of the internal pair (R, U) solving the modified equations (4,5).

Important observation

It follows from the energy bound (6) that for a.a. ω (with respect to P, the Loeb measure)

$|U(\tau,\omega)|$ is finite and so $\overline{U}(\tau,\omega) = U(\tau,\omega)$ for all τ

and

for *almost all* times τ, $||U(\tau,\omega)||$ is finite.

The importance is that for $U \in \mathbf{H}_N$

- if $|U(\tau)|$ is finite then $U(\tau)$ is weakly nearstandard
- if $||U(\tau)||$ is finite then $U(\tau)$ is strongly nearstandard.

Before we can take standard parts we need two further properties of the evolution of the internal density $(R(\tau), U(\tau)) =$ (density, velocity).

Lemma 2 *For almost all ω the function $R(\tau)U(\tau)$ is weakly S-continuous; that is, if $\sigma, \tau \in {}^*[0, T]$ with $\sigma \approx \tau$ then $R(\sigma)U(\sigma) \approx R(\tau)U(\tau)$ weakly in $\mathbf{H}$.*

This is proved by showing that $R(\tau)U(\tau)$ is the solution of an internal weak stochastic integral, for which the corresponding weak integral is shown to be S-continuous in a conventional way. (We would like to have $U(\tau)$ weakly S-continuous; the weaker condition proved in this Lemma is the reason for the weaker definition of solution.)

The second result is:

Lemma 3 *For almost all ω, whenever $||U(\sigma)||, ||U(\tau)||$ are finite (which happens for almost all times τ, σ) and $\sigma \approx \tau$ then $U(\sigma) \approx U(\tau)$ strongly in $\mathbf{H}$.*

This means that the standard part will be "almost continuous" in the strong topology. The proof is technical, using Lemma 2.

Solving the stochastic non-homogeneous Navier-Stokes equations.

Theorem 2 (Main Existence Theorem) *Suppose that* $u_0 \in \mathbf{H}$ *and* $\rho_0 \in L^\infty(D)$ *with* $0 < m \leq \rho_0(x) \leq M$*, and* f, g *satisfy appropriate growth and continuity conditions. Then there is a weak solution* (ρ, u) *to the stochastic nonhomogeneous Navier-Stokes equations with*

$$\mathbb{E}\left(\sup_{t\leq T} |u(t)|^2 + \nu \int_0^T ||u(t)||^2 \, dt\right) < E$$

and for almost all ω*, for all* t

$$m \leq \rho(t,x) \leq M \quad \text{for almost all} \quad x$$

Proof. (Outline) Take $R_0 \in {}^*C^1(D)$ with $R_0 \approx \rho_0$ in the weak* topology (possible since $C^1(D)$ is dense in $L^1(D)$.
Let $(U(\tau), R(\tau))$ be the solution to the modified hyperfinite dimensional Galerkin equations as above with (4–5) as defined in the previous section, with $U(0) = \mathrm{Pr}\,{}^*u(0)$ and $R(0) = R_0$. For almost all ω we have the conclusions of the previous lemmas.

Definition of u.

$$u(t,\omega) = {}^\circ U(\tau,\omega)$$

for a.a. ω, and for $t \approx \tau$ for which $||U(\tau,\omega)|| < \infty$. Then $U(\tau,\omega)$ is a lifting of $u(t,\omega)$, and $U(\cdot,\omega)$ is an SL^2 lifting for a.a. ω.

Definition of ρ

For a.a. ω, we have $R(\tau,\omega) \in {}^*L^\infty(D)$ and $|R(\tau,\omega)| \leq M$ for all τ so we can take the standard part ${}^\circ R(\tau,\omega)$ (in the weak* topology). The internal density equation for R + the fact that $\sup_{\tau \leq T} |U(\tau,\omega)|$ is finite is used to show that $R(\cdot,\omega)$ is weak* S-continuous for a.a. ω, so we define :

$$\rho(t,\omega) = {}^\circ R(\tau,\omega)$$

for any $\tau \approx t$. Then $\rho \in L^\infty([0,T] \times D \times \Omega)$ and $m \leq \rho(t,x,\omega) \leq M$ for a.a. (t,x,ω).

It is relatively routine to show that the pair (ρ,u) is a solution to the equations. The tools are Loeb-Bochner integration theory for the deterministic terms and the extension of Anderson's stochastic integration theory to the continuous time and infinite dimensional setting. ■

Regularity in dimension 2

In the 2D setting (i.e. a fluid moving in a bounded domain in the plane) there is more regularity to the solution, provided g has a little more regularity.

Theorem 3 *Suppose that* $d = 2$ *and the initial condition* $u_0 \in \mathbf{V}$ *and* (ρ, u) *is the solution to the stochastic non-homogeneous Navier-Stokes equations constructed above. Suppose further that* $g : [0,t] \times \mathbf{V} \to L(\mathbf{H}, \mathbf{V})$ *and* $|g(t,u)|_{\mathbf{H},\mathbf{V}} \leq a(t)(1 + ||u||)$*. Then almost surely:*

(a) $\sup_{t \in [0,T]} ||u(t)|| + \int_0^T |Au(t)|^2 dt < \infty$ *where* $A = -\Delta$*;*

(b) $u(t)$ *is strongly continuous in* $\mathbf{H}$ *and weakly continuous in* $\mathbf{V}$*;*

(c) the equation for $u(t,\omega)$ *holds for* **all** $T_0 \leq T$*.*

The deterministic nonhomogeneous Navier-Stokes equations

Putting $g = 0$ throughout the above proof simplifies and gives a new (simpler?) proof of existence (and regularity if $d = 2$) for the deterministic nonhomogeneous incompressible Navier-Stokes equations.

For the additional regularity when $d = 2$, we can achieve a little more:

Theorem 4 *Suppose that $d = 2$ and the initial condition $u_0 \in \mathbf{V}$ and $(\rho(t), u(t))$ is the solution to the deterministic non-homogeneous Navier-Stokes equations constructed by taking $g = 0$ in the previous theorem. Then*

$$\sup_{t\in[0,T]} ||u(t)|| + \int_0^T |Au(t)|^2 dt + \int_0^T |u_t(t)|^2 \, dt < \infty$$

where u_t denotes the time derivative $\dfrac{du}{dt}$;

Bibliography

S.N.Antonsev, A.V.Kazhikhov and V.N.Monakhov, *Boundary value problems in mechanisms of nonhomogeneous fluids,* Elsevier Science Publishers, Amsterdam, 1990. English translation of *Kraevye Zadachi Mekhaniki Neodnorodnykh Zhidkostei,* Nauka Publishers, Novosibirsk, 1983 (in Russian).

J. L. Boldrini, M. A. Rojas-Medar and E. Fernández-Cara, Semi-Galerkin approximation and strong solutions to the equations of the nonhomogeneous asymmetric fluids, *J. Math. Pures Appl.* **82** (2003), 1499–1525

M. Capiński and N.J. Cutland, A simple proof of existence of weak and statistical solutions of Navier–Stokes equations, *Proc. Royal. Soc. London* A **436** (1992), 1-11.

M. Capiński & N.J. Cutland, Stochastic Navier-Stokes equations, *Applicandae Mathematicae* **25**(1991), 59-85.

M.Capiński & N.J.Cutland, *Nonstandard Methods for Stochastic Fluid Mechanics*, Series on Advances in Mathematics for Applied Sciences, Vol 27, World Scientific, Singapore 1995.

M. Capiński & N.J. Cutland, Existence of global stochastic flow and attractors for Navier–Stokes equations, *Probab. Theory Relat. Fields* **115**(1999), 121-151.

Y. Cho, H. Kim, Unique solvability for the density-dependent Navier–Stokes equations, *Nonlinear Analysis* **59** (2004), 465 – 489

H.J. Choe, H. Kim, Strong solutions of the Navier–Stokes equations for non-homogeneous incompressible fluids, *Comm. Partial Differential Equations* **28** (2003) 1183–1201.

N.J.Cutland, Loeb measure theory, in *Developments in nonstandard mathematics*, Eds. N.J.Cutland, F.Oliveira, V.Neves, J.Sousa-Pinto, Pitman Research Notes in Mathematics Vol. **336**, Longman 1995, pp.151–177.

N.J.Cutland, *Loeb Measures in Practice - Recent Advances*, Springer Lecture Notes in Mathematics **1751**(2000), Springer, Berlin, vii+111 pp.

N.J. Cutland and H.J. Keisler, Global attractors for 3D stochastic Navier-Stokes equations, *J. Dynamics and Diff. Equations* **16**(2004), 205-266.

N.J. Cutland and H.J. Keisler, Attractors and neo-attractors for 3D stochastic Navier-Stokes equations, *Stochastics and Dynamics*,**5**(4) (2005) 487–533.

B.E. Enright, *A Nonstandard Approach to the Stochastic Nonhomogeneous Navier-Stokes Equations*, PhD Thesis, University of Hull, UK, 1999.

A.Ichikawa, Stability of semilinear stochastic evolution equations, *Journal of Mathematical Analysis and Applications* **90** (1982), 12–44.

A.V.Kazhikhov, Solvability of the initial and boundary-value problem for the equations of motion of an inhomogeneous viscous incompressible fluid. *Sov. Phy. Dokl, Vol* 19, No 6 (1974) 331-332. English translation of the paper found in *Dokl. Akad. Nauk SSSR*, 216 No 6 (1974) 1240-1243.

J.U. Kim, Weak solutions of an initial boundary value problem for an incompressible viscous fluid with nonnegative density. *SIAM J. Math. Anal.* **18(**1**)** **(**1987), 89-96.

R. Salvi, The equations of viscous incompressible non-homogeneous fluids: on the existence and regularity, *J. Austral. Math. Soc. Ser. B* **33**(1991), 94–110.

J. Simon, Ecoulement d'um fluide non homogène avec une densité initiale s'annulant, *C. R. Acad. Sci. Paris Ser. A* **15** (1978), 1009–1012.

J. Simon, Nonhomogeneous viscous incompressible fluids: existence of velocity, density, and pressure, *SIAM J. Math. Anal.* **21**(5)(1990),1093-1117.

R.Temam, *Navier-Stokes Equations and Nonlinear Functional Analysis,* SIAM, 1983.

H.F.Yashima, *Equations de Navier-Stokes Stochastiques non homogenes et applications,* Thesis, Scuola Normale Superiore, Pisa (1992).