NONHOMOGENEOUS STOCHASTIC NAVIER-STOKES EQUATIONS

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Topic #4: *Nonstandard Methods in the study of Navier-Stokes equations and in Mathematical Physics*

[Joint work with Brendan Enright.[1]]

We discuss the solution of the **non-homogeneous** stochastic Navier-Stokes equations:

$$\begin{aligned}\rho\frac{\partial u}{\partial t}+\rho &< u,\nabla > u=\nu\Delta u-\nabla p+\rho f(t,u)+\rho g(t,u)\frac{dw}{dt}\\ \operatorname{div} u &= 0,\ \ u|_{\partial D}=0,\ \ u|_{t=0}=u_0\end{aligned} \tag{1}$$

$$\frac{\partial\rho}{\partial t}+< u,\nabla > \rho=0,\ \ \rho|_{t=0}=\rho_0 \tag{2}$$

for the velocity field of a viscous incompressible fluid in a bounded domain $D\subset\mathbb{R}^d$ ($d=2,3$) with density ρ that is not constant. The term p is the pressure; f represents external forces and the term gdw (where w is a Wiener process) represents additional random forces. Weak solutions for the **deterministic** equations (that is, with $g\equiv 0$) were first found by Kazhikhov [4] (see also [1]) in space dimensions $d=2,3$. A stochastic version with additive noise (that is, with $g\equiv\mathbf{1}$) was solved by Yashima [5].

We will outline the construction of solutions to the above equations with a general external force and **multiplicative** noise, using an extension of the Loeb space methods first used to solve them in the homogeneous (i.e. $\rho=$ constant) case [2, 3]. The role played by nonstandard methods is two-fold.

(1) Finite-dimensional approximations are "easily" solved. So we can take a hyperfinite dimensional approximate solution, and then its standard part is a candidate for a solution. For the deterministic equations this provides a considerable simplification of the basic existence result of [4].

(2) For the stochastic equations, a rich space is needed especially in dimension $d=3$. The hyperfinite dimensional stochastic equations are again "easily" solvable, with an internal adapted space carrying the solution. The standard part of this is carried on the corresponding adapted Loeb space, and it is a solution to the stochastic equations above. These are the first known solutions for the case of multiplicative noise. The solutions display more regularity in the 2D case.

References

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