GENERALIZED FUNCTIONS AND INFINITESIMALS

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It has been widely recognized since half a century that there will never exist a nonlinear theory of generalized functions, in any mathematical context. The aim of this note is to show the converse and to invite the reader to participate to the debate and consequences at an unexpectedly elementary level. The above paradox appears as another instance of the historical controversy on the existence of infinitesimals in mathematics [A,Lu,Me].

1 – Prerequisites.

If f, g are functions of class C^1 on $\mathbb{R}$ let us recall the integration by parts formula

$$\int_a^b f'(x)g(x)dx = -\int_a^b f(x)g'(x)dx + f(b)g(b) - f(a)g(a).$$

If φ is a C^1 function on $\mathbb{R}$ such that φ is null outside a bounded set (“$\varphi(-\infty)=0=\varphi(+\infty)$”) it becomes

$$\int f'(x)\varphi(x)dx = -\int f(x)\varphi'(x)dx.$$

The Sobolev-Schwartz concept of "distribution" [So,Sch1] consists in interpreting f ’ as the linear map

$$\varphi \;\; \alpha \;\; -\int f(x)\varphi'(x)dx$$

which makes sense even if f is not differentiable (provided it permits the integration which is a rather weak property). If Ω is an open set in $\mathbb{R}^N$ we denote by $D'(\Omega)$ the vector space of all distributions on Ω; we do not need to enter into this concept. Elements of $D'(\Omega)$ have many properties of the C^∞ functions on Ω concerning differentiation.

In short: the concept of distributions permits to differentiate freely rather irregular functions (that are not differentiable in the classical sense) at the price that their partial derivatives are objects (distributions) that are not usual functions. The typical example is: let H be the Heaviside function defined by: $H(x) = 0$ if $x < 0$, $H(x) = 1$ if $x > 0$ (and H(0) unspecified). H is not differentiable at $x = 0$ (in the classical sense) because of the discontinuity there. Its derivative (in the sense of distributions) is the “Dirac delta function” δ: $\delta(x)=0$ if $x\neq 0$, $\delta(0)$ “infinite” so that

$$\int_{-\infty}^{+\infty} \delta(x)dx = \int_{-\infty}^{+\infty} H'(x)dx = [H]_{-\infty}^{+\infty} = 1-0 = 1.$$

The above explains calculations in which physicists differentiate functions that, like H, cannot be differentiated in the classical sense. Physicists not only differentiate irregular functions, they also mix differentiation and multiplication by treating formally irregular functions as if they were C^∞ functions. But Schwartz proved in 1954 [Sch1 p10 of the 1966 edition]: “A general multiplication of distributions is impossible in any theory [of generalized functions], possibly different from distribution theory, where there exists a differentiation and a Dirac delta function”. More precisely Schwartz’s theorem states [Sch2]: there does not exist an algebra A such that :

1) the algebra $C^0(\mathbb{R})$ (of all continuous functions on $\mathbb{R}$) is a subalgebra of A and the function $x \;\; \alpha \;\; 1$ is the unit element in A.

2) there exists a linear map $D : A \to A$, ("differentiation") such that

D reduces to the usual differentiation on C^1 functions

$D(uv) = Du.v + u.Dv \quad \forall u,v \in A$

$D \circ D(x \mapsto |x|) \neq 0.$

Notice that $D|x|$ should be -1 for x<0 and +1 for x>0, and thus $\int D \circ D(|x|)dx$ should be equal to 2 ; hence the non existence result claimed above. The above claim means that numerous calculations of physicists are irremediably meaningless from the mathematical viewpoint.

2 - Nonlinear generalized functions.

25 years ago [Co1,2,3] I found a differential algebra $\mathcal{G}(\Omega)$ (i.e. an algebra with internal partial derivatives : $\frac{\partial}{\partial x_i}\mathcal{G}(\Omega) \subset \mathcal{G}(\Omega)$) in the situation

$$C^\infty(\Omega) \subset D'(\Omega) \subset \mathcal{G}(\Omega)$$

in which the inclusion $D'(\Omega) \subset \mathcal{G}(\Omega)$ is canonical (i.e. free from arbitrary choices) and

- the partial derivatives $\frac{\partial}{\partial x_i}$ in $\mathcal{G}(\Omega)$ induce those in $D'(\Omega)$
- the multiplication in $\mathcal{G}(\Omega)$ induces on $C^\infty(\Omega)$ the usual multiplication of C^∞ functions.

There are slightly different variants of $\mathcal{G}(\Omega)$; starting from the classical differential algebra $C^\infty(\Omega)$, they are obtained according to the pattern of the construction of $\mathbb{R}$ from $\mathbb{Q}$ by the method of Cauchy sequences of rational numbers :

$\mathcal{A}$= an algebra of appropriate families $(f_i)_{i\in I}$ of C^∞ functions on Ω ;
$\mathcal{I}$ = an ideal of $\mathcal{A}$ made of those families $(f_i)_{i\in I}$ "close to zero" as "$i \to \infty$" in the index set I;
$\mathcal{G}(\Omega) = \mathcal{A}/\mathcal{I}$ (= quotient of the algebra $\mathcal{A}$ by the ideal $\mathcal{I}$).

The objects in $\mathcal{G}(\Omega)$ can be treated as C^∞ functions on Ω (but not always exactly like C^∞ functions, which explains various inconsistencies encountered by physicists from "formal" calculations).

The above sounds inconsistent with the Schwartz impossibility result: at least one of the assumptions in Schwartz's theorem should not hold. The assumption that does not hold is "$C^0(\Omega)$ is a subalgebra of $\mathcal{G}(\Omega)$" (although $C^\infty(\Omega)$ is a subalgebra of $\mathcal{G}(\Omega)$).

Let f,g be two continuous functions on Ω ; one has two products: the classical one f.g and a new one (in $\mathcal{G}(\Omega)$) denoted by $f \odot g$, which in general are different elements of $\mathcal{G}(\Omega)$: $f.g \neq f \odot g$. But they are not so much different since $\forall \varphi \in C_c^\infty(\Omega)$ (i.e. φ infinitely differentiable with compact support) the integral

$$\left| \int_\Omega (f.g - f \odot g)(x)\ \varphi(x)\, dx \right|$$

(which makes sense naturally in the $\mathcal{G}$ context) is a "generalized real number", nonzero but less than r for any real number r>0: in short it is a nonzero infinitesimal real number: infinitesimal numbers appear here: they were not invited, not welcome but imposed ! This is a strong connection with nonstandard analysis in a broad sense.

The $\mathcal{G}$ theory shows a perfect coherence with classical mathematics thanks to these infinitesimals: if in $\mathcal{G}$ you drop new objects such as $\delta^2, \delta^3, \ldots$ and if you identify G_1, $G_2 \in \mathcal{G}(\Omega)$ if $\forall \varphi\ \mathcal{C}_c^\infty(\Omega)$ $\int_\Omega (G_1 - G_2)(x)\,\varphi(x)dx$ is infinitesimal then you obtain $D'(\Omega)$, but you have lost the structure of an algebra ($D'(\Omega)$ is only a vector space). Therefore the Schwartz non-existence result is based on the refusal of infinitesimals.

Let us show why the classical product has to be – infinitesimally-changed.

$\rightarrow$Compute the integral

$$I=\int_{\mathbb{R}} \left(H^2(x)-H(x)\right) H'(x)\ dx$$

which we assume to be issued from a convenient idealization in classical physics, where H denotes the Heaviside step function and H' its derivative (the Dirac delta distribution). H may be considered as an idealization (for the needed sake of simplicity) of a $\mathcal{C}^1$ function with a jump from the value 0 to the value 1 in a very small interval around x = 0. Thus classical calculations are justified:

$$I=\left[\frac{H^3}{3}-\frac{H^2}{2}\right]_{x=-\infty}^{x=+\infty}=\frac{1}{3}-\frac{1}{2}-0=-\frac{1}{6}\ .$$

This suggests that $H^2 \neq H$ (since $I \neq 0$): H^2 and H differ at x = 0, precisely where H' takes an "infinite value", and this undefined form $0\times\infty$ gives here the value $-\frac{1}{6}$ after integration. Therefore the classical formula $H^2=H$ has to be considered as erroneous in a context suitable to compute I. But it holds in the sense that $\forall \varphi \in C_c^\infty(\mathbb{R})$ $\int (H^2(x)-H(x))\,\varphi(x)dx$ is infinitesimal. We note this as a weak equality $H^2 \approx H$. In $\mathcal{G}(\Omega)$ we note $G_1 \approx G_2$ if $\forall \varphi \in \mathcal{C}_c^\infty(\Omega)$ $\int_\Omega (G_1 - G_2)(x)\,\varphi(x)dx$ is infinitesimal. If T_1, T_2 are two distributions on Ω then $T_1 \approx T_2$ in $\mathcal{G}(\Omega)$ implies $T_1 = T_2$ in $D'(\Omega)$. Thus H^2 above (i.e. the square of H in $\mathcal{G}(\Omega)$) is not a distribution although it is infinitely close to H.

For physical applications it will be basic to have in mind that although there is only one Heaviside distribution, there is an infinity of Heaviside like objects in $\mathcal{G}(\mathbb{R})$; all of them are called "Heaviside generalized functions". Of course the same holds for their derivatives: "Dirac delta generalized functions", see [Co7, Bia p150, Co4 p47].

$\rightarrow$The above (i.e. different Heaviside functions) is also very concretely imposed by physics: even at an obvious qualitative level depiction of an elasto-plastic shock wave requires very different Heaviside functions for different physical variables see [Co6, Bia p120, Co4 p106].

$\rightarrow$Here is a simplified version of Schwartz's proof in which we assume that the algebra of step functions is a subalgebra of A (instead of $\mathcal{C}^0(\mathbb{R})$, so as to permit a much simpler proof). In the classical algebra of step functions, therefore in A from our specific assumption:

$$H^2 = H \qquad \text{and} \qquad H^3 = H.$$

By differentiation (we note DH = H'): $2HH' = H'$ and $3H^2H' = H'$.
Since $H^2 = H$ in the algebra A: $3H^2H' = 3HH'$, thus $3HH' = H'$.

Thus we have at the same time that $HH' = \frac{1}{2}H'$ and $HH' = \frac{1}{3}H'$, which implies $H' = 0$. ■

In $\mathcal{G}(\mathbb{R})$, one has $H^2 \approx H$ and $H^3 \approx H$ which gives by differentiation $2HH' \approx H'$ and $3H^2H' \approx H'$ but H^2H' is not weakly equal to HH': the weak equality $\approx$ is not coherent with multiplication, which fortunately stops the above calculation before its end.

3 – <u>A use of infinitesimals in continuum mechanics.</u>

About 1980 there appeared projectiles that destroyed in one shot all existing models of battle tanks, thus implying the need to design new armour. Impacts last only a few microseconds: it is impossible to perform detailed experiments and therefore numerical simulations are indispensable to design the requested new armour. "Trivial discretizations" – such as $f'(x) \# \frac{f(x+h)-f(x)}{h}$ – need to be complemented by "artificial viscosity" which in turn makes the numerical schemes degenerate too quickly. This shows the need to use good quality schemes – here Godunov schemes – for the system of solid mechanics. Godunov schemes are based on explicit (possibly with minor auxiliary numerical calculations) solutions of the Riemann problem; the Riemann problem is the particular case of the Cauchy problem when the initial data $x \alpha\ u_0(x)$ is constant on both sides of a discontinuity (u_0 looks like the Heaviside function H). Godunov's scheme in one space dimension is described in [Co4] ; numerical schemes are given in [LR1] in one and several space dimension.

The equations to solve are the equations of continuum mechanics for solids, see [Co4 p14]. A very simplified model used to expose the method is the system of 3 equations:

$$\begin{aligned}
&\rho_t + (\rho u)_x = 0\\
&(\rho u)_t + (\rho u^2)_x = \tau_x\\
&\tau_t + u\tau_x = u_x
\end{aligned}$$

where $\rho=\rho(x,t)$=volumic mass, $u=u(x,t)$=velocity, $\tau=\tau(x,t)$=stress ; indices t,x mean partial derivatives $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}$ respectively. The first equation is the equation of mass conservation, the second one is the equation of momentum conservation, the third one is a state law of elastic solids in fast deformation (with coefficient chosen equal to 1). There appears a product of the kind $H.\delta$ due to the term $u\tau_x$, when one seeks a solution of the Riemann problem. Indeed $H.\delta$ does not make sense within the distributions, so one works in the $\mathcal{G}$ setting. In the $\mathcal{G}$ setting one has to consider the infinity of Heaviside like objects to represent a discontinuity of a physical variable w [Co7, Bia p150, Co4 p47]:

$$w(x,t) = w_l + \left(w_r - w_l\right) H_w\ (x-ct)$$

with H_w a Heaviside generalized function. This formula expresses that the physical variable w takes the value w_λ if x<ct and the value w_r if x>ct (= discontinuity travelling at constant speed c). Therefore the product $u\tau_x$ involves the product $H_u.H_\tau'$ that can take very different values depending on H_u and H_τ. In this sense the product $u\tau_x$ is ambiguous in the $\mathcal{G}$ context, in absence of additional information.

<u>First idea</u>: state all 3 equations of the simplified model with = in $\mathcal{G}$ (i.e. the algebraic equality in $\mathcal{G}$). One proves there does not exist a discontinuous solution [Co4 p40, 60]. This is not acceptable !

<u>Second idea</u>: state all 3 equations with the weak equality $\approx$ in $\mathcal{G}$. One proves there are infinitely many different solutions for a given initial condition [Co4 p69-70]; this is not acceptable!

<u>Third idea</u>: choose an intermediate statement on physical ground. Physicists have observed that shock waves have an "infinitesimal width" of the order of magnitude of a few hundred crystalline meshes. One can therefore isolate (by thought) small volumes inside this width where conservation laws apply. This suggests to state them with (algebraic) = in $\mathcal{G}$. On the other hand the state law (3rd equation) has been checked only on a material at rest, i.e. on both sides of the shock wave: by analogy with $H^2 \approx H$ this suggests to state it with the weak equality $\approx$ in $\mathcal{G}$. Thus one states in $\mathcal{G}$ the system of 3 equations as:

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2)_x = \tau_x$$
$$\tau_t + u\tau_x \approx u_x .$$

Then one obtains a well defined (unique for given initial data) solution of the Riemann problems [Co4 p72]. This method has been used in [LR1] and other instances (for contact discontinuities – that are not shock waves – one states the state laws with = in $\mathcal{G}$ since there is no fast deformation)

4 – <u>Infinitesimals in general relativity.</u>

Rotating black holes produce an electro-magnetic field (Reissner-Nordström). For extremely fast rotation (ultrarelativistic limit) one has the rather unexpected result that this electromagnetic field vanishes but its energy-momentum tensor does not. This "absurdity" should be clarified: has it its origin in a mathematical mistake or does it point out a breakdown of physics ? Rigorous calculations in the $\mathcal{G}$ setting ([Ste1,Vi]) show that the field looks like $\sqrt{\delta}$, the square root of a Dirac delta generalized function, while its energy momentum tensor (involving the square of the field) is δ-like. Since $\sqrt{\delta}\approx 0$ the field is infinitesimal but non zero, thus permitting its energy momentum tensor to be non vanishing. Then the physically unsatisfactory situation of a vanishing field with non zero energy momentum tensor is mathematically perfectly clarified: the paradox was due to a lack of rigor in the mathematical calculations.

In suitable coordinates impulsive gravitational waves can be represented as follows [Gr], [St2], [K1], [K2]: the space time is flat except for a hypersurface u=0 where a δ-like impulse modelling a gravitational shock wave is located. They are treated by the "scissors and paste method" of Penrose: space time is divided into two halves by removal of the hypersurface u=0, then the two halves are joined together in a specific way (Penrose junction conditions). The corresponding mathematical calculations involve nonlinear generalized functions (ill defined products of distributions within distribution theory) due to the nonlinearity of Einstein equations and the presence of a Dirac delta function in the space-time metric. It is shown in [Gr], [St2], [K1], [K2] that these calculations represent a well defined diffeomorphism in the $\mathcal{G}$ sense.

Einstein equations are nonlinear and are classically stated in the C^2 setting. But their solutions are very often less regular than C^2 (discontinuous, δ-like,…). Geroch and Traschen [Ge] have proved that distribution theory cannot be used for a description of gravitational sources which are not supported on a space-time submanifold of codimension more than one: physically interesting sources like cosmic strings or point particles are strictly excluded. This fact and the above examples show the need for a Riemannian geometry based on the nonlinear generalized functions [Gr]. It allows one to sort the numerous singularities of solutions to Einstein equations into

- mathematical ones that make sense in the $\mathcal{G}$ setting (therefore where physics is OK) : above examples
- non mathematical ones where a new physics (= quantum gravity) would be needed : center of a black hole, time 0 of the big bang…

In short general relativity appears to make deep use of infinitesimals.

5 – Questions.

The nonlinear theory of generalized functions is inevitably based on infinitesimals, it applies to physics and of course also to mathematics motivated by physics (may be more than 400 articles on applications: Math.Sci.Net, Zbltt, Arxiv). Therefore this theory contributes to the debate on existence of infinitesimals in mathematics [A,Lu,Me]. Furthermore:

1. Should it be considered as some kind of nonstandard analysis? elements of answer are: use of infinitesimals, compatibility of both theories [Hos,T,O p258].
2. What improvements nonlinear generalized functions could bring to nonstandard analysis? may be enlargement of its field of applications.
3. What improvements nonstandard analysis could bring to nonlinear generalized functions? may be tools from mathematical logics, axiom of choice, transfer and saturation principles.

Appendix 1. We give a very elementary definition of $\mathcal{G}(\Omega)$ in a simplified case (no canonical inclusion of $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$). Let $I=]0,1]$ (the value 1 is unimportant, $\varepsilon \in]0,1]$ will be as small as desired)

$\mathcal{A} = \{(u_\varepsilon)_{\varepsilon\in I} \in (C^\infty(\Omega))^I$ such that $\forall K \subset\subset \Omega, \forall \alpha \in IN^n, \exists N \in IN$ with $\sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N})$ as $\varepsilon \to 0\}$ ($K \subset\subset \Omega$ means "K compact subset of Ω", $IN = \{0,1,2,...\}$)

$\mathcal{I} = \{(u_\varepsilon)_{\varepsilon\in I} \in A$ such that $\forall K \subset\subset \Omega$ $\forall q \in IN$ $\sup_{x\in K} |u_\varepsilon(x)| = O(\varepsilon^q)$ as $\varepsilon \to 0\}$

$\mathcal{G}_s(\Omega) = \mathcal{A}/\mathcal{I}$ (the index s stands for "simplified" or "special" in contrast with the "full" algebra $\mathcal{G}(\Omega)$ [Gr]).

From a nice remark due to Grosser [Gr p11] we do not need to introduce ∂^α in the definition of $\mathcal{I}$. The inclusion $C^\infty(\Omega) \subset \mathcal{G}_s(\Omega)$ is obtained by choosing $u_\varepsilon = f$ $\forall\varepsilon$ if $f \in C^\infty(\Omega)$. The inclusion $C^0(\Omega) \subset \mathcal{G}_s(\Omega)$ (non canonical) is more technical since it has to agree with the inclusions $\mathcal{C}^\infty(\Omega)\subset \mathcal{C}^0(\Omega)$ and $\mathcal{C}^\infty(\Omega) \subset \mathcal{G}_s(\Omega)$ see [Gr p12-25], (or elsewhere in the literature). A non-standard version of this definition is given in Hoskins-Sousa Pinto [Hos chap.6].

Appendix 2. In a smooth classical physical situation introduce a plausible irregularity: consider the medium is made of different layers (=use of step functions to model it), or consider a population concentrated at a point (= use of a Dirac delta function to model it). The models often show products of the kind H.δ [Hoe1, Hu ,O p161-164] or δ^2 [O p170-180, Co8]. Anybody can find examples and solve them (no need to enter into general relativity or quantum mechanics), some of them possibly of interest. An immediate example to be reproduced in a number of situations: introduce a weight function $\chi(x)=1+\delta(x-(a+b)/2)$ into the two basic examples (brachistochron and catenoid) in the calculus of variations: minimization of the integrals

$$B=\int_a^b \chi(x)((1+u'(x)^2)/u(x))^{1/2}\,dx \quad , \quad C=\int_a^b \chi(x).u(x).(1+u'(x)^2)^{1/2}\,dx$$

leads to ODEs (Euler-Lagrange) involving products of distributions (the function χ might model a danger or a price mainly concentrated at x= (a+b)/2).

Whether such models deserve a detailed study depends on their use in engineering: §3, hydrodynamics [Ba,Hu,Ber], hurricanes [LR2], earthquakes [Hoe2,3], or in physics: §4, [Gr1,Vi, recent papers in Arxiv by Kunzinger, Steinbauer,Vickers and coauthors] Schwarzschild and Kerr spacetimes, ultrarelastivistic black holes, geodesics in irregular spacetimes, cosmic strings.

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[A] L.Arkeryd. Nonstandard Analysis.Amer.Math.Monthly 112,2005,p926-928

[Ba] R.Baraille-G.Bourdin-F;Dubois-A.Y.LeRoux. Une version a pas fractionnaires …Comptes Rendus Acad.Sci.Paris 314,1,1992,p147-152.

[Be] F.Berger-J.F.Colombeau-M.Moussaoui. Dirac measures solutions… Comptes Rendus Acad.Sci.Paris.316,1,1993, p989-994.

[Bia] H.A. Biagioni. A Nonlinear Theory of Generalized Functions. Springer Lecture Notes in Math 1421 Springer Verlag 1990.

[Co1]J.F. Colombeau. New Generalized Functions and Multiplication of Distributions. North Holland, Amsterdam, 1984.

[Co2]J.F. Colombeau. Elementary Introduction to New Generalized Functions. North Holland, Amsterdam, 1985.

[Co3]J.F. Colombeau. A multiplication of distributions. J.Math. Ana. Appl. 94, p96-115, 1983.

[Co4]J.F. Colombeau. Multiplication of distributions. Springer Lecture Notes in Maths 1532. Springer Verlag 1992.

[Co5]J.F. Colombeau. Multiplication of distributions. Bull. AMS. 23,2, 1990, p251-268.

[Co6]J.F. Colombeau.The elastoplastic shock problem. J. Math Phys 30, 90, 1989, p2273-2279.

[Co7]J.F. Colombeau, AY Le Roux, A Noussair, B. Perrot. Microscopique profiles… SIAM J. Num. Ana. 26,4, 1989, p871-883.

[Co8]J.F.Colombeau, M.Oberguggenberger. Hyperbolic system with …Comm in PDEs 15,7, 1990, p905-938.

[Ge]R. Geroch, J. Traschen. Strings and other ... Phys. Rev D, 36 (4), p1017-1031, 1987.

[Gr]M. Grosser, M. Kunzinger, M. Oberguggenberger, R. Steinbauer. Geometric theory of generalized functions with applications to general relativity. Kluwer, Dordrecht-Boston-New York, 2001.

[Hoe1] G. Hoermann,L.Oparnica. Distributional solutions concepts….Arxiv Math 2006.

[Hoe2] G.Hoermann, M.de Hoop. Geophysical modelling with… .Arxiv Math 2001.
[Hoe3] G.Hoermann , M. de Hoop. Detection of wave front set….Arxiv 2001.
[Hos]R.F. Hoskins, J. Sousa-Pinto. Distributions…Ellis-Horwood, New York-London, 1994.
[Hu] J.Hu. The Riemann problem….Comm.Math. Phys. 194,1, p191-205,1998.
[K1]M. Kunzinger, R Steinbauer. A rigourous solution concept ... J. Math. Phys. 40, p1479-1489, 1999.
[K2]M. Kunzinger, R Steinbauer. A note on the Penrose … Class. Quant. Grav. 16, p1255-1264, 1999.
[LR1]A.Y Le Roux.Phd thesises held at the University of Bordeaux 1 between 1985 and 1992, to be found from the university library: Cauret 1986, Adamczewski 1986, De Luca 1989, Arnaud 1990.Baraille 1992.
[LR2] A.Y.LeRoux,M.N.LeRoux,J.A.Marti . A mathematical model for hurricanes . Comptes Rendus Acad.Sci.Paris 339,1, p313-316,2004.
[Lu] W.A.J.Luxemburg. What is nonstandard Analysis?.Amer.Math.Monthly????p38-67
[Me] F.A.Medvedev. Non-standard Analysis and…Amer .Math.Monthly. 1998, p659-664.
[O]M. Oberguggenberger. Multiplication of distributions… Pitman Research Notes in Math 259, 1992, Longman.
[Sch1]L. Schwartz .Théorie des Distributions, Hermann, Paris, numerous editions.
[Sch2]L. Schwartz.Sur l'impossibilité de la multiplication …, Comptes Rendus Acad. Sci. Paris 239, 1954, p847-848.
[So]S.L. Sobolev. Méthode nouvelle ..., Math. Sbornik 1, 1936, p39-71.
[St1]R. Steinbauer. The ultra relastivistic limit of the Reissner-Nordström field... J. Math. Phys. 38, p1614-1622, 1998.
[St2]R. Steinbauer.Geodesic and geodesic deviation… J. Math. Phys. 39, p2201-2212, 1998.
[T]T.D. Todorov. Colombeau's generalized functions … in Stankovic, Pap, Pilipovic, Vladimirov editors. Generalized Functions, Convergence Structures, and their Applications, Plenum Press, New York 1988, p327-339.
[Vi]J.A. Vickers. Nonlinear generalized functions in general relativity ; in Grosser, Hörman, Kunzinger, Oberguggenberger editors, Nonlinear Theory of Generalized Functions, Chapman and Hall Research Notes in Math Vol 401 p275-290, 1999.