Convergence of discrete schemes for the Perona-Malik equation

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Abstract

We prove the convergence, up to a subsequence, of the spatial semidiscrete scheme for the one-dimensional Perona-Malik equation $u_t = (\phi'(u_x))_x$, $\phi(p) := \frac{1}{2}\log(1+p^2)$, when the initial datum $\overline{u}$ is 1-Lipschitz out of a finite number of jump points, and we characterize the problem satisfied by the limit solution. In the more difficult case when $\overline{u}$ has a whole interval where $\phi''(\overline{u}_x)$ is negative, we construct a solution by a careful inspection of the behaviour of the approximating solutions in a space-time neighbourhood of the jump points. The limit solution u we obtain is the same as the one obtained by replacing $\phi(\cdot)$ with the truncated function $\min(\phi(\cdot), 1)$, and it turns out that u solves a free boundary problem. The free boundary consists of the points dividing the region where $|u_x| > 1$ from the region where $|u_x| \leq 1$. Finally, we consider the full space-time discretization (implicit in time) of the Perona-Malik equation, and we show that, if the time step is small with respect to the spatial grid h, then the limit is the same as the one obtained with the spatial semidiscrete scheme. On the other hand, if the time step is large with respect to h, then the limit solution equals $\overline{u}$, i.e., the standing solution of the convexified problem.

1 Introduction

The Perona-Malik equation is a nonlinear forward-backward parabolic equation introduced in [11] with the aim of recostructing an image on a computer. In one spatial

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dimension the equation is the gradient flow of the nonconvex functional

$$F^{\phi}(u) = \int_I \phi(u_x)\ dx, \qquad \phi(p) := \frac{1}{2}\log(1+p^2), \tag{1.1}$$

$I := (0,1)$, and reads as

$$u_t = (\phi'(u_x))_x. \tag{1.2}$$

We couple (1.2) with the initial condition

$$u(0) = \overline{u}, \tag{1.3}$$

and periodic boundary conditions on ∂I. Since $\phi'' > 0$ in the "stable" region $ST_\phi := (-1,1)$ of ϕ and $\phi'' < 0$ in the "locally unstable" region $LUS_\phi := \mathbb{R} \setminus [-1,1]$ of ϕ, equation (1.2) is forward parabolic when $u_x(x,t)$ belongs to ST_ϕ, while it becomes backward parabolic, and therefore ill-posed, when $u_x(x,t) \in LUS_\phi$. We define

$$\mathrm{stab}(u(\cdot,t)) := \{x \in I : u_x(x,t) \in \overline{ST_\phi}\} = \{x \in I : |u_x(x,t)| \leq 1\},$$

$$\mathrm{unstab}(u(\cdot,t)) := \{x \in I : u_x(x,t) \in LUS_\phi\} = \{x \in I : |u_x(x,t)| > 1\},$$

and $J_{u(t)} \subset I$ the set of jump points of $u(t)$. Due to instabilities and lack of regularity that one can expect in $\mathrm{unstab}(\overline{u})$ (effectively observed in numerical experiments [12], [9], [13], [4], see also Figure 1) the (even local) existence of a solution to (1.1), (1.3) in some functional class becomes a nontrivial question, when the initial datum $\overline{u}$ is *transcritical*, i.e., when both $\mathrm{stab}(\overline{u})$ and $\mathrm{unstab}(\overline{u})$ have nonempty interior. Various notions of solutions and regularizations have been proposed for (1.2), see for instance [1], [4], [7], [13] (and [6], [14] for other ill-posed equations). It is worth noting that, in general, it is to be expected that different regularizations lead to completely different results. From the mathematical point of view, one interesting problem is the identification of the limit of the approximate solutions as the regularization parameters go to zero; to our best knowledge, this is a largely open question, at the origin of the present paper. Indeed, we are here interested in studying the limit behaviour of the spatial semidiscretized schemes and of the full space-time discretizations for (1.2), under periodic boundary conditions. As we shall see, the semidiscretized schemes and the full space-time discretizations can lead to completely different limits, which is a strong indication of the unstable nature of equation (1.2).
Our first results (Theorems 4.4 and 4.6) identify the limit along a subsequence of the functions u^h which are the solutions of the spatial semidiscretization of (1.2). i.e.,

$$\dot{u}^h = D_h^+(\phi'(D_h^-(u^h))), \qquad u^h(0) = \overline{u}^h, \tag{1.4}$$

h the grid size, $D_h^{\pm}$ as in (2.4), $\overline{u}^h \to \overline{u}$ as in Section 3.1, provided the initial datum $\overline{u}$ has a pointwise unstable region. This means that $\overline{u}$ is smooth with $\overline{u}_x \in \overline{ST_\phi}$ in a finite number of adjacent intervals, and $\overline{u}$ jumps at the separation points: intuitively, unstab($\overline{u}$) reduces to a finite number of points $0 < \overline{a}_1 < \cdots < \overline{a}_m < 1$, which are the jump points of $\overline{u}$ (i.e. unstab($\overline{u}$) $= J_{\overline{u}}$). We write $\overline{u} \in \mathcal{P}^\phi(I)$ to denote such a class of initial data. We prove that, if $\overline{u} \in \mathcal{P}^\phi(I)$, then $u(t)$ is I-periodic and solves

$$\begin{cases} u(t) \in \mathcal{P}^\phi(I), \quad J_{u(t)} \subseteq J_{\overline{u}}, & \\ u_t = (\phi'(u_x))_x & \text{in } \bigcup_{t\geq 0}\left(I \setminus J_{u(t)}\right) \times \{t\}, \\ \lim_{y \to \overline{a}_j} u_x(y,t) = 0 & \text{for a.e. } t > 0 \text{ such that } \overline{a}_j \in J_{u(t)}, \\ u(0) = \overline{u}. & \end{cases} \tag{1.5}$$

This result means that, provided $u_x(t)$ belongs to the stable region of ϕ and no jumps disappear or re-appear, then u solves the natural forward parabolic equation, coupled with zero Neumann boundary conditions at the interior jump points. In order to derive (1.5) we need a strong compactness property of the space-time gradients of the functions u^h (see Proposition 3.3) which allows to pass to the limit in the nonlinear terms of the approximating equations, and to control the boundary terms which give raise to the interior Neumann conditions. Note also that the piecewise linear approximations $\overline{u}^h$, used as initial data in the semidiscretization scheme, are effectively transcritical, since unstab($\overline{u}^h$) is not pointwise and consists of isolated intervals of length h. Hence, the assumption that the initial datum $\overline{u}$ has a pointwise unstable region does not prevent the discretized problem to be influenced by the backward parabolic character of the equation.

From (1.5) we have that the jumps of $u(t)$ are among the jumps of $\overline{u}$, and that some of the jumps of $u(t)$ can disappear at some time $t^* > 0$. Note carefully that Theorem 4.4 does not exclude that, once a jump has disappeared at $t = t^*$, it may reappear again among the jumps of $u(s)$ at some time s larger than t^*. Namely, we cannot exclude the situation

$$\overline{a}_j \in J_{u(t)} \text{ for } t \in [0, t^*), \qquad \overline{a}_j \notin J_{u(t^*)}, \qquad \overline{a}_j \in J_{u(s)} \text{ for some } s > t^*. \tag{1.6}$$

and pathological behaviours of the above sort may happen several times (see Remark 4.1).

This is related to the possibility that two parts of the graph of $u(t)$ where $|u_x| \leq 1$ and that separately flow from the two sides of a jump point $\overline{a}_j$, collide at $\overline{a}_j$ at t^*. In general one could expect that, after the collision, the two branches merge in a unique larger branch that continues the flow keeping the condition $|u_x| \leq 1$ (or even $|u_x| < 1$) without creating a new jump point (located necessarily at $\overline{a}_j$). However, we cannot exclude that the larger branch develops the jump at $\overline{a}_j$ for some $s > t^*$ (as in [5] it is possible to show that solutions of Theorem 4.4 are not unique). Note that if this happens, the nondecreasing property of the set-valued map $t \to$ stab($u(t)$) is violated (while it is known to be always true that $t \to$ stab($u^h(t)$) is monotone for any $h > 0$, see [8], [10] and Proposition 2.3 (d)).

Let us now pass to describe the more interesting case when $\overline{u}$ is transcritical; in our notation $\overline{u} \in \mathcal{A}^\phi(I)$ (see Section 2.1). The identification of the limit of the spatial semidiscrete schemes in this case seems to be a difficult problem, related to the understanding of the (probably quite complicated) quick formation of microstructures in unstab($\overline{u}$). In Example 5.1 we show that, as an indication of the presence of this phenomenon, the various derivatives of the discretized solution change alternatively their sign (at fixed h). In Theorem 5.2 we show that, given $\overline{u} \in \mathcal{A}^\phi(I)$, it is possible to find a sequence (δ_k) of positive numbers converging to zero, and functions $\overline{u}_{\delta_k} \in \mathcal{P}^\phi(I)$ converging to $\overline{u}$ as $k \to +\infty$ and such that the corresponding solutions (given by Theorem 4.6) converge to a function u such that $u(t) \in \mathcal{A}^\phi(I)$ for all $t \in [0, +\infty)$ which solves

$$\int_{I\times(0,+\infty)} u_t \, \psi \; dx \; dt + \int_{\cup_{t\geq 0}(\mathrm{stab}(u(t))\times\{t\})} \phi'(u_x)\psi_x \; dx \; dt = 0, \qquad u(0) = \overline{u}, \qquad (1.7)$$

for any test function ψ. The most delicate part in the proof relies on the choice of $\overline{u}_{\delta_k}$ in the open intervals $(\overline{a}_j, \overline{b}_j)$ of unstab($\overline{u}$) and on the control of the behaviour of the corresponding solution u_{δ_k} (given by Theorem 4.6) in $(\overline{a}_j, \overline{b}_j)$. Roughly speaking, in $(\overline{a}_j, \overline{b}_j)$ the initial datum $\overline{u}$ is approximated with a staircase function $\overline{u}_{\delta_k}$ of size approximately δ_k, in such a way that, whenever a collision of two adjacent branches of u_{δ_k} appears at some jump point $\overline{x}$ at some time t, such a collision happens with nonzero relative velocity, and in such a way that $\overline{x} \notin J_{u_{\delta_k}(s)}$ for all $s > t$. Note carefully that Theorem 5.2 is valid only for the specific approximating sequence $(\overline{u}_{\delta_k})$ of $\overline{u}$; no assertion is given for a generic $L^2(I)$-approximation of $\overline{u}$.
It is interesting to remark that (1.7) is the same limit equation obtained in [5] for a different function ϕ. More precisely, if we denote by ϕ_* the *truncated* function

$$\phi_*(p) := \min(\phi(p), 1), \qquad p \in \mathbb{R},$$

then the solution to (1.2) for a transcritical $\overline{u}$ is the same as the one obtained by considering the gradient flow of the (nonsmooth) functional F_{ϕ_*}. The qualitative properties of solutions of the gradient flow of F_{ϕ_*} have been discussed in [5]; here we only recall that the unstable region of $u(t)$ stay still, and can be invaded by the stable one, giving origin to a free boundary problem. Or, also, jumps may appear at the points subdividing the stable region of $\overline{u}$ from the unstable one, even if $\overline{u}$ is of class $\mathcal{C}^\infty$. See Figure 1 where one can observe that, for the Perona-Malik equation, the unstable region of $\overline{u}$ is gradually eroded by the stable one.

It is interesting to make more precise the previous comment that different regularizations lead to different solutions in the limit. The starting remark is that applying the implicit time discretization (or minimizing movement method [2]) to the functional F^ϕ (no discretization in space) leads in the limit as the time step λ^{-1} goes to zero, to the standing solution $\overline{u}$, independently of the presence of stable or unstable regions of $\overline{u}$. The reason is that minimizing F^ϕ is equivalent to minimize its lower semicontinuous envelope, i.e., $F_{\phi^{**}}$, where ϕ^{**} is the convexification of ϕ. Since in the present case

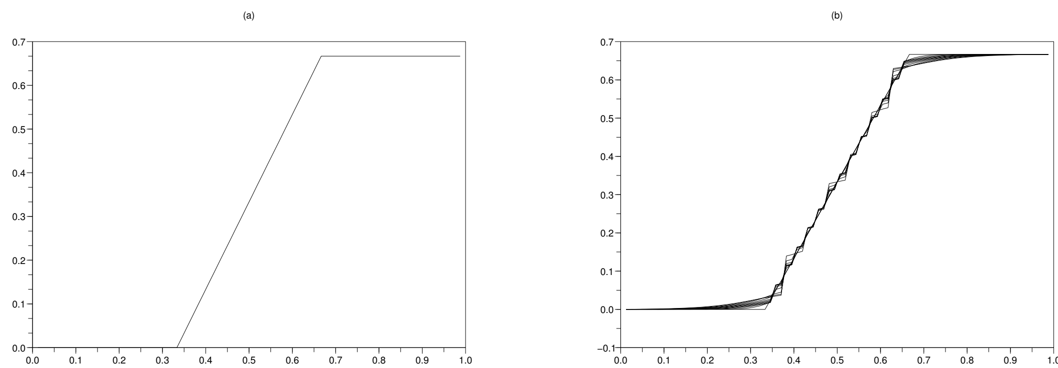

Figure 1: (a): transcritical initial datum; (b) short-time evolution

$\phi^{**} \equiv 0$, we deduce that the implicit time discretization scheme leads, in the limit, to the standing solution $\overline{u}$ for all times. Such a solution is of course completely different from the function u of Theorem 4.6; note that the remark applies also when $\mathrm{unstab}(\overline{u})$ is empty, a case in which it is reasonable to expect the solution to (1.2) (now forward parabolic everywhere) to be well defined and nontrivial. Therefore, the use of such a type of implicit time discretization may be questionable. This observation suggests that, if we approximate (1.2) with a full space-time discretization (implicit in time) described by the two parameters h and λ, provided λ^{-1} goes to zero sufficiently fast with respect to h, the limit solution should be the standing solution $\overline{u}$. On the other hand, by Theorem 4.6 it is natural to expect that, if h goes to zero sufficiently fast with respect to λ, the solution u in (1.7) should be reached in the limit $(h, \lambda) \to (0, +\infty)$. We substantiate these assertions with Propositions 6.1, 6.2. Classifying all possible limits of the completely discrete schemes as the grid size and the time step go to zero independently is beyond the scope of the present paper.
After this paper was finished, is has been pointed out to us that Theorem 4.4 was independently proved in [10, Corollary 2.11 and Remark 2.12].

2 Preliminaries

$BV(I)$ is the space of functions with bounded variation in I, see [3]. Unless otherwise specified, any $u \in BV(I)$ is identified with its representative defined pointwise everywhere as $u(x) = (u(x_+) + u(x_-))/2$ for any $x \in I$, where $u(x_-)$ and $u(x_+)$ are respectively the left and the right limit. If $u(x_+) \neq u(x_-)$ we say that x is a jump point of u and J_u indicates the set of jump points of u. We set $u(0) := u(0_+)$ and $u(1) := u(1_-)$.

The functional F^ϕ in (1.1) is defined on the whole of $BV(I)$ as

$$F^\phi(u) := \int_I \phi(u_x^a)\ dx, \qquad u \in BV(I), \tag{2.1}$$

where u_x^a is the absolutely continuous part of the distributional derivative of u.
If u depends on $(x,t) \in I \times (0,T)$, we write $u(t)(\cdot) = u(\cdot, t) = u(t)$.
$AC^2([0,+\infty); L^2(I))$ is the space of absolutely continuous functions u from $[0,+\infty)$ to $L^2(I)$ such that $u_t \in L^2(I \times (0,+\infty))$, see for instance [2].
Given an integer $N > 0$ and $i \in \{0, \ldots, N\}$, we divide $\overline{I}$ in N intervals of length $h := 1/N$ (the grid-size). Since we work in the I-periodic setting, throughout the paper we *identify* the node 0 with the node N (hence $N+1$ with 1 and -1 with $N-1$).
Sequences $(g^h)_h$ of functions will be simply denoted by (g^h).
We denote by $\mathcal{C}^1_\#$ the class of the functions $\psi \in \mathcal{C}^1(\mathbb{R} \times [0,+\infty))$ which are 1-periodic in x for all $t \geq 0$ and $\psi(\cdot, t) \equiv 0$ for all t large enough.
$\lfloor \cdot \rfloor$ is the lower integer part.
By $\bar{c}$ we indicate a positive constant independent of h, the value of which may vary from line to line.

2.1 The classes $\mathcal{P}^\phi(I)$ and $\mathcal{A}^\phi(I)$

We denote by $\mathcal{P}^\phi(I)$ the class of all *piecewise Lipschitz functions with pointwise unstable set*, i.e., those functions $u \in BV(I)$ with $u(0) = u(1)$ such that J_u is finite and, if C is a connected component of $I \setminus J_u$, then u is Lipschitz in C with $u_x \in \overline{ST_\phi}$ (i.e., $|u_x| \leq 1$) almost everywhere in C. For such a function u we set

$$\mathrm{unstab}(u) = J_u.$$

We denote by $\mathcal{A}^\phi(I) \subset BV(I)$ the set of all ϕ -admissible functions. Precisely, an I-periodic function u belongs to $\mathcal{A}^\phi(I)$ if there exist a natural number $m \geq 0$ and real numbers

$$0 < a_1 \leq b_1 < \ldots < a_m \leq b_m < 1 \tag{2.2}$$

such that
u is Lipschitz with $u_x \in \overline{ST_\phi}$ almost everywhere in each connected component of $I \setminus \bigcup_{j=1}^m [a_j, b_j]$;
if $j \in \{1, \ldots, m\}$ and $a_j < b_j$, then u is monotone in $[a_j, b_j]$ and

$$|u(x) - u(y)| > |x - y| \qquad \text{for } x, y \in [a_j, b_j],\ x \neq y. \tag{2.3}$$

It may happen that $[a_j, b_j]$ reduces to a point: if $a_j = b_j$ for some $j \in \{1, \ldots, m\}$ then we require $a_j \in J_u$.

Remark 2.1. Recalling that $u(x) = (u(x_-) + u(x_+))/2$, if $a_j < b_j$, $a_j \in J_u$, and if u is nondecreasing in $[a_j, b_j]$, then $u(a_j) < \lim_{x \downarrow a_j} u(x)$.
Note also that $b_i < a_{i+1}$ in (2.2), hence there cannot be two adjacent intervals where u is monotone and (2.3) holds.

For a function $u \in \mathcal{A}^\phi(I)$ we set

$$\text{unstab}(u) = \bigcup_{j=1}^{m} [a_j, b_j], \qquad \text{stab}(u) = I \setminus \text{unstab}(u).$$

2.2 Spatial discretizations

$PL_h(I)$ is the N-dimensional vector subspace of $\text{Lip}(I)$ of all piecewise linear functions defined on the grid. Note that $PL_h(I)$ is not contained in $\mathcal{A}^\phi(I)$.
$PC_h(I)$ is the N-dimensional vector subspace of $L^2(I)$ of all left-continuous piecewise constant functions on the grid.
Given $u \in PL_h(I)$ (resp. $u \in PC_h(I)$) we denote with $u_1, \ldots, u_N$ the coordinates of u with respect to the basis of the hat (resp. flat) functions, and $u \in PL_h(I)$ will be identified with $(u_1, \ldots, u_N) \in \mathbb{R}^N$, where $u_i := u(ih)$, $i = 1, \ldots, N$ (and $u_0 = u_N$). $PL_h(I)$ is endowed with the L^2-norm $\|u\|^2_{PL_h(I)} = h \sum_{i=1}^N |u_i|^2$.
We define the linear map $D_h^- : PL_h(I) \to PC_h(I)$ and its adjoint $D_h^+ : PC_h(I) \to PL_h(I)$ as

$$(D_h^- u)_i = \frac{1}{h}(u_i - u_{i-1}), \qquad (D_h^+ w)_i = \frac{1}{h}(w_{i+1} - w_i), \qquad i \in \{1, \ldots, N\}. \tag{2.4}$$

If $u \in PL_h(I)$ and $i \in \{1, \ldots, N\}$, then $(D_h^- u)_i = u_x$ in $((i-1)h, ih)$.
The restriction F_h of F^ϕ to $PL_h(I)$ is a smooth function of N variables and reads as

$$F_h(u) = h \sum_{i=1}^N \phi\left((D_h^- u)_i\right) = h \sum_{i=1}^N \phi\left(\frac{u_i - u_{i-1}}{h}\right), \qquad u \in PL_h(I). \tag{2.5}$$

We indefferently use the notation $\phi'((D_h^- u)_i)$ or $\phi'((D_h^- u))_i$.

Remark 2.2. The $L^2(I)$-gradient flow of F_h on $PL_h(I)$ is expressed by the following system of nonlinear ordinary differential equations:

$$\dot{u}_i = -\frac{1}{h}\frac{\partial F_h}{\partial u_i} = \frac{1}{h}\left\{\phi'\left(\frac{u_{i+1} - u_i}{h}\right) - \phi'\left(\frac{u_i - u_{i-1}}{h}\right)\right\} = \left(D_h^+ \phi'(D_h^- u)\right)_i, \tag{2.6}$$

$i \in \{1, \ldots, N\}$, with the periodicity condition $u_0 = u_N$.

Useful properties of solutions to the gradient system of F_h in the space $PL_h(I)$ are described by the following proposition.

Proposition 2.3. *For $h \in (0,1)$ we have $F_h \in \mathcal{C}^\infty(PL_h(I)) \cap \mathcal{C}^{1,1}(PL_h(I))$. Therefore the Cauchy problem*

$$\dot{u}_i = -\frac{1}{h}\frac{\partial F_h}{\partial u_i}(u) \quad \text{for} \quad i \in \{1, \ldots, N\}, \qquad u(0) = \overline{u} \in PL_h(I), \tag{2.7}$$

has a unique solution $\nu^h \in \mathcal{C}^\infty([0,+\infty); PL_h(I))$. Moreover

(a) the function $t \in [0,+\infty) \to F_h\big(\nu^h(t)\big)$ is nonincreasing, and

$$\|\nu^h(t_2) - \nu^h(t_1)\|_{L^2(I)} \le \big(F_h(\nu^h(t_1))\big)^{1/2} |t_2 - t_1|^{1/2}, \qquad 0 \le t_1 \le t_2 < +\infty; \tag{2.8}$$

(b) the function $t \in [0,+\infty) \to \sup_{x \in I} \nu^h(x,t)$ (resp. $t \in [0,+\infty) \to \inf_{x\in I} \nu^h(x,t)$) is nonincreasing (resp. nondecreasing);

(c) the function $t \in [0,+\infty) \to \|\nu^h_x(t)\|_{L^1(I)}$ is nonincreasing;

(d) if $\overline{u} \in PL_h(I) \cap \mathcal{A}^\phi(I)$, then

$$0 \le t_1 \le t_2 \Rightarrow \mathrm{stab}(\nu^h(t_1)) \subseteq \mathrm{stab}(\nu^h(t_2));$$

(e) if $|\nu^h_x(x,t)| \le 1$ at some $(x,t) \in I \times [0,+\infty)$, then $|\nu^h_x(x,t+\tau)| < 1$ for any $\tau > 0$.

Proof. The regularity, existence and uniqueness assertions follow from the analiticity of ϕ and of the maps $t \in [0,+\infty) \to (D_h^- \nu^h(t))_i$, together with the boundedness of ϕ'. Since D_h^- is a linear operator we have

$$\begin{aligned} \frac{d}{dt}F_h(\nu^h) &= h\sum_{i=1}^N \phi'((D_h^-\nu^h))_i (D_h^- \frac{d}{dt}\nu^h)_i \, dx \\ &= -h\sum_{i=1}^N (D_h^+\phi'((D_h^-\nu^h)))_i \dot{\nu}_i^h = -h\sum_{i=1}^N (\dot{\nu}_i^h)^2 = -\big\|\dot{\nu}^h\big\|^2_{L^2(I)} \le 0. \end{aligned} \tag{2.9}$$

Using Hölder's inequality and (2.9) we get

$$\|\nu^h(t_2) - \nu^h(t_1)\|_{L^2(I)} \le |t_2 - t_1|^{1/2} \Big(\int_{t_1}^{t_2} \big\|\dot{\nu}^h\big\|^2_{L^2(I)} \, d\tau\Big)^{1/2} \le \big(F_h\big(\nu^h(t_1)\big)\big)^{1/2} |t_2 - t_1|^{1/2}.$$

We say that $i \in \{1, \ldots, N\}$ is a relative maximum for the function $\nu^h(t)$ if $\nu^h(t)_i \ge \max\{\nu^h(t)_{i-1}, \nu^h(t)_{i+1}\}$. Recalling (2.6), we have $\dot{\nu}^h(t)_i \le 0$ if i is a relative maximum. Since the case of relative minima is similar, (b) follows.
If we define

$$S_i(t) := \begin{cases} \mathrm{sign}(\nu^h(t)_{i+1} - \nu^h(t)_i) & \text{if } \nu^h(t)_{i+1} \ne \nu^h(t)_i \\ \mathrm{sign}\,(\dot{\nu}_h(t)_{i+1} - \dot{\nu}_h(t)_i) & \text{if } \nu^h(t)_{i+1} = \nu^h(t)_i \end{cases} \qquad t \in [0,+\infty),$$

then following [5, Lemma 5.3, Theorem 5.4] we have

$$\frac{d}{dt^+}\|\nu_x^h\|_{L^1(I)} = h\sum_{i=1}^{N} \frac{d}{dt^+}\left|\frac{\nu_{i+1}^h - \nu_i^h}{h}\right| = \sum_{i=1}^{N} S_i\left(\dot\nu_{i+1}^h - \dot\nu_i^h\right) = \sum_{i=1}^{N}\left(S_{i-1} - S_i\right)\dot\nu_i^h,$$

where d/dt^+ is the right derivative. Hence to prove (c) it is enough to show that $(S_{i-1} - S_i)\,\dot\nu_i^h \leq 0$ for any $i \in \{1,\dots,N\}$, and this can be done as in the proof of [5, Theorem 5.4 (d)], as well as the proof of assertions (d) and (e). See also [10]. $\square$

3 Convergence of the spatial semidiscrete scheme

We approximate (1.2), (1.3) with (2.6) with well prepared initial data in the following sense.

3.1 Approximation of $\overline{u}$, definition of u^h and the limit u

Let

$$\overline{u} \in \mathcal{P}^\phi(I), \qquad J_{\overline{u}} = \{\overline{a}_1,\dots,\overline{a}_m\}, \tag{3.1}$$

and $h \in (0,1)$. If we define

$$\overline{u}^h \in PL_h(I) \cap \mathcal{A}^\phi(I)$$

as $\overline{u}_i^h := \overline{u}(ih)$ for $i \in \{0,\dots,N\}$, a direct verification (see [5, Lemma 6.1]) yields that: there exist mesh points $0 < \overline{a}_1^h < \dots < \overline{a}_m^h < 1$ with $\overline{a}_j \in [\overline{a}_j^h, \overline{a}_j^h + h]$ for any $j \in \{1,\dots,m\}$, such that

$$\mathrm{unstab}(\overline{u}^h) = \bigcup_{j=1}^{m} [\overline{a}_j^h, \overline{a}_j^h + h], \tag{3.2}$$

$$\lim_{h\to 0}\|\overline{u}^h - \overline{u}\|_{L^2(I)} = 0 = \lim_{h\to 0}\left(\|\overline{u}^h\|_{BV(I)} - \|\overline{u}\|_{BV(I)}\right), \qquad \lim_{h\to 0} F_h(\overline{u}^h) = F^\phi(\overline{u}). \tag{3.3}$$

Note that $\overline{u}^h$ does not necessarily belong to $\mathcal{P}^\phi(I)$: indeed, jump points of $\overline{u}$ correspond to intervals of length h where the slope of $\overline{u}^h$ belongs to LUS_ϕ.

Definition 3.1. *For an initial datum as in* (3.1) *and the corresponding sequence* $(\overline{u}^h)$ *described above, we denote by* $u^h \in \mathcal{C}^\infty([0,+\infty); PL_h(I))$ *the solution of the Cauchy problem*

$$\dot u^h(t) = D_h^+(\phi'(D_h^- u(t))), \qquad u^h(0) = \overline{u}^h. \tag{3.4}$$

From (2.9), (3.3) and Proposition 2.3 it follows that

$$\sup_{t>0}\sup_h F_h(u^h(t)) + \sup_h \|\dot u^h\|_{L^2((0,+\infty);L^2(I))} + \sup_h \|u^h\|_{L^\infty((0,+\infty);BV(I))} < +\infty. \tag{3.5}$$

As a consequence, if we define

$$\mathcal{X} := L^\infty((0,+\infty)); BV(I)) \cap AC^2([0,+\infty)); L^2(I)),$$

we have that the sequence (u^h) admits a (not relabelled) subsequence weakly converging in $H^1((0,T); L^2(I))$ for any $T > 0$ and weakly* in $L^\infty((0,+\infty)); BV(I))$ (we shortly say that (u^h) weakly converges in $\mathcal{X}$) to a function

$$u \in \mathcal{X}. \tag{3.6}$$

Moreover $u^h(t) \to u(t)$ weakly* in $BV(I)$, with $\|u(t)\|_{BV(I)} \leq \|\overline{u}\|_{BV(I)}$ for any $t \geq 0$.

3.1.1 Some qualitative properties of u

Denote by $L(\overline{a}_j)$ the half-line in space-time defined as

$$L(\overline{a}_j) := \{\overline{a}_j\} \times [0,+\infty). \tag{3.7}$$

Proposition 3.2. *Let $\overline{u}$ be as in* (3.1). *Then the function u in* (3.6) *satisfies*

$$u(t) \in \mathcal{P}^\phi(I), \qquad J_{u(t)} \subseteq J_{\overline{u}} \qquad \forall t \in [0,+\infty), \tag{3.8}$$

and $|u(\overline{a}_{j_+}, \cdot) - u(\overline{a}_{j_-}, \cdot)| \in \mathcal{C}^0([0,+\infty))$ for any $j \in \{1,\dots,m\}$.

Proof. Proposition 2.3 (d) implies that $\mathrm{stab}(\overline{u}^h) \subseteq \mathrm{stab}(u^h(t))$ for any $t \geq 0$ and any $h \in (0,1)$, therefore $u^h(\cdot,t)$ is one-Lipschitz in each connected component of $\mathrm{stab}(\overline{u}^h)$. Let K be any interval compactly contained in $I \setminus J_{\overline{u}}$. Then $u^h(\cdot,t)$ is one-Lipschitz in K for h small enough, hence (recall Proposition 2.3 (b)) the sequence (u^h) admits a subsequence uniformly converging in K to $u(\cdot,t)$. It follows that $u(\cdot,t)$ is one-Lipschitz in each connected component of $I \setminus J_{\overline{u}}$, and (3.8) follows.
Let $\delta := \min_{j=0,\dots,m}(\overline{a}_{j+1} - \overline{a}_j) > 0$, where $\overline{a}_0 := 0$ and $\overline{a}_{m+1} := 1$. As a consequence of (3.8), we have that $u(\cdot,s)$ is one-Lipschitz in $(\overline{a}_j - \delta, \overline{a}_j)$ and in $(\overline{a}_j, \overline{a}_j + \delta)$ for any $s \geq 0$. Fix $t, s \geq 0$, $s \neq t$, and $x, x' \in (\overline{a}_j - \delta, \overline{a}_j + \delta)$ with $x < \overline{a}_j < x'$. We have

$$\begin{aligned}
|u(\overline{a}_{j_+}, t) - u(\overline{a}_{j_-}, t)| &\leq 2\delta + |u(x,t) - u(x',t)|, \\
|u(x,s) - u(x',s)| &\leq |u(\overline{a}_{j_+}, s) - u(\overline{a}_{j_-}, s)| + 2\delta,
\end{aligned}$$

so that

$$\begin{aligned}
& |u(x,s) - u(x,t)| + |u(x',s) - u(x',t)| \geq |u(x,t) - u(x',t)| - |u(x,s) - u(x',s)| \\
\geq \quad & |u(\overline{a}_{j_+}, t) - u(\overline{a}_{j_-}, t)| - |u(\overline{a}_{j_+}, s) - u(\overline{a}_{j_-}, s)| - 4\delta.
\end{aligned}$$

From (3.6) it follows that $u(y,\cdot)$ is continuous on $[0,+\infty)$ for almost every $y \in I$, therefore we can choose x and x' continuity points. Letting $s \to t$ in the above inequalities, interchanging the role of s and t, and using the arbitrariness of δ, it follows that $|u(\overline{a}_{j_+}, \cdot) - u(\overline{a}_{j_-}, \cdot)|$ is continuous on $[0,+\infty)$ for all $j \in \{1,\dots,m\}$. $\square$

As a consequence of Proposition 3.2 we have

$$\bigcup_{t \geq 0} \big(J_{u(t)} \times \{t\}\big) \;=\; \bigcup_{j=1}^{m} \big\{(\overline{a}_j, t) \in L(\overline{a}_j) : u(\overline{a}_{j_+}, t) \neq u(\overline{a}_{j_-}, t)\big\}. \tag{3.9}$$

3.1.2 Strong compactness of the sequence $(\phi'(u_x^h))$

In order to characterize the limit function u, some form of strong compactness of the sequence of gradients (u_x^h) is required. This is the content of the following proposition (see also [10, Proposition 3.3 and Theorem 2.10] for a similar result).

Proposition 3.3. *Let $\overline{u}$ be as in* (3.1) *and let K be an interval compactly contained in* $\mathrm{stab}(\overline{u})$*. Then the functions u^h in Definition 3.1 enjoy the following property:*

$$\forall \epsilon \in (0,1) \quad \text{the sequence } \Big(\frac{d}{dt}\phi'(u_x^h)\Big) \text{ is bounded in } L^2(K \times (\epsilon, +\infty)). \tag{3.10}$$

Proof. Define $v^h \in PC_h(I)$ and $w_h \in PC_h(I)$ as

$$v_i^h := (D_h^- u^h)_i, \qquad w_i^h := \phi'(v_i^h), \qquad i \in \{1, \dots, N\}.$$

From (3.4) it follows

$$\dot{v}_i^h = (D_h^- \dot{u}^h)_i = (D_h^- D_h^+ \phi'(v^h))_i = (D_h^- D_h^+ w^h)_i, \qquad i \in \{1, \dots, N\}, \tag{3.11}$$

with periodic boundary conditions.
Let $j, k \in \mathbb{N}$ be such that

$$[(j-k)h, (j+k)h] \subset\subset \mathrm{stab}(\overline{u}).$$

For $h \in (0,1)$ sufficiently small we have

$$[(j-k)h, (j+k)h] \subset \mathrm{stab}(\overline{u}^h),$$

and from Proposition 2.3 (d), there exists $h_0 > 0$ such that

$$[(j-k)h, (j+k)h] \subset \mathrm{stab}(u^h(t)) \qquad t > 0, \ h \in (0, h_0). \tag{3.12}$$

Multiplying both sides of (3.11) by $\dot{w}_i^h$ and summing over i yields

$$\begin{aligned} \sum_{i=j-k}^{j+k} \dot{v}_i^h \dot{w}_i^h &= \sum_{i=j-k}^{j+k} (D_h^- D_h^+ w^h)_i \ \dot{w}_i^h \\ &= -\sum_{i=j-k}^{j+k-1} (D_h^+ w^h)_i (D_h^+ \dot{w}^h)_i + \frac{1}{h}(D_h^+ w^h)_{j+k} \dot{w}_{j+k}^h - \frac{1}{h}(D_h^+ w^h)_{j-k-1} \dot{w}_{j-k}^h \quad (3.13) \\ &= -\frac{1}{2}\frac{d}{dt} \sum_{i=j-k}^{j+k-1} \big((D_h^+ w^h)_i\big)^2 + \frac{1}{h}(D_h^+ w^h)_{j+k} \dot{w}_{j+k}^h - \frac{1}{h}(D_h^+ w^h)_{j-k-1} \dot{w}_{j-k}^h. \end{aligned}$$

Take a smooth function f with the following properties:

$$f : [-1, 1] \to \mathbb{R}, \qquad f(0) = 0, \qquad f' = \sqrt{\phi''} \quad \text{in } \overline{ST_\phi} = [-1, 1]. \tag{3.14}$$

Then, for $i \in \{(j-k)h, \dots, (j+k)h\}$,

$$\dot w_i^h = \phi''(v_i^h)\dot v_i^h = \sqrt{\phi''(v_i^h)}\,\frac{d}{dt}f(v_i^h), \qquad \dot v_i^h \dot w_i^h = \phi''(v_i^h)(\dot v_i^h)^2 = \big(\frac{d}{dt}f(v_i^h)\big)^2. \quad (3.15)$$

As a consequence, using also (3.13) and the equality $\dot u^h = D_h^+ w^h$, it follows

$$\begin{aligned}\sum_{i=j-k}^{j+k} \big(\frac{d}{dt}f(v_i^h)\big)^2 &= -\frac{1}{2}\frac{d}{dt}\sum_{i=j-k}^{j+k-1}\big((D_h^+ w^h)_i\big)^2 + \frac{1}{h}(D_h^+ w^h)_{j+k}\dot w_{j+k}^h - \frac{1}{h}(D_h^+ w^h)_{j-k-1}\dot w_{j-k}^h \\ &= -\frac{1}{2}\frac{d}{dt}\sum_{i=j-k}^{j+k-1}(\dot u_i^h)^2 + \frac{1}{h}\dot u_{j+k}^h \dot w_{j+k}^h - \frac{1}{h}\dot u_{j-k-1}^h \dot w_{j-k}^h. \qquad (3.16)\end{aligned}$$

Integrating both sides of (3.16) on the time interval $(\epsilon, +\infty)$, and recalling (2.9), we get

$$\begin{aligned}& h\sum_{i=j-k}^{j+k}\int_\epsilon^{+\infty} \big(\frac{d}{dt}f(v_i^h)\big)^2\,dt \\ \le\ & \frac{h}{2}\sum_{i=j-k}^{j+k-1}(\dot u_i^h(\epsilon))^2 + \int_\epsilon^{+\infty}(\dot u_{j+k}^h \dot w_{j+k}^h - \dot u_{j-k-1}^h \dot w_{j-k}^h)\,dt \qquad (3.17) \\ \le\ & \frac{h}{2}\sum_{i=j-k}^{j+k-1}(\dot u_i^h(\epsilon))^2 + \|\dot u_{j+k}^h\|_{L^2((\epsilon,+\infty))}\,\|\dot w_{j+k}^h\|_{L^2((\epsilon,+\infty))} \\ & +\|\dot u_{j-k-1}^h\|_{L^2((\epsilon,+\infty))}\,\|\dot w_{j-k}^h\|_{L^2((\epsilon,+\infty))}.\end{aligned}$$

Observe that, given $n \in \mathbb{N}$ and a function $\eta \in L^1((0,+\infty))$, there exists $\epsilon_n \in (\frac{1}{2n}, \frac{1}{n})$ such that $\eta(\epsilon_n) \le 2n\int_{\frac{1}{2n}}^{\frac{1}{n}} \eta\, ds \le 2cn \le \frac{2c}{\epsilon_n}$, $c := \|\eta\|_{L^1((0,+\infty))}$. Therefore, since by (3.5) the function $h\sum_{i=1}^N (\dot u_i^h(t))^2$ is uniformly bounded in $L^1((0,+\infty))$ with respect to h, there exists $\epsilon' \in [\epsilon/2, \epsilon]$ such that

$$h\sum_{i=j-k}^{j+k-1}(\dot u_i^h(\epsilon'))^2 \le \frac{\overline{c}}{\epsilon'}.$$

Hence, replacing ϵ by ϵ' (still denoted by ϵ) and using (3.15), (3.17), (3.12) and the bound $|\phi''| \le 1$ in $\overline{ST_\phi}$, we get

$$\begin{aligned} h\sum_{i=j-k}^{j+k}\big\|\frac{d}{dt}f(v_i^h)\big\|^2_{L^2((\epsilon,+\infty))} \le\ & \frac{\overline{c}}{\epsilon} + \|\dot u_{j+k}^h\|_{L^2((0,+\infty))}\,\big\|\frac{d}{dt}f(v_{j+k}^h)\big\|_{L^2((\epsilon,+\infty))} \\ & + \|\dot u_{j-k-1}^h\|_{L^2((0,+\infty))}\,\big\|\frac{d}{dt}f(v_{j-k}^h)\big\|_{L^2((\epsilon,+\infty))}. \qquad (3.18)\end{aligned}$$

Define now for $k \geq 1$

$$\alpha_k^h := h \sum_{i=j-k+1}^{j+k-1} \|\frac{d}{dt} f(v_i^h)\|^2_{L^2((\epsilon,+\infty))} - \frac{\overline{c}}{\epsilon}, \qquad \beta_k^h := \|\dot{u}_{j+k}^h\|_{L^2((0,+\infty))} + \|\dot{u}_{j-k-1}^h\|_{L^2((0,+\infty))}.$$

In this notation, (3.18) implies $\sqrt{h}\alpha_{k+1}^h \leq 2\beta_k^h \sqrt{\alpha_{k+1}^h - \alpha_k^h}$. Hence

$$\alpha_{k+1}^h - \alpha_k^h \geq h \; \frac{(\alpha_{k+1}^h)^2}{4(\beta_k^h)^2} \geq h \; \frac{\alpha_k^h \alpha_{k+1}^h}{4(\beta_k^h)^2} \, . \tag{3.19}$$

If we set $\gamma_k^h := -1/\alpha_k^h$, (3.19) yields

$$\gamma_{k+1}^h - \gamma_k^h \geq h \; \frac{1}{4(\beta_k^h)^2} \, . \tag{3.20}$$

Recall that Jensen's inequality implies $\zeta(\frac{1}{m}\sum_{i=1}^m (\beta_i^h)^2) \leq \frac{1}{\ell}\sum_{i=1}^\ell \zeta((\beta_i^h)^2)$ for a convex function ζ. If we choose $\zeta(x) = \frac{1}{x}$ for $x > 0$, for $k \geq k_0 - 1 \geq 1$ inequality (3.20) gives

$$\gamma_k^h \geq \gamma_{k_0}^h + \frac{h}{4} \sum_{i=k_0}^{k-1} \frac{1}{(\beta_i^h)^2} \geq \gamma_{k_0}^h + \frac{h^2}{4}(k-k_0)^2 \Big(h \sum_{i=k_0}^{k-1} (\beta_i^h)^2 \Big)^{-1} \geq \gamma_{k_0}^h + h^2 \; \frac{(k-k_0)^2}{c},$$

where $c = 8h \sum_{i=1}^N \|\dot{u}_i^h\|^2_{L^2((0,+\infty))}$ is uniformly bounded with respect to h.
From the previous inequality we deduce

$$\alpha_k^h \geq \Big(\frac{1}{\alpha_{k_0}^h} - h^2 \; \frac{(k-k_0)^2}{c} \Big)^{-1} . \tag{3.21}$$

Since

$$\alpha_k^h < +\infty \qquad \forall k \leq \lfloor \frac{\mathrm{dist}\,(hj, \mathrm{unstab}(\overline{u}))}{h} \rfloor, \tag{3.22}$$

it follows that the right hand side of (3.21) must be positive for all k for which (3.22) is valid. Hence

$$\alpha_{k_0}^h = h \sum_{i=j-k_0+1}^{j+k_0-1} \|\frac{d}{dt} f(v_i^h)\|^2_{L^2((\epsilon,+\infty))} - \frac{\overline{c}}{\epsilon} \leq c \, \Big(\mathrm{dist}\,(hj, \mathrm{unstab}(\overline{u})) - h(k_0+1) \Big)^{-2}, \tag{3.23}$$

for all k_0 with $1 \leq k_0 \leq \lfloor \frac{1}{h} \mathrm{dist}\,(hj, \mathrm{unstab}(\overline{u})) \rfloor$.

Let us choose $j, k \in \mathbb{N}$ such that

$$\begin{aligned} K \;&\subset\; [h(j-k+1), h(j+k-1)] \subset [h(j-k-1), h(j+k+1)] \\ &\subset\; K^+_{\delta/2} := \{x \in I : \mathrm{dist}(x, K) < \delta/2\}, \end{aligned}$$

where $\delta := \mathrm{dist}(K, \mathrm{unstab}(\overline{u})) > 0$. Observe that

$$\mathrm{dist}(hj, \mathrm{unstab}(\overline{u})) - h(k+1) \geq \mathrm{dist}(K^+_{\delta/2}, \mathrm{unstab}(\overline{u})) = \delta/2.$$

From (3.23) and the fact $|\phi''| \leq 1$ (recall (3.12)) it then follows

$$\begin{aligned} & h \sum_{i=j-k+1}^{j+k-1} \| \frac{d}{dt} \phi'(v_i^h) \|^2_{L^2((\epsilon,+\infty))} \leq h \sum_{i=j-k+1}^{j+k-1} \| \frac{d}{dt} f(v_i^h) \|^2_{L^2((\epsilon,+\infty))} \qquad (3.24) \\ \leq & \ \frac{\overline{c}}{\epsilon} + c \left(\mathrm{dist}\,(hj, \mathrm{unstab}(\overline{u})) - h(k+1) \right)^{-2} \leq \frac{\overline{c}}{\epsilon} + \frac{4c}{\delta^2}. \end{aligned}$$

Therefore (3.10) is proved. □

Corollary 3.4. *Let $\overline{u}$ be as in* (3.1) *and let u^h be as in Definition 3.1. Then the sequence (u^h_x) has a subsequence converging to u_x almost everywhere in $I \times (0, +\infty)$.*

Proof. Fix an interval K compactly contained in $\mathrm{stab}(\overline{u})$, and let $n \in \mathbb{N}$. Define $w^h := \phi'(D_h^- u^h)$. From Propositions 3.2 and 3.3 it follows

$$\sup_h \left(\| \frac{d}{dt} w^h \|_{L^2(K \times (\frac{1}{n}, +\infty))} + \| D_h^+ w^h \|_{L^2(I \times (0,+\infty))} \right) < +\infty. \qquad (3.25)$$

Therefore, if we denote by $\widetilde{w}^h(t)$ the piecewise linear function having the same values as $w^h(t)$ at the nodes of the grid, we have that $(\widetilde{w}^h)$ has a subsequence converging in $L^2(K \times (0, +\infty))$, as $h \to 0$. From this, $\|w^h - \widetilde{w}^h\|_{L^2(K\times(0,+\infty))} \leq h \|D_h^+ w^h\|_{L^2(K \times (0,+\infty))}$ and (3.25) it follows that (w^h) has a subsequence converging in $L^2(K \times (0, +\infty))$. The assertion then follows by letting $n \to \infty$, invading $\mathrm{stab}(\overline{u})$ with a sequence of intervals compactly contained in $\mathrm{stab}(\overline{u})$, and using the fact that ϕ' is strictly increasing in ST_ϕ. □

Remark 3.5. Notice that, if K is an interval compactly contained in $\mathrm{stab}(u(\bar{t}))$ for some $\bar{t} > 0$, reasoning as in the proof of Corollary 3.4 we obtain the estimate

$$\sup_h \left(\| \frac{d}{dt} w^h \|_{L^2(K \times (\bar{t} + \frac{1}{n}, +\infty))} + \| D_h^+ w^h \|_{L^2(I \times (0,+\infty))} \right) < +\infty. \qquad (3.26)$$

Remark 3.6. An inspection of the proof of Proposition 3.3 shows that the supremum on the left hand side of (3.25) (resp. of (3.26)) depends on $\overline{u}$ only through $\mathrm{dist}(K, \mathrm{unstab}(\overline{u}))$ (resp. $\mathrm{dist}(K, \mathrm{unstab}(u(\bar{t})))$), and through $\|\overline{u}\|_{BV(I)}$ and $F^\phi(\overline{u})$.

4 The limit problem satisfied by u when $\overline{u} \in \mathcal{P}^\phi(I)$

We want to identify the limit equation satisfied by u in (3.6), provided $\overline{u}$ is as in (3.1). To better understand the next results, an observation is in order.

Remark 4.1. In general, some jump points of $u(t)$ among the points $\overline{a}_1, \dots, \overline{a}_m$ may disappear during the flow, and may possibly subsequently re-appear. Indeed, there exist, locally around $x = 1/2$, translating solutions which are convex, even with respect to $1/2$ and one-Lipschitz, that can be approximated with piecewise linear translatory solutions τ^h with the following properties:
τ^h are convex and one-Lipschitz in a left neighbourhood of $1/2$ and in a right neighbourhood of $1/2 + h$;
τ^h have, in $(1/2, 1/2 + h)$, slope diverging as $h \to 0$ of order smaller than $1/h$.
Hence, the approximating solutions have an unstable region containing the point $1/2 + h/2$ for all times, which disappears in the limit $h \to 0$. Suitably modifying such approximating functions into functions $\widetilde{\tau}^h$, it is reasonable to expect that a jump point located at $1/2$ may appear at a positive time t. This is possible since, in general, the maximum principle in $\mathrm{unstab}(\widetilde{\tau}^h(t))$ can be violated, and therefore the slope of $\widetilde{\tau}^h(t)$ in $(1/2, 1/2 + h)$ may increase and become of the order $1/h$, thus originating a jump in the limit $h \to 0$ at time t. Note that, at the discrete level, the unstable region $(1/2, 1/2 + h)$ of $\widetilde{\tau}^h(t)$ is always present, also for times before t.

Definition 4.2. *We say that* $R(j_1, j_2, s_1, s_2) = [\overline{a}_{j_1}, \overline{a}_{j_2}] \times [s_1, s_2]$ *is a nice space-time rectangle for* u *if* $0 \le s_1 < s_2$, $j_1, j_2 \in \{1, \dots, m\}$, $\overline{a}_{j_1} < \overline{a}_{j_2}$ *and*

$$(\overline{a}_{j_1}, \overline{a}_{j_2}) \text{ is a connected component of } \mathrm{stab}(u(t)) \qquad \forall t \in [s_1, s_2].$$

Remark 4.3. Proposition 3.2 implies that there exist nice space-time rectangles for u. Moreover, if $R(j_1, j_2, s_1, s_2)$ is such a rectangle,

$$\exists c > 0 \ : \ |u(\overline{a}_{j_k+}, t) - u(\overline{a}_{j_k-}, t)| \ge c, \qquad k \in \{1, 2\}, \ t \in [s_1, s_2]. \tag{4.1}$$

Theorem 4.4. *Let* $\overline{u} \in \mathcal{P}^\phi(I)$ *and let* u *be as in* (3.6). *Let* $R = R(j_1, j_2, s_1, s_2)$ *be a nice space-time rectangle for* u. *Then*

$$\int_R u_t \, \psi \; dx \; dt + \int_R \phi'(u_x) \psi_x \; dx \; dt = 0 \qquad \forall \psi \in \mathcal{C}^1_\#. \tag{4.2}$$

Therefore, u *solves*

$$\begin{cases} u_t = \phi''(u_x) u_{xx} & \text{in } R, \\ \\ \displaystyle\lim_{y \to \overline{a}_{j_k}, \ (y,t) \in R} u_x(y, t) = 0 & \text{a.e. } t \in (s_1, s_2), \ k = 1, 2. \end{cases} \tag{4.3}$$

Proof. Let $\psi^h : [0, +\infty) \to PL_h(I)$ be defined as $\psi^h(t)_i := \psi(ih, t)$, $i \in \{0, \dots, N\}$. Observe that $\psi^h \to \psi$ in $\mathrm{Lip}(\overline{I} \times [0, +\infty))$ as $h \to 0$. Let $c > 0$ be such that (4.1) holds. Set

$$\mathrm{I}^h(t) \ := \ h \sum_{i = 2 + \overline{a}^h_{j_1}/h}^{\overline{a}^h_{j_2}/h} \phi'\big((D_h^- u^h(t))_i\big)(D_h^- \psi^h(t))_i, \qquad t \in [s_1, s_2], \tag{4.4}$$

which is a sort of discretized version of the second addendum on the left hand side of (4.2). Then, omitting the dependence on t in the notation when no confusion is possible,

$$
\begin{aligned}
\mathrm{I}^h(t) &= -h \sum_{i=1+\overline{a}^h_{j_1}/h}^{\overline{a}^h_{j_2}/h} \left(D^+_h \phi'(D^-_h u^h)\right)_i \psi^h_i \\
&\quad + \phi'\left((D^-_h u^h)_{1+\overline{a}^h_{j_2}/h}\right) \psi^h_{\overline{a}^h_{j_2}/h} - \phi'\left((D^-_h u^h)_{1+\overline{a}^h_{j_1}/h}\right) \psi^h_{1+\overline{a}^h_{j_1}/h} \\
&= -h \sum_{i=1+\overline{a}^h_{j_1}/h}^{\overline{a}^h_{j_2}/h} \dot{u}^h_i \psi^h_i \qquad (4.5) \\
&\quad + \phi'\left((D^-_h u^h)_{1+\overline{a}^h_{j_2}/h}\right) \psi^h_{\overline{a}^h_{j_2}/h} - \phi'\left((D^-_h u^h)_{1+\overline{a}^h_{j_1}/h}\right) \psi^h_{1+\overline{a}^h_{j_1}/h} \\
&=: \mathrm{I}^h_1(t) + \mathrm{I}^h_2(t) + \mathrm{I}^h_3(t).
\end{aligned}
$$

Note that

$$|\mathrm{I}^h_2(t)| + |\mathrm{I}^h_3(t)| \leq \overline{c}\, h, \qquad t \in [s_1, s_2], \tag{4.6}$$

since ψ^h are uniformly bounded with respect to h, $\phi'(p) = \mathcal{O}(1/p)$ as $p \to \infty$, and by (4.1)

$$|(D^-_h u^h(t))_{1+\overline{a}^h_{j_k}/h}| \geq \overline{c} h^{-1}, \qquad k \in \{1, 2\},\ t \in [s_1, s_2],$$

for h small enough. Integrating (4.6) in time, we get

$$\lim_{h \to 0} \int_{s_1}^{s_2} \left(\mathrm{I}^h_2(t) + \mathrm{I}^h_3(t)\right) dt = 0. \tag{4.7}$$

Moreover, since $\dot{u}^h \rightharpoonup u_t$ in $L^2(I \times (0, +\infty))$ as $h \to 0$ (see (3.5)) and the functions ψ^h are uniformly bounded, from (4.5) we get

$$\lim_{h \to 0} \int_{s_1}^{s_2} \mathrm{I}^h_1(t)\, dt = -\int_R u_t \psi \, dxdt. \tag{4.8}$$

From (4.7) and (4.8) it then follows

$$\lim_{h \to 0} \int_{s_1}^{s_2} \mathrm{I}^h(t)\, dt = -\int_R u_t \psi \, dxdt. \tag{4.9}$$

Recalling that R is a nice space-time rectangle, by Corollary 3.4 it follows that the sequence (u^h_x) converges to u_x almost everywhere in R. Since (ψ^h) converges to ψ in $\mathrm{Lip}(\overline{I} \times [0, +\infty))$, we can integrate (4.4) in time and pass to the limit as $h \to 0$ to obtain

$$\lim_{h \to 0} \int_{s_1}^{s_2} \mathrm{I}^h(t)\, dt = \int_R \phi'(u_x) \psi_x \, dxdt. \tag{4.10}$$

Then (4.2) follows from (4.9) and (4.10). $\square$

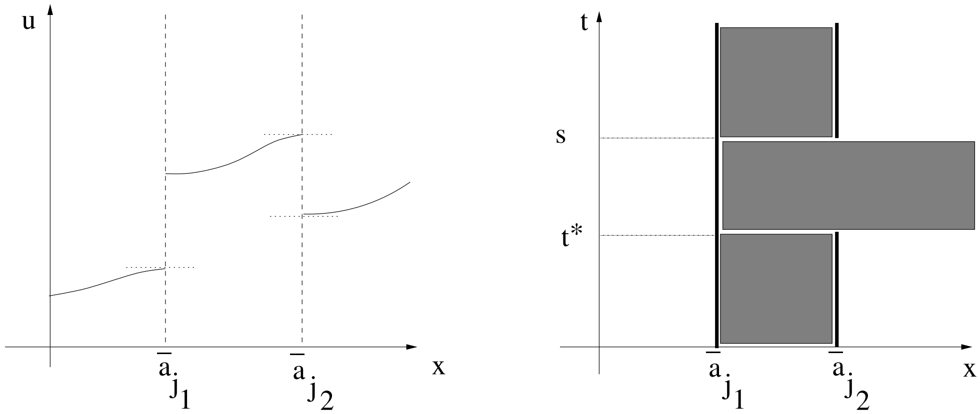

Figure 2: A possible example when two branches of the solution u (adjacent to $\overline{a}_{j_2}$) collide and then detach. We have $J_{u(t)} = \{\overline{a}_{j_1}, \overline{a}_{j_2}\}$ for all $t \in [0, t^*)$, $J_{u(t)} = \overline{a}_{j_1}$ for all $t \in (t^*, s)$, and $J_{u(t)} = \{\overline{a}_{j_1}, \overline{a}_{j_2}\}$ for all $t \in (s, +\infty)$. Hence $\cup_{t\geq 0}(J_{u(t)} \times \{t\})$ is the union of the solid vertical segments contained in $L(\overline{a}_{j_1})$ and $L(\overline{a}_{j_2})$ respectively (see (3.7)). The gray boxes are examples of nice space-time rectangles.

Remark 4.5. If there exists an interval $(\overline{b}_j, \overline{a}_{j+1}) \subset \mathrm{stab}(\overline{u})$ where $\overline{u} \in \mathcal{P}^\phi(I)$ takes the constant value α and $\{\overline{b}_j, \overline{a}_{j+1}\} \subset J_{\overline{u}}$, then

$$u(t) \equiv \alpha \quad \text{on } (\overline{b}_j, \overline{a}_{j+1}) \qquad \forall t \in (0, \tau),$$

where $\tau := \inf\{t \in (0, +\infty) : \overline{b}_j \notin J_{u(t)} \text{ or } \overline{a}_{j+1} \notin J_{u(t)}\} > 0$.

Differently from Theorem 4.4, the next theorem states a global property of u in $I \times (0, +\infty)$.

Theorem 4.6. *Let $\overline{u} \in \mathcal{P}^\phi(I)$ and let u be as in (3.6). Then*

$$\int_{I\times(0,+\infty)} u_t \, \psi \, dxdt + \int_{I\times(0,+\infty)} \phi'(u_x^a) \, \psi_x \, dxdt = 0 \qquad \forall \psi \in \mathcal{C}^1_\#, \tag{4.11}$$

where $u_x^a(\cdot, t)$ is the absolutely continuous part of the distributional derivative of $u(\cdot, t)$ with respect to x.

Proof. Let (ψ^h) be as in the proof of Theorem 4.4. For any $h \in (0, 1)$ and any $j \in \{1, \dots, m\}$ define $T_j^h := \sup\left\{t \geq 0 : \mathrm{unstab}(u^h(t)) \cap [\overline{a}_j^h, \overline{b}_j^h] \neq \emptyset\right\}$. Observe that $T_j^h \in (0, +\infty]$, $[\overline{a}_j^h, \overline{b}_j^h] = \mathrm{unstab}(u^h(t)) \cap [\overline{a}_j^h, \overline{b}_j^h]$ for all $t \in [0, T_j^h)$, and

$$\overline{b}_j^h - \overline{a}_j^h = h \qquad \forall t \in [0, T_j^h). \tag{4.12}$$

Set also $[\overline{a}_j^h, \overline{b}_j^h] = \emptyset$ for all $t \in (T_j^h, +\infty)$.
Possibly extracting a (not relabelled) subsequence, we can assume that $\lim_{h\to 0} T_j^h \to T_j \in (0, +\infty]$ for any $j \in \{1, \dots, m\}$ (the strict positivity of T_j can be proved using arguments similar to those used in the proof of Proposition 3.2, see also [5]).

Let

$$\mathrm{I}^h(t) := h \sum_{i:((i-1)h,ih)\subset \mathrm{stab}(u^h(t))} \phi'\left((D_h^- u^h(t))_i\right)(D_h^-\psi^h(t))_i, \qquad t\in[0,+\infty). \tag{4.13}$$

Define, for $t > 0$, $\mathfrak{I}^h(t) := \{j \in \{1,\dots,m\} : T_j^h > t\}$, $\overline{a}_{m+1}^h = \overline{a}_1^h$, and $\mathfrak{I}(0) := \{1,\dots,m\}$. Using also (2.6) we have

$$\begin{aligned}
\mathrm{I}^h(t) &= -h \sum_{i:((i-1)h,ih)\subset \mathrm{stab}(u^h(t))} \left(D_h^+\phi'(D_h^- u^h)\right)_i \psi_i^h \\
&\quad + \sum_{j\in\mathfrak{I}^h(t)} \Big(\phi'(u_x^h(\overline{a}_{j+}^h,t))\ \psi^h(\overline{a}_j^h,t) - \phi'(u_x^h(\overline{b}_{j+}^h,t))\ \psi^h(\overline{b}_j^h,t)\Big) \\
&= -h \sum_{i:((i-1)h,ih)\subset \mathrm{stab}(u^h(t))} \dot{u}_i^h\ \psi_i^h \\
&\quad + \sum_{j\in\mathfrak{I}^h(t)} \Big(\phi'(u_x^h(\overline{a}_{j+}^h,t))\ \psi^h(\overline{a}_j^h,t) - \phi'(u_x^h(\overline{b}_{j+}^h,t))\ \psi^h(\overline{b}_j^h,t)\Big).
\end{aligned}$$

Inserting inside the summation above the expressions

$$h\dot{u}^h(\overline{b}_j^h,t) = \phi'(u_x^h(\overline{b}_{j+}^h,t)) - \phi'(u_x^h(\overline{b}_{j-}^h,t)),$$

we obtain

$$\begin{aligned}
\mathrm{I}^h(t) &= -h\sum_{i=1}^N \dot{u}_i^h\ \psi_i^h \\
&\quad + \sum_{j\in\mathfrak{I}^h(t)} \Big[\phi'(u_x^h(\overline{a}_{j+}^h,t))\ \psi^h(\overline{a}_j^h,t) - \phi'(u_x^h(\overline{b}_{j-}^h,t))\ \psi^h(\overline{b}_j^h,t)\Big] \\
&:= \mathrm{I}_1^h(t) + \mathrm{I}_2^h(t)
\end{aligned}$$

(note the sum over the whole set of indices, and the presence of $\overline{b}_{j-}^h$ inside the parentheses). By (4.12) we have $\phi'(u_x^h(\overline{a}_{j+}^h,t)) = \phi'(u_x^h(\overline{b}_{j-}^h,t))$, and $|\psi^h(\overline{a}_j^h,t) - \psi^h(\overline{b}_j^h,t)| \le \overline{c}\, h$. Therefore, using the boundedness of ϕ' it follows

$$|\mathrm{I}_2^h(t)| \le \overline{c}\ h. \tag{4.14}$$

Integrating in time, and using an argument similar to the one leading to (4.8), we deduce

$$\lim_{h\to 0}\int_0^{+\infty} \mathrm{I}_1^h(t)\ dt = -\int_0^{+\infty}\int_I u_t\psi\ dxdt. \tag{4.15}$$

From (4.14) and (4.15) it then follows

$$\lim_{h\to 0}\int_0^{+\infty} \mathrm{I}^h(t)\ dt = -\int_{I\times(0,+\infty)} u_t\psi\ dxdt, \tag{4.16}$$

and from Proposition 3.3 and (4.13) we get

$$\lim_{h\to 0} \int_0^{+\infty} \mathrm{I}^h(t)\ dt = \int_{\mathrm{stab}(\overline{u})\times(0,+\infty)} \phi'(u_x)\psi_x\ dxdt = \int_{I\times(0,+\infty)} \phi'(u_x^a)\psi_x\ dxdt. \tag{4.17}$$

Putting together (4.15) and (4.17), we obtain (4.11). □

We point out that results similar to Theorems 4.4 and 4.6 have been independently proved in [10, Corollary 2.11 and Remark 2.12].

Remark 4.7. From (4.11) and the strict monotonicity of ϕ' on $[-1,1]$, it follows that

$$u_x(\overline{a}_{j_+}, t) = u_x(\overline{a}_{j_-}, t) \qquad \text{a.e. } t \in (0,+\infty), \quad j \in \{1,\dots,m\}. \tag{4.18}$$

In particular, condition (4.18) is valid on the "phantom" segments contained in $L(\overline{a}_j)$, i.e., at those points $(\overline{a}_j, t)$ with $t \in (t^*, s)$ in (1.6) (and at the remaining points of $L(\overline{a}_j)$ equality (4.18) is fulfilled, since u satisfies zero Neumann boundary conditions from the left and the right of $\overline{a}_j$).

Remark 4.8. Assume that the phenomenon of attaching-detaching appears, i.e., let $\overline{a}_j$ and t^* be as in (1.6). As a consequence of Proposition 2.3 (d) and the convergence of (u^h) to u, it follows that if $u(\overline{a}_{j_-}, t) < u(\overline{a}_{j_+}, t)$ for $t \in (0, t^*)$, necessarily $u(\overline{a}_{j_-}, \tau) \le u(\overline{a}_{j_+}, \tau)$ for all $\tau \in (t^*, +\infty)$. Similarly, if $u(\overline{a}_{j_-}, t) > u(\overline{a}_{j_+}, t)$ for $t \in (0, t^*)$, necessarily $u(\overline{a}_{j_-}, \tau) \ge u(\overline{a}_{j_+}, \tau)$ for all $\tau \in (t^*, +\infty)$.

5 The limit problem satisfied by u when $\overline{u} \in \mathcal{A}^\phi(I)$

The aim of this section is to provide a notion of solution to (1.2), (1.3) when $\overline{u} \in \mathcal{A}^\phi(I)$ is transcritical.

We first show an example which indicates the *existence of the wrinkling phenomenon* in a very short time scale for an initial datum having an unstable region with nonempty interior. Unlike the rest of the paper, for simplicity in the example we deal with Dirichlet boundary conditions. We use for simplicity the notation $u_i^h(t)$ in place of $u^h(t)_i$.

Example 5.1. Let $\beta > 1$, so that $\phi''(\beta) < 0$. Consider the solution u^h of (2.6) for $i \in \{1,\dots,N-1\}$, with $u^h(0) = 0$ and $u^h(1) = \beta(1-h)$, with transcritical initial condition

$$\overline{u}^h(x) := \begin{cases} 0 & x \in [0,h] \\ \beta(x-h) & x \in [h, 1-h], \\ \beta(1-2h) & x \in [1-h, 1], \end{cases} \tag{5.1}$$

which can be considered as an approximation of a (some sort of) solution of (1.2) and (1.3) with $\overline{u}(x) = \beta x$ under Dirichlet boundary conditions $u(0,t) = 0$ and $u(1,t) = \beta$.

As a consequence of (2.6) and (5.1), we have

$$\begin{cases} \dot{u}_1^h(0^+) = \phi'(\beta)/h > 0, & \dot{u}_{N-1}^h(0^+) = -\dot{u}_1^h(0^+) < 0, \\ \dot{u}_i^h(0^+) = 0 & \text{for } i \in \{2, \dots, N-2\}. \end{cases} \tag{5.2}$$

Similarly, from

$$h^2 \ddot{u}_i^h = \phi'' \left(\frac{u_{i+1}^h - u_i^h}{h} \right) (\dot{u}_{i+1}^h - \dot{u}_i^h) - \phi'' \left(\frac{u_i^h - u_{i-1}^h}{h} \right) (\dot{u}_i^h - \dot{u}_{i-1}^h), \tag{5.3}$$

it follows

$$\begin{cases} \ddot{u}_2^h(0^+) = \phi''(\beta)\dot{u}_1^h(0^+) < 0, & \ddot{u}_{N-2}^h(0^+) = -\ddot{u}_2^h(0^+) > 0, \\ \ddot{u}_i^h(0^+) = 0 & \text{for } i \in \{3, \dots, N-3\}. \end{cases} \tag{5.4}$$

More generally,

$$\begin{cases} \left(\dfrac{d}{dt}\right)^i u_i^h(t)_{|t=0^+} = \dfrac{1}{h^2}\phi''(\beta) \left(\dfrac{d}{dt}\right)^{i-1} u_{i-1}^h(t)_{|t=0^+} \\ \left(\dfrac{d}{dt}\right)^i u_{N-i}^h(t)_{|t=0^+} = -\left(\dfrac{d}{dt}\right)^i u_i^h(t)_{|t=0^+} = \dfrac{1}{h^2}\phi''(\beta) \left(\dfrac{d}{dt}\right)^{i-1} u_{N-i+1}^h(t)_{|t=0^+} \end{cases} \tag{5.5}$$

where, if N is odd, $i \in \{4, \dots, \frac{N-1}{2}\}$ and, if N is even, $i \in \{4, \dots, \frac{N}{2}\}$, and

$$\left(\frac{d}{dt}\right)^j u_i^h(t)_{|t=0^+} = \left(\frac{d}{dt}\right)^j u_{N-i}^h(t)_{|t=0^+} = 0 \qquad \forall\, j \leq i-1.$$

Note that $\left(\frac{d}{dt}\right)^i u_i^h(t)_{|t=0^+}$ has alternate signs (and if N is even $\left(\frac{d}{dt}\right)^{N/2} u_{N/2}^h(t)_{|t=0^+} = 0$). This is an indication of the emergence of the so-called wrinkling phenomenon, see Figures 3, 1.

We now prove our result concerning a notion of global solution to (1.2), (1.3) when $\overline{u}$ is transcritical.

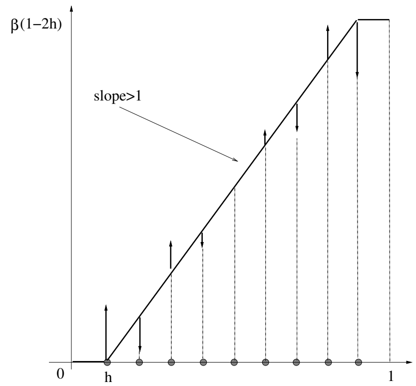

Figure 3: Example 5.1: the i-th arrow starting from the left indicates the initial i-th derivative of the solution; all previous initial derivatives vanish. Note the alternate signs

Theorem 5.2. *Let $\overline{u} \in \mathcal{A}^\phi(I) \setminus \mathcal{P}^\phi(I)$. Then there exist a sequence $(\delta_k) \subset (0,1)$ converging to zero as $k \to +\infty$, and a sequence $(\overline{u}_{\delta_k}) \subset \mathcal{P}^\phi(I)$ with*

$$\sup_k \left(\|\overline{u}_{\delta_k}\|_{BV(I)} + F^\phi(\overline{u}_{\delta_k}) \right) \leq \|\overline{u}\|_{BV(I)} + F^\phi(\overline{u}) < +\infty, \tag{5.6}$$

converging in $L^2(I)$ to $\overline{u}$ as $k \to +\infty$, such that, if we denote by u_{δ_k} the solution described in Theorem 4.6 having initial datum $\overline{u}_{\delta_k}$, then (u_{δ_k}) has a subsequence weakly converging in $\mathcal{X}$ to a function $u \in \mathcal{X}$ which satisfies

$$\begin{cases} u(t) \in \mathcal{A}^\phi(I) \qquad \text{for a.e. } t \in (0,+\infty), \\ \displaystyle\int_{I\times(0,+\infty)} u_t\, \psi \; dx\; dt + \int_{\bigcup_{t\geq 0}\left(\mathrm{stab}(u(t))\times\{t\}\right)} \phi'(u_x)\psi_x \; dx \; dt = 0 \qquad \forall \psi \in \mathcal{C}^1_\#, \\ u(0) = \overline{u}. \end{cases} \tag{5.7}$$

Proof. Denote by $\bigcup_{i\in\mathcal{I}}[\overline{a}_i, \overline{b}_i]$ the union of the closures of the connected components of $\mathrm{unstab}(\overline{u})$ having nonempty interior, namely $\overline{a}_i < \overline{b}_i$ for all $i \in \mathcal{I}$. Define $\overline{u}_\delta := \overline{u}$ in $I \setminus \cup_{i\in\mathcal{I}}[\overline{a}_i, \overline{b}_i]$, and $\overline{u}_\delta$ to be a staircase function of horizontal size δ, $\delta = \frac{\overline{b}_i - \overline{a}_i}{n}$ for some $n \in \mathbb{N}$, see Figure 4. Note that the functions $\overline{u}_\delta$ converge to $\overline{u}$ in $L^2(I)$ as $\delta \to 0$, and

$$\sup_\delta \left(\|\overline{u}_\delta\|_{BV(I)} + F^\phi(\overline{u}_\delta) \right) \leq \|\overline{u}\|_{BV(I)} + F^\phi(\overline{u}) < +\infty. \tag{5.8}$$

Moreover $\overline{u}_\delta \in \mathcal{P}^\phi(I)$, so that we can consider the solution u_δ given by Theorem 4.6 (obtained as a weak limit in $\mathcal{X}$ of u^h_δ as $h \to 0$) in correspondence of the initial datum $\overline{u}_\delta$. Observe also that by (3.5) and (5.8) it follows that the functions u_δ are uniformly bounded in $\mathcal{X}$ with respect to δ, therefore (u_δ) has a (not relabelled) subsequence weakly converging in $\mathcal{X}$ to a function $u \in \mathcal{X}$.

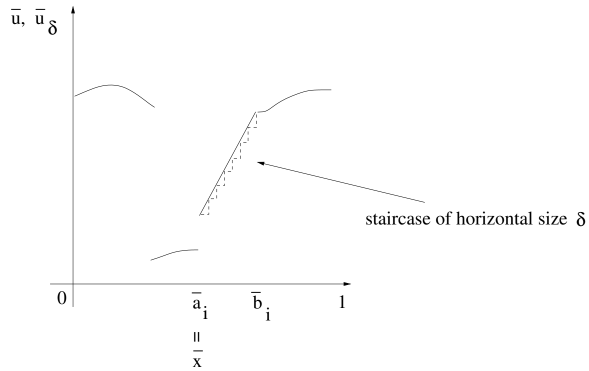

Figure 4: The function $\overline{u}$ and the functions $\overline{u}_\delta$ in Theorem 5.2

For all $x \in J_{\overline{u}_\delta}$ we define $T_\delta(x) > 0$ as

$$T_\delta(x) := \sup \mathcal{J}_\delta(x), \qquad \mathcal{J}_\delta(x) := \{t \in (0, +\infty) : x \in J_{u_\delta(t)}\}. \tag{5.9}$$

The set $\mathcal{J}_\delta(x)$ may fail to be connected, since adjacent branches of $\mathrm{stab}(u_\delta)$ at x can collide during the flow and then detach at some subsequent time. In particular, it may also happen that, at some finite positive time t the point x does not belong to $J_{u_\delta(t)}$ and still $T_\delta(x) = +\infty$.
For any $i \in \mathcal{I}$ let $n_i^\delta \in [1, (\overline{b}_i - \overline{a}_i + \delta)/\delta]$ be the number of jump points of $\overline{u}_\delta$ in $[\overline{a}_i, \overline{b}_i]$, which we denote by

$$\overline{y}_{i,j}^\delta \in [\overline{a}_i, \overline{b}_i], \qquad j \in \{1, \dots, n_i^\delta\}, \tag{5.10}$$

ordered as $\overline{y}_{i,j_1}^\delta < \overline{y}_{i,j_2}^\delta$ if $j_1 < j_2$.
Claim. It is possible to choose a sequence (δ_k) converging to zero as $k \to +\infty$ in such a way that $\mathcal{J}_{\delta_k}(x)$ is connected for any $x \in J_{\overline{u}_{\delta_k}}$ and any $k \in \mathbb{N}$, i.e.

$$x \in J_{\overline{u}_{\delta_k}} \qquad \Longrightarrow \begin{cases} x \in J_{u_{\delta_k}(t)} & \forall t \in (0, T_{\delta_k}(x)), \ \forall k \in \mathbb{N}, \\ x \notin J_{u_{\delta_k}(t)} & \forall t \in (T_{\delta_k}(x), +\infty), \ \forall k \in \mathbb{N}. \end{cases} \tag{5.11}$$

Fix $\delta > 0$. Given $x \in J_{\overline{u}_\delta}$ define

$$t_\delta(x) := \inf \{t \in (0, +\infty) : x \in \mathrm{stab}(u_\delta(t))\},$$

with the usual convention that $\inf \emptyset = +\infty$. We have $0 < t_\delta(x) \le T_\delta(x)$ and we want to show that we can pick $\delta > 0$ in such a way that

$$t_\delta(x) = T_\delta(x) \qquad \forall x \in J_{\overline{u}}. \tag{5.12}$$

If $t_\delta(x) = +\infty$ there is nothing to prove. Therefore, we can assume that $x \in J_{\overline{u}_\delta}$ is such that

$$t_\delta(x) < +\infty. \tag{5.13}$$

Let

$$\mathcal{B} := \{x \in J_{\overline{u}_\delta} : t_\delta(x) \le t_\delta(y) \text{ for all } y \in J_{\overline{u}_\delta}\}.$$

From Remark 4.5 it follows that

$$\mathcal{B} \subset \bigcup_{i \in \mathcal{I}} \{\overline{y}^\delta_{i,1}, \overline{y}^\delta_{i,n_i}\}.$$

Namely, points in $\mathcal{B}$ are among the extremal jump points of $\overline{u}_\delta$ in each $[\overline{a}_i, \overline{b}_i]$. Let us fix the attention on one of the elements $\overline{x} \in [\overline{a}_i, \overline{b}_i]$ of $\mathcal{B}$, and assume without loss of generality that $\overline{x} = \overline{y}^\delta_{i,1}$ and that $\overline{u}_\delta$ is nondecreasing in $[\overline{a}_i, \overline{b}_i]$. Recall from Remark 2.1 that

$$\overline{u}_\delta(\overline{x}_-) < \overline{u}_\delta(\overline{x}_+). \tag{5.14}$$

Hence, using Remark 4.8,

$$u_\delta(\overline{x}_-, t) \le u_\delta(\overline{x}_+, t) \qquad \forall t \ge 0. \tag{5.15}$$

By the choice of $\overline{x}$, we have $u_\delta(t) = \overline{u}_\delta$ in $[\overline{x}, \overline{y}^\delta_{i,n_i}]$ for all $t \in [0, t_\delta(\overline{x})]$, which implies

$$\lim_{x \downarrow \overline{x}} (u_\delta)_t\,(x, t_\delta(\overline{x})) = 0.$$

On the other hand, using (5.13) and provided we slightly reduce $\overline{u}_\delta(\overline{x}_+) - \overline{u}_\delta(\overline{x}_-)$, we may assume that the collision of the two branches happens with nonzero velocity, namely

$$\lim_{x \uparrow \overline{x}} (u_\delta)_t\,(x, t_\delta(\overline{x})) > 0 \tag{5.16}$$

(the limit in (5.16) exists by the smoothness up to the boundary of the solution of (4.3)) Therefore, at the time $t_\delta(\overline{x})$, we have the collision of the flowing branch of the graph of $u_\delta(t)$ on the left of $\overline{x}$ (coming from below), with the horizontal segment on the right; such a collision takes place with nonzero relative velocity. Moreover, recalling (1.5) and (5.16), in a left neighbourhood of $\overline{x}$ we have that

$$\text{the function } u_\delta(\cdot, t_\delta(\overline{x})) \text{ is uniformly convex.} \tag{5.17}$$

Indeed, if we consider (on the left of $\overline{x}$) the solution v of the equation in (4.3) with zero Neumann boundary conditions, we have that $v(t) = u_\delta(t)$ for $t < t_{\delta(\overline{x})}$, and therefore by continuity $v(t) = u_\delta(t)$ also at $t = t_\delta(\overline{x})$.
Define

$$u^* := u_\delta(\overline{x}, t_\delta(\overline{x})).$$

We divide the proof of (5.12) into two steps.
Step 1. There exists a sequence (s_n) of times, with $s_n \downarrow t_\delta(\overline{x})$, such that

$$(u_\delta)_x(\cdot, s_n) < 0 \qquad \text{somewhere in } \left(\overline{x}, \overline{y}^\delta_{i,2}\right), \quad \forall n \in \mathbb{N}. \tag{5.18}$$

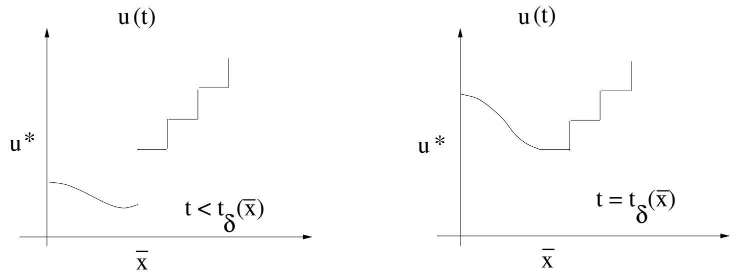

Figure 5: A collision at time $t_\delta(\overline{x})$. The branch on the left of $\overline{x}$ has positive vertical velocity at $\overline{x}$ at the time $t_\delta(\overline{x})$

To prove such an assertion we distinguish two cases.
Case 1. There exists a sequence (s_n) with $s_n \downarrow t_\delta(\overline{x})$ such that

$$u_\delta(\overline{x}_-, s_n) > u^*. \tag{5.19}$$

In this case, there exists a sequence $(\widetilde{s}_n)$ with $\widetilde{s}_n \in (t_\delta(\overline{x}), s_n]$ which satisfies the assertion of step 1. Indeed, if not, the solution is nondecreasing in $(\overline{x}, \overline{y}^\delta_{i,2})$, hence it assumes its maximum at the right extremum $\overline{y}^\delta_{i,2}$. We can now use the maximum principle for the solution u_δ (in the interval $(\overline{x}, \overline{y}^\delta_{i,2})$) of the forward strictly parabolic equation in (4.3) with zero Neumann condition at the right extremum $\overline{y}^\delta_{i,2}$ and deduce, being u^* a solution of the problem,

$$u_\delta(\cdot, s_n) \le u_\delta(\cdot, t_\delta(\overline{x})) = \overline{u}_\delta(\cdot) = u^* \qquad \text{in } \left(\overline{x}, \overline{y}^\delta_{i,2}\right). \tag{5.20}$$

On the other hand from (5.15) and (5.19) we have

$$u_\delta(\overline{x}_+, s_n) \ge u_\delta(\overline{x}_-, s_n) > u^*,$$

which contradicts (5.20).
In the remaining case we concentrate on a left neighbourhood of $\overline{x}$.
Case 2. There exists $\tau > 0$ such that

$$u_\delta(\overline{x}_-, s) \le u^* \qquad \forall s \in [t_\delta(\overline{x}), t_\delta(\overline{x}) + \tau]. \tag{5.21}$$

In this case, we have

$$\left(u_\delta\right)_x(\overline{x}_-, t) < 0 \quad \text{for a set of times } t \in [t_\delta(\overline{x}), t_\delta(\overline{x}) + \tau] \text{ of positive measure.} \tag{5.22}$$

Indeed, assume by contradiction that

$$\left(u_\delta\right)_x(\overline{x}_-, t) \ge 0 \quad \text{for a.e. } t \in (t_\delta(\overline{x}), t_\delta(\overline{x}) + \tau). \tag{5.23}$$

Recalling also (5.17) we can choose
a left neighbourhood U of $\overline{x}$,
a time $\widetilde{\tau} \in (0, \tau)$,
a function ϑ defined in $U \times [t_\delta(\overline{x}), t_\delta(\overline{x}) + \widetilde{\tau}]$ having the following properties:
$\vartheta(\cdot, t_\delta(\overline{x})) < u_\delta(\cdot, t_\delta(\overline{x}))$ in $U \times \{t_\delta(\overline{x})\}$ and $\vartheta(\overline{x}_-, t_\delta(\overline{x})) = u_\delta(\overline{x}, t_\delta(\overline{x}))$,
ϑ is a smooth (sub)solution of (1.2) in $U \times [t_\delta(\overline{x}), t_\delta(\overline{x}) + \widetilde{\tau}]$, has zero Neumann boundary condition at $\{\overline{x}\} \times [t_\delta(\overline{x}), t_\delta(\overline{x}) + \widetilde{\tau}]$ and translates with positive vertical velocity of the order $\frac{1}{2} \lim_{x \uparrow \overline{x}} (u_\delta)_t(x, t_\delta(\overline{x}))$.
By (5.23) and the maximum principle it follows that $u_\delta(\overline{x}_-, s) \geq \theta(\overline{x}_-, s) > u^*$ for all $s \in (t_\delta(\overline{x}), t_\delta(\overline{x}) + \tau']$ for some $\tau' \in (0, \widetilde{\tau})$, which falsifies (5.21). Hence (5.22) is proved.

Recalling from Remark 4.7 that $(u_\delta)_x(\overline{x}_-, t) = (u_\delta)_x(\overline{x}_+, t)$ for almost every $t \in (t_\delta(\overline{x}), t_\delta(\overline{x}) + \tau)$, it follows that necessarily (5.22) holds also with $\overline{x}_-$ replaced by $\overline{x}_+$. This implies (5.18).

Step 2. Conclusion of the proof of (5.12).
From step 1 and the weak convergence in $\mathcal{X}$ of (u^h_δ) to u_δ, it follows that the functions $u^h_\delta(s_n)$ are strictly decreasing somewhere on $\left(\overline{x}, \overline{y}^\delta_{i,2}\right)$ for h small enough (possibly depending on n).
We now claim that

$$\overline{x} \in \operatorname{stab}\left(u^h_\delta(s_n)\right). \tag{5.24}$$

To prove the claim, assume that

$$\{\overline{x}, \overline{y}^\delta_{i,2}\} \subset \operatorname{unstab}\left(u^h_\delta(s)\right) \qquad \forall s \in (0, s_n). \tag{5.25}$$

Then u^h_δ is monotone nondecreasing in $(\overline{x}, \overline{y}^\delta_{i,2})$ for all $s \in (0, s_n)$. Indeed, if this is false, there is $\overline{s} \in (0, s_n)$ such that $u^h_\delta(s)$ is strictly decreasing somewhere in $(\overline{x}, \overline{y}^\delta_{i,2})$, hence it has is a minimum point in $(\overline{x}, \overline{y}^\delta_{i,2})$, which contradicts Proposition 2.3 (b).
It follows that (5.25) is false, and therefore either $\overline{x} \in \operatorname{stab}(u^h_\delta(s_n))$ or $\overline{y}^\delta_{i,2} \in \operatorname{stab}(u^h_\delta(s_n))$. Since the first point which disappears among $\overline{x}, \overline{y}^\delta_{i,2}$ is $\overline{x}$, (5.24) is proved.
Therefore, by Proposition 2.3 (d), we have $\overline{x} \in \operatorname{stab}\left(u^h_\delta(s)\right)$ for all $s \geq s_n$, which in turn implies $\overline{x} \in \operatorname{stab}(u_\delta(s))$ for all $s > t_\delta(\overline{x})$, hence (5.12) holds with $x = \overline{x}$. By iterating the previous argument, we can choose a sequence (δ_k) converging to 0 such that all jumps of $\overline{u}_{\delta_k}$ have mutual distance of order δ_k in each interval $[\bar{a}_i, \bar{b}_i]$, and whenever a jump of u_{δ_k} disappears, the collision of the two branches happens with nonzero velocity. By a diagonal procedure, we get (5.11) and the proof of the claim is concluded.

Define

$$T^{\delta_k}_{i,j} := T_{\delta_k}(\overline{y}_{i,j}).$$

Then for any $i \in \mathcal{I}$ there exists two indices $\underline{\ell}^{\delta_k}_i, \overline{\ell}^{\delta_k}_i \in \{1, \ldots, n^{\delta_k}_i\}$, with $\underline{\ell}^{\delta_k}_i \leq \overline{\ell}^{\delta_k}_i$, such that $T^{\delta_k}_{i,j}$ is strictly increasing and finite in j for $j \in \{1, \ldots, \underline{\ell}^{\delta_k}_i\}$, $T^{\delta_k}_{i,j}$ is constantly equal to $+\infty$ for $j \in \{\underline{\ell}^{\delta_k}_i + 1, \ldots, \overline{\ell}^{\delta_k}_i - 1\}$, and $T^{\delta_k}_{i,j}$ is strictly decreasing and finite in j for $j \in \{\overline{\ell}^{\delta_k}_i, \ldots, n^{\delta_k}_i\}$. See Figure 6.

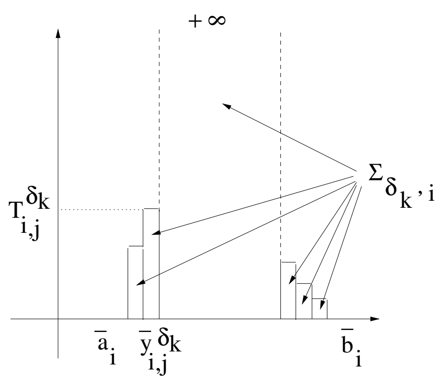

Figure 6: Definition of $\Sigma_{\delta_k,i}$ in (5.26)

Up to a subsequence, denote by $\underline{c}_i \in [\overline{a}_i, \overline{b}_i]$ (resp. $\overline{c}_i \in [\overline{a}_i, \overline{b}_i]$) the limit of the points $h\underline{\ell}_i^{\delta_k}$ (resp. $h\overline{\ell}_i^{\delta_k}$) as $h \to 0$.
Define

$$\Sigma_{\delta_k,i} := \bigcup_{j=1}^{n_i^{\delta_k}-1} [\overline{y}_{i,j}^{\delta_k}, \overline{y}_{i,j+1}^{\delta_k}] \times [0, \min(T_{i,j}^{\delta_k}, T_{i,j+1}^{\delta_k})), \qquad \Sigma_{\delta_k} := \bigcup_{i\in\mathcal{I}} \Sigma_{\delta_k,i}. \tag{5.26}$$

Note that $\min(T_{i,j}^{\delta_k}, T_{i,j+1}^{\delta_k}) = T_{i,j}^{\delta_k}$ (resp. $= T_{i,j+1}^{\delta_k}$) for $j \in \{1, \ldots, \underline{\ell}_i^{\delta_k}\}$ (resp. for $j \in \{\overline{\ell}_i^{\delta_k}, \ldots, n_i^{\delta_k}\}$.
For $i \in \mathcal{I}$ we also define the function $\varphi_{\delta_k,i} : I \to [0, +\infty]$ as

$$\varphi_{\delta_k,i}(x) := \begin{cases} \min(T_{i,j}^{\delta_k}, T_{i,j+1}^{\delta_k}) & \text{if } x \in [\overline{y}_{i,j}^{\delta_k}, \overline{y}_{i,j+1}^{\delta_k}] \\ 0 & \text{otherwise.} \end{cases}$$

From the monotonicity properties of $T_{i,j}^{\delta_k}$ it follows that the function $\varphi_{\delta_k,i}(\cdot)$ is nondecreasing in $[\overline{y}_{i,1}^{\delta_k}, \underline{\ell}_i^{\delta_k}]$, nonincreasing in $[\overline{\ell}_i^{\delta_k}, \overline{y}_{i,n_i^{\delta_k}}^{\delta_k}]$, and constantly equal to 0 on $I \setminus [\overline{y}_{i,1}^{\delta_k}, \overline{y}_{i,n_i^{\delta_k}}^{\delta_k}]$. As a consequence, possible passing to a subsequence, for all $i \in \mathcal{I}$ the functions $\varphi_{\delta_k,i}$ converge pointwise as $k \to +\infty$ to a nonnegative function φ_i, with $\varphi_{i|(\overline{a}_i,\underline{c}_i)} \in BV_{\rm loc}((\overline{a}_i, \underline{c}_i))$ and $\varphi_{i|(\overline{c}_i,\overline{b}_i)} \in BV_{\rm loc}((\overline{c}_i, \overline{b}_i))$, and with the property that φ_i is nondecreasing in $(\overline{a}_i, \underline{c}_i)$, identically equal to $+\infty$ in $(\underline{c}_i, \overline{c}_i)$, and nonincreasing in $(\overline{c}_i, \overline{b}_i)$. Moreover, each set $\Sigma_{\delta_k,i}$ converges in the $L^1_{\rm loc}$ topology to

$$\Sigma_i := \{(x,t) \in [\overline{a}_i, \overline{b}_i] \times [0, +\infty) : 0 \le t < \varphi_i(x)\}$$

as $k \to +\infty$.

We are now in the position to conclude the proof of the theorem. By definition, we have

$$(u_{\delta_k})_x = (u_{\delta_k})_t = 0 \qquad \text{a.e. in } \Sigma_{\delta_k}. \tag{5.27}$$

It follows that $u_{\delta_k} = \overline{u}_{\delta_k}$ in Σ_{δ_k}, which implies

$$u = \overline{u} \qquad \text{in } \Sigma := \bigcup_{i=1}^{m} \Sigma_i \tag{5.28}$$

(where $\overline{u}(x,t) := \overline{u}(x)$ for any $t \geq 0$). Moreover

$$\Sigma \subseteq \Big(\bigcup_{i \in \mathcal{I}} [\overline{a}_i, \overline{b}_i]\Big) \times [0,+\infty) \subseteq \mathrm{unstab}(\overline{u}) \times [0,+\infty). \tag{5.29}$$

From (5.28) and (5.29) it follows

$$\Sigma \subseteq \bigcup_{t \geq 0} \big(\mathrm{unstab}(u(t)) \times \{t\}\big). \tag{5.30}$$

On the other hand the fact that u_{δ_k} is one-Lipschitz in $(I \times [0,+\infty)) \setminus \Sigma_{\delta_k}$ implies that u is one-Lipschitz in $\big(I \times [0,+\infty)\big) \setminus \overline{\Sigma}$, i.e.,

$$\bigcup_{t \geq 0} \big(\mathrm{stab}(u(t)) \times \{t\}\big) \subseteq \big(I \times [0,+\infty)\big) \setminus \overline{\Sigma}.$$

Hence $\bigcup_{t \geq 0} \big(\mathrm{unstab}(u(t)) \times \{t\}\big) \subseteq \overline{\Sigma}$. Therefore from (5.30)

$$\Sigma \subseteq \bigcup_{t \geq 0} \big(\mathrm{unstab}(u(t)) \times \{t\}\big) \subseteq \overline{\Sigma}. \tag{5.31}$$

Note that the particular form of the functions φ_i implies that $\overline{\Sigma} \setminus \Sigma$ has zero Lebesgue measure. Hence from the inclusions in (5.31) it follows that the three sets Σ, $\overline{\Sigma}$, and $\bigcup_{t \geq 0} \big(\mathrm{unstab}(u(t)) \times \{t\}\big)$ have the same Lebesgue measure.
Recalling (4.11) and (5.27), for all $\psi \in \mathcal{C}^1_{\#}$ we have

$$\int_{I \times (0,+\infty)} (u_{\delta_k})_t \ \psi \ dx \ dt + \int_{\big(\bigcup_{t \geq 0} (\mathrm{stab}(u_{\delta_k}(t)) \times \{t\})\big) \setminus \Sigma_{\delta_k}} \phi' \left((u_{\delta_k})_x\right) \psi_x dx \ dt = 0\,. \tag{5.32}$$

In view of the weak convergence in $\mathcal{X}$ of the sequence (u_{δ_k}) to u, we have

$$\lim_{k \to +\infty} \int_{I \times (0,+\infty)} (u_{\delta_k})_t \ \psi \ dxdt = \int_{I \times (0,+\infty)} u_t \psi \ dxdt. \tag{5.33}$$

It remains to pass to the limit as $k \to +\infty$ in the second addendum on the right hand side of (5.32). Since $u_{\delta_k}(t) \in \mathcal{P}^{\phi}(I)$ for any $t \geq 0$, it follows that

$$\big(I \times (0,+\infty)\big) \setminus \bigcup_{t \geq 0} \big(\mathrm{stab}(u_{\delta_k}(t)) \times \{t\}\big)$$

has zero Lebesgue measure. Hence $\left(\bigcup_{t\geq 0}\mathrm{stab}(u_{\delta_k}(t))\times\{t\}\right)\setminus\Sigma_{\delta_k}$ converges in $L^1_{\mathrm{loc}}(I\times(0,+\infty))$ to $(I\times(0,+\infty))\setminus\Sigma$, which has the same Lebesgue measure of $(I\times(0,+\infty))\setminus\cup_{t\geq 0}(\mathrm{unstab}(u(t))\times\{t\})$.
We deduce that

$$\left(\bigcup_{t\geq 0}\mathrm{stab}(u_{\delta_k}(t))\times\{t\}\right)\setminus\Sigma_{\delta_k} \text{ converges in } L^1_{\mathrm{loc}}(I\times+\infty) \text{ to } \bigcup_{t\geq 0}\left(\mathrm{stab}(u(t))\times\{t\}\right).$$

Observe that if R is a compact rectangle with $R\subset\subset\bigcup_{t\geq 0}(\mathrm{stab}(u(t))\times\{t\})$, then R is contained in $\left(\bigcup_{t\geq 0}\mathrm{stab}(u_{\delta_k}(t))\times\{t\}\right)\setminus\Sigma_{\delta_k}$, for k large enough. Reasoning as in Corollary 3.4, and recalling Remarks 3.5 and 3.6, we get that the functions $\phi'\left((u_{\delta_k})_x\right)$ are uniformly bounded in $H^1_{\mathrm{loc}}\left(\bigcup_{t\geq 0}\mathrm{stab}(u(t))\times\{t\}\right)$ with respect to k, which implies that, passing to a further subsequence, $(u_{\delta_k})_x\to u_x$ almost everywhere in $\bigcup_{t\geq 0}(\mathrm{stab}(u(t))\times\{t\})$. Passing to the limit in (5.32) as $k\to+\infty$, and using the Dominated Convergence Theorem, we obtain that

$$\lim_{k\to+\infty}\int_{\left(\bigcup_{t\geq 0}(\mathrm{stab}(u_{\delta_k}(t))\times\{t\})\right)\setminus\Sigma_{\delta_k}}\phi'\left((u_{\delta_k})_x\right)\psi_x dx\ dt=\int_{\bigcup_{t\geq 0}(\mathrm{stab}(u(t))\times\{t\})}\phi'\left(u_x\right)\psi_x dx\ dt. \tag{5.34}$$

Then (5.7) follows from (5.33) and (5.34). □

Note that, thanks to the inclusion $u\in AC^2(0,+\infty);L^2(I))$, it is not possible that the functions φ_i vanish identically.

6 Convergence of the space-time discrete scheme

The space-time discretization of (1.2) can be obtained by using the minimizing movements method (see [2]), which is a generalization of the usual implicit Euler scheme. We apply the method to F_h (extended to $+\infty$ in $L^2(I)\setminus PL_h(I)$), therefore we define $G:(1,+\infty)\times L^2(I)\times L^2(I)\to[0,+\infty]$ as

$$G(\lambda,v,w):=F_h(v)+\frac{\lambda}{2}\int_I|v-w|^2\ dx, \tag{6.1}$$

where $\lambda>1$ corresponds to the inverse of the time step. Given $h\in(0,1)$, consider a function $w^h:(1,+\infty)\times\mathbb{N}\to PL_h(I)$ such that $w^h(\lambda,0)=\overline{u}^h$ for any $\lambda\in(1,+\infty)$, and take

$$w^h(\lambda,k+1)\in\mathrm{argmin}\left\{G\left(\lambda,\cdot,w^h(\lambda,k)\right)\right\},\qquad \lambda\in(1,+\infty),\ k\in\mathbb{N}. \tag{6.2}$$

Minimizers w^h of $G(\lambda,\cdot,w^h(\lambda,k))$ in (6.2) in general are not unique, however they satisfy the Euler-Lagrange equation

$$\frac{1}{h}\frac{\partial F_h}{\partial u_i}(w^h(\lambda,k+1))+\lambda(w^h(\lambda,k+1)-w^h(\lambda,k))_i=0,\qquad i\in\{1,\ldots,N\}. \tag{6.3}$$

We denote by

$$u^{h,\lambda}(t) := (\lfloor \lambda t\rfloor + 1 - \lambda t)w^h(\lambda, \lfloor \lambda t\rfloor) + (\lambda t - \lfloor \lambda t\rfloor)w^h(\lambda, \lfloor \lambda t\rfloor + 1) \qquad (6.4)$$

the piecewise linear time interpolation between $w^h(\lambda, \lfloor \lambda t\rfloor)$ and $w^h(\lambda, \lfloor \lambda t\rfloor + 1)$. Equalities (6.3) and (6.4) imply that, up to a countable set of times, $u^{h,\lambda}$ is a solution of

$$\begin{cases} \dot u^{h,\lambda}(t)_i = -\lambda\big(w^h(\lambda, \lfloor \lambda t\rfloor) + w^h(\lambda, \lfloor \lambda t\rfloor + 1)\big)_i = -\dfrac{1}{h}\dfrac{\partial F_h}{\partial u_i}(u^{h,\lambda}(t + s_\lambda(t))) \\ \qquad = (D_h^+(\phi'(D_h^- u^{h,\lambda}(t + s_\lambda(t)))))_i \qquad \text{for } i \in \{1, \dots, N\}, \\ u^{h,\lambda}(0) = \overline{u}^h, \end{cases} \qquad (6.5)$$

where $t + s_\lambda(t) = (\lfloor \lambda t\rfloor + 1)/\lambda \in [t, t + 1/\lambda]$, and therefore the Borel function s_λ satisfies

$$s_\lambda(t) \in [0, 1/\lambda] \qquad \forall t \geq 0. \qquad (6.6)$$

The next result says that if the time step $1/\lambda$ goes to zero sufficiently fast with respect to the mesh size h, then the space-time discretized solutions of (1.2) converge to the function u in (3.6).

Proposition 6.1. *Let $\overline{u} \in \mathcal{P}^\phi(I)$ and let u be as in (3.6). Let $(\lambda_h) \subset (1, +\infty)$ be a sequence satisfying*

$$\lambda_h \geq e^{1/h^\alpha} \quad \text{for any } \alpha > 2. \qquad (6.7)$$

Let u^{h,λ_h} be defined as in (6.4). Then

$$\forall T > 0 \qquad \lim_{h\to 0} u^{h,\lambda_h} = u \quad \text{in} \quad L^\infty((0,T); L^2(I)). \qquad (6.8)$$

Proof. Define

$$e(t) := \|u^{h,\lambda_h}(t) - u^h(t)\|^2_{L^2(I)} \qquad t \geq 0.$$

From (2.6) and the boundedness of ϕ' it follows

$$|(D_h^+(\phi'(D_h^- u^h)))_i| \leq \overline{c}\, h^{-1}, \qquad i \in \{1, \dots, N\}. \qquad (6.9)$$

Moreover,

$$|\frac{\partial}{\partial u_j}(D_h^+(\phi'(D_h^-(u^h))))_i| \leq \overline{c}\, h^{-2}, \qquad i, j \in \{1, \dots, N\}. \qquad (6.10)$$

From (6.10) we obtain, up to a countable set of times,

$$\begin{aligned} \frac{\dot e(t)}{2} &= \langle u^h(t) - u^{h,\lambda_h}(t), D_h^+\phi'(D_h^- u^h(t)) - D_h^+(\phi'(D_h^- u^{h,\lambda_h}(t + s_{\lambda_h}(t))))\rangle_{L^2(I)} \\ &\leq \overline{c}\, h^{-2}\, \|u^h(t) - u^{h,\lambda_h}(t)\|_{L^2(I)}\, \|u^h(t) - u^{h,\lambda_h}(t + s_{\lambda_h}(t))\|_{L^2(I)} \\ &\leq \overline{c}\, h^{-2} e(t) + \overline{c}\, h^{-2}\sqrt{e(t)}\|u^{h,\lambda_h}(t) - u^{h,\lambda_h}(t + s_{\lambda_h}(t))\|_{L^2(I)}. \end{aligned}$$

Therefore, using (6.6) and (6.9) we have

$$\frac{\dot{e}(t)}{2} \leq \overline{c}\; h^{-2} e(t) + \overline{c}^2 h^{-3} \lambda_h^{-1} \sqrt{e(t)} \leq \overline{c}\; h^{-2} \sqrt{e(t)} \big(\sqrt{e(t)} + h^{-1} \lambda_h^{-1} \big).$$

Solving the Cauchy problem

$$\frac{\dot{f}}{\overline{c}\; h^{-2} \sqrt{f} \big(\sqrt{f} + h^{-1} \lambda_h^{-1} \big)} = 1, \qquad f(0) = 0,$$

we obtain $f(t) = \frac{1}{h^2 \lambda_h^2} \big(e^{\frac{\overline{c}\; h^{-2} t}{2}} - 1 \big)^2$. Then $\lim_{h \to 0} \sup_{t \in [0,T]} e(t) \leq \lim_{h \to 0} \sup_{t \in [0,T]} f(t)$, and the latter limit is zero provided (6.7) holds, and (6.8) follows. □

On the other hand, if the time step $1/\lambda$ goes to zero too slowly with respect to the mesh size h, then the full discretized solutions converge to the standing solution $\overline{u}$, as proved in the next proposition, where we allow the initial datum also to have an unstable set with nonempty interior.

Proposition 6.2. *Let $\overline{u} \in \mathcal{A}^\phi(I)$, with $\overline{u}$ of class $\mathcal{C}^1$ in the closure of each connected component of* stab$(\overline{u})$. *Let $(\lambda_h) \subset (1, +\infty)$ be a sequence converging to $+\infty$ as $h \to 0$ satisfying*

$$\lambda_h \leq h^{-\gamma} \quad \text{for some } \gamma \in (0, 1/2). \tag{6.11}$$

Then

$$\forall T > 0 \qquad \lim_{h \to 0} u^{h, \lambda_h} = \overline{u} \quad \text{in } L^\infty((0,T); L^2(I)).$$

Proof. Let $\gamma \in (0, 1/2)$ and fix $\alpha \in (\gamma, 1)$. We first show that

$$G(\lambda_h, w^h(\lambda_h, 1), \overline{u}) \leq \overline{c} \left(h^{1-\alpha} \log(1/h) + \lambda_h h^{2\alpha} \right). \tag{6.12}$$

Let us consider a piecewise linear function $\overline{v}_h$, which approximates $\overline{u}$ in $L^2(I)$, having alternatively

$$\text{slope} \;\; \beta_h = \mathcal{O}(h^{1-\alpha}) \quad \text{on an interval of length } h^\alpha,$$

and

$$\text{slope} \;\; \beta_h^* = \mathcal{O}(h^{\alpha - 1}) \quad \text{on an interval of length } h,$$

see Figures 7, 8.
Notice that

$$\phi'(\beta_h) = \mathcal{O}(h^{\alpha-1}), \qquad \phi'(\beta_h^*) = \mathcal{O}(h^{\alpha-1}), \tag{6.13}$$

and

$$h^{-\alpha} \left(h \beta_h^* + h^\alpha \beta_h \right) = \mathcal{O}(1). \tag{6.14}$$

Moreover, from (2.1) and the expression of ϕ it follows

$$F_h(\overline{v}_h) = \mathcal{O}\Big(h^{-\alpha} \big(h \phi(\beta_h^*) + h^\alpha \phi(\beta_h) \big) \Big) = \mathcal{O}\left(h^{1-\alpha} \log(1/h) \right). \tag{6.15}$$

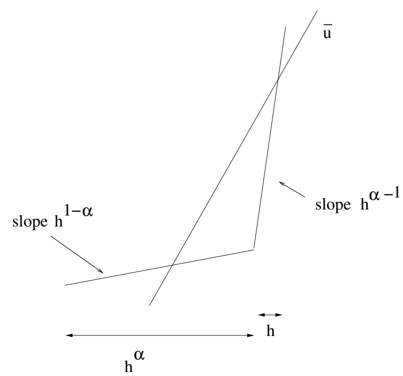

Figure 7: The local shape of the function $\overline{v}_h$. The function $\overline{u}$ is linear of slope larger than one

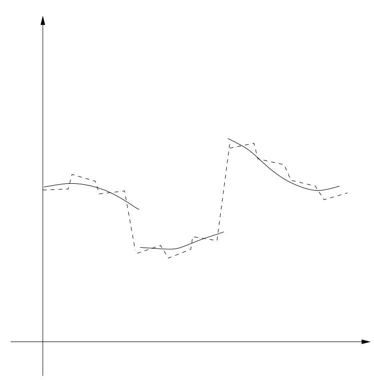

Figure 8: The function $\overline{u}$ (solid curves) and the functions $\overline{v}_h$ (dashed segments)

By the assumptions on $\overline{u}$ we also get

$$\|\overline{v}_h - \overline{u}\|^2_{L^2(I)} = \mathcal{O}(h^{2\alpha}). \tag{6.16}$$

Then (6.12) follows from (6.15) and (6.16), since

$$G(\lambda_h, w^h(\lambda_h, 1), \overline{u}) \leq G(\lambda_h, \overline{v}_h, \overline{u}).$$

Note that the right hand side on (6.12) converges to 0 as $h \to 0$, since $\lambda_h \leq h^{-\gamma}$ and $\gamma < 2\alpha$.
Since the function $k \to F_h(w^h(\lambda_h, k))$ is nonincreasing, from (6.12) we get

$$F_h(w^h(\lambda_h, k)) \leq \overline{c}\left(h^{1-\alpha}\log(1/h) + \lambda_h h^{2\alpha}\right), \tag{6.17}$$

hence, using also (6.16),

$$\begin{aligned} G(\lambda_h, w^h(\lambda_h, k+1), w^h(\lambda_h, k)) &\leq G(\lambda_h, w^h(\lambda_h, k), w^h(\lambda_h, k)) \\ &\leq \overline{c}\left(h^{1-\alpha}\log(1/h) + \lambda_h h^{2\alpha}\right), \end{aligned} \tag{6.18}$$

for all $k \in \mathbb{N}$. Recalling (6.4), from (6.17) and (6.18) we obtain

$$\begin{aligned} \|u^{h,\lambda_h}(t) - \overline{u}\|_{L^2(I)} &\leq \sum_{k=0}^{\lfloor \lambda_h t \rfloor} \|w^h(\lambda_h, k+1) - w^h(\lambda_h, k)\|_{L^2(I)} \\ &\leq \sum_{k=0}^{\lfloor \lambda_h t \rfloor} \sqrt{\frac{2G(\lambda_h, w^h(\lambda_h, k+1), w^h(\lambda_h, k))}{\lambda_h}} \\ &\leq \overline{c}\, t\, \sqrt{\lambda_h h^{1-\alpha}\log(1/h) + \lambda_h^2 h^{2\alpha}}, \end{aligned} \tag{6.19}$$

which converges to 0 provided that $\gamma < \min\{1-\alpha, \alpha\}$. The thesis now follows taking $\alpha = 1/2$. $\square$

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