THE ABSOLUTE ARITHMETIC CONTINUUM AND THE UNIFICATION OF ALL NUMBERS GREAT AND SMALL

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Topic #1: *Nonstandard Theories and Models, and Foundations of Nonstandard Methods.*

In his monograph On Numbers and Games [1976], J. H. Conway introduced a real-closed field containing the reals and the ordinals as well as a vast array of less familiar numbers. Indeed, this particular real-closed field, which Conway calls No, is so remarkably inclusive that, subject to the proviso that numbers–construed here as members of ordered "number" fields–be individually definable in terms of sets of von Neumann-Bernays-Gdel set theory with Global Choice, henceforth NBG, it may be said to contain "All Numbers Great and Small." In this respect, No bears much the same relation to ordered fields that the system of real numbers bears to Archimedean ordered fields. However, in addition to its distinguished structure as an ordered field, No has a rich hierarchical structure that (implicitly) emerges from the recursive clauses in terms of which it is defined. This algebraico-tree-theoretic structure, or simplicity hierarchy, as we have called it [1994], depends upon No's (implicit) structure as a lexicographically ordered binary tree and arises from the fact that the sums and products of any two members of the tree are the simplest possible elements of the tree consistent with No's structure as an ordered group and an ordered field, respectively, it being understood that x is simpler than y just in case x is a predecessor of y in the tree. In a number of earlier works [Ehrlich 1987; 1989; 1992], we suggested that whereas the real number system should merely be regarded as constituting an Archimedean arithmetic continuum, the system of surreal numbers may be regarded as a sort of absolute arithmetic continuum (modulo NBG). In this paper, we will outline some of the properties of the system of surreal numbers that emerged in [Ehrlich 1988; 1992; 1994; 2001] which lend credence to this thesis, and draw attention to some important respects in which the theory of surreal numbers may be regarded as vast generalization of Cantor's theory of ordinals, a generalization which also provides a setting for Abraham Robinson's [1961] infinitesimal approach to analysis as well as for the profound and largely overlooked non-Cantorian theories of the infinite (and infinitesimal) pioneered by Giuseppe Veronese [1891], Tullio Levi-Civita [1892; 1898], David Hilbert [1899] and Hans Hahn [1907] in connection with their work on non-Archimedean ordered algebraic and geometric systems and by Paul du Bois-Reymond [1870-71; 1882], Otto Stolz [1883], G. H. Hardy [1910; 1912] and Felix Hausdorff [1909] in connection with their work on the rate of growth of real functions.

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