A Functional Characterization of Nonstandard Models - Marco Forti[1]

A main feature of nonstandard models of Analysis is the existence of a canonical extension ${}^*f : {}^*\mathbb{R} \to {}^*\mathbb{R}$ of any function $f : \mathbb{R} \to \mathbb{R}$. The *nonstandard models* preserve those properties of the standard structure which are currently being considered (*Transfer Principle*). Among various "elementary" presentations of the nonstandard methods given in [1], we choose here a "functional approach" aimed to show that *a few clear, natural, purely functional* conditions are all what is needed for the strongest requirements of nonstandard models.

We consider a superset ${}^*X$ of X together with an operator $* : X^X \to {}^*X^{{}^*X}$, which "preserves compositions and diagonal". Assume that $0, 1 \in X$, and let $\chi : X \times X \to \{0, 1\}$ be the characteristic function of the diagonal, so as to have $(\chi \circ (f, g))(x) = 1 \iff f(x) = g(x)$. Call ${}^*X$ a *functional extension* of X if the following conditions are fulfilled for all $\xi \in {}^*X$ and all $f, g : X \to X$:

1. ${}^*g({}^*f(\xi)) = {}^*(g \circ f)(\xi)$,
2. ${}^*(\chi \circ (f, g))(\xi) = \begin{cases} 1 & \textit{if} \quad {}^*f(\xi) = {}^*g(\xi) \\ 0 & \textit{otherwise} \end{cases}$

The main result of [2] isolates a simple necessary and sufficient condition for obtaining a true *nonstandard model* of X, namely

Theorem *Let ${}^*X$ be a functional extension of X. Then ${}^*X$ is isomorphic to a limit ultrapower $X^I/\mathcal{D}|\mathcal{E}$ if and only if ${}^*X$ is accessible, i.e.*

3. *for all $\xi, \eta \in {}^*X$ there are $f, g : X \to X$ and $\zeta \in {}^*X$ s.t. ${}^*f(\zeta) = \xi$, ${}^*g(\zeta) = \eta$.*

Moreover ${}^*X$ is isomorphic to an ultrapower $X^X/\mathcal{U}$ if and only if there exists $\zeta \in {}^*X$ such that any $\xi \in {}^*X$ is equal to ${}^*f(\zeta)$ for suitable $f : X \to X$.

By Keisler's Theorem, ${}^*X$ is a nonstandard extension of X if and only if it is isomorphic to a limit ultrapower of X. So one can extend all n-ary functions and relations and obtain the full Transfer Principle for all first order properties. However we can avoid any appeal to the ultrapower construction. In fact the properties 1-3 alone allow for a *unique, unambiguous, "parametric"* definition of the extension ${}^*\varphi$ of each n-ary function $\varphi : X^n \to X$, namely

$$ {}^*\varphi(\xi_1, \ldots, \xi_n) = {}^*(\varphi \circ (f_1, \ldots, f_n))(\zeta), $$

where $f_i : X \to X$ and $\zeta \in {}^*X$ are such that ${}^*f_i(\zeta) = \xi_i$ for $i = 1, \ldots, n$. After extending n-ary relations by means of the corresponding characteristic functions in n variables, any *accessible functional extension* ${}^*X$ of X becomes a *complete nonstandard model.* The Transfer Principle for all elementary properties can be proved directly by induction on the complexity of the formula expressing the property.

References

[1] V. Benci, M. Di Nasso, M. Forti - The Eightfold Path to Nonstandard Analysis in *Nonstandard Methods and Applications in Mathematics* (N.J. Cutland, M. Di Nasso, D.A. Ross, eds.), L.N. in Logic **25**, A.S.L. 2006.

[2] M. Forti - A functional characterization of complete elementary extensions. (submitted).

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