INFINITE PATHS AND CLIQUES IN RANDOM GRAPHS

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ABSTRACT. We study the thresholds for the emergence of various properties in random subgraphs of \((\mathbb{N},<)\). In particular, we give sharp sufficient conditions for the existence of (finite or infinite) cliques and paths in a random subgraph. No specific assumption on the probability, such as independency, is made. The main tools are a topological version of Ramsey theory, exchangeability theory and elementary ergodic theory.

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1. INTRODUCTION

Let \(G = (\mathbb{N}, \mathbb{N}^{(2)})\) be the directed graph over \(\mathbb{N}\) with set of edges \(\mathbb{N}^{(2)} := \{(i,j) \in \mathbb{N}^2 : i < j\}\). Let us randomly choose some of the edges of \(G\): that is, we associate to the edge \((i,j)\) a measurable set \(X_{i,j} \subseteq \Omega\), where \((\Omega, \mathcal{A}, \mu)\) is a base probability space. Assuming \(\mu(X_{i,j}) \geq \lambda\) for each \((i,j)\), we then ask whether the resulting random subgraph \(X\) of \((\mathbb{N}, \mathbb{N}^{(2)})\) contains an infinite path:

**Problem 1.** Let \((\Omega, \mathcal{A}, \mu)\) be a probability space. Let \(\lambda > 0\) and for all \((i,j) \in \mathbb{N}^{(2)}\), let \(X_{i,j}\) be a measurable subset of \(\Omega\) with \(\mu(X_{i,j}) \geq \lambda\). Is there an infinite increasing sequence \(\{n_i\}_{i \in \mathbb{N}}\) such that \(\bigcap_{n \geq n_i \in \mathbb{N}} X_{n_i, n_i+1}\) is non-empty?

More formally, a random subgraph \(X\) of a directed graph \(G = (V_G, E_G)\) (with set of edges \(E_G \subset V_G \times V_G\)), is a measurable function \(X : \Omega \to 2^{E_G}\) where \(\Omega = (\Omega, \mathcal{A}, \mu)\) is a probability space, and \(2^{E_G}\) is the powerset of \(E_G\), identified with the set of all functions from \(E_G\) to \(\{0,1\}\) (with the product topology and the \(\sigma\)-algebra of its Borel sets). For each \(x \in \Omega\), we identify \(X(x)\) with the subgraph of \(G\) with vertices \(V_G\) and edges \(X(x)\). Given \(e \in E_G\), the set \(X_e := \{x \in \Omega \mid e \in X(x)\}\) represents the event that the random graph \(X\) contains the edge \(e \in E_G\). The family \((X_e)_{e \in E_G}\) determines \(X\) putting:
\( \mathcal{X}(x) = \{ e \in E_G \mid x \in \mathcal{X}_e \} \). So a random subgraph of \( G \) can be equivalently defined as a function from \( E_G \) to \( 2^\Omega \) assigning to each \( e \in E_G \) a measurable subset \( \mathcal{X}_e \) of \( \Omega \).

As in classic percolation theory, we wish to estimate the probability that \( \mathcal{X} \) contains an infinite path, in terms of a parameter \( \lambda \) that bounds from below the probability \( \mu(\mathcal{X}_e) \) that an edge \( e \) belongs to \( \mathcal{X} \). Note that it is not a priori obvious that the existence of an infinite path has a well-defined probability, since it corresponds to the uncountable union of the sets \( \bigcap_{i \in \mathbb{N}} \mathcal{X}_{i,k,i,k+1} \) over all strictly increasing sequences \( i : \mathbb{N} \to \mathbb{N} \). However, it turns out that it belongs to the \( \mu \)-completion of the \( \sigma \)-algebra generated by the \( \mathcal{X}_{i,j} \). It has to be noticed that the analogy with classic bond percolation is only formal, the main difference being that in the usual percolation models (see for instance [G:89]) the events \( \mathcal{X}_{i,j} \) are supposed independent, whereas in the present case the probability distribution is completely general, i.e. we do not impose any restriction on the events \( \mathcal{X}_{i,j} \) (and on the probability space \( \Omega \)).

Problem 1 has been originally proposed by P. Erdős and A. Hajnal in [EH:64], and a complete answer was already given by D. H. Fremlin and M. Talagrand in the very interesting paper [FT:85], where other related problems are also considered. In particular, when the probability space \( (\Omega, \mu) \) is the interval \([0,1]\) equipped with the Lebesgue measure, they show that the threshold for the existence of infinite paths is \( \lambda = 1/2 \). One of the main goals of this paper is to present a different method which, in particular, allows us to recover the result of [FT:85]. Our approach is a reduction to the following dual problem.

**Problem 2.** Given a directed graph \( F \), determine the minimal \( \lambda_c \) such that, whenever \( \inf_{e \in \mathbb{N}^2} \mu(\mathcal{X}_e) > \lambda_c \), there is a graph morphism \( f : \mathcal{X}(x) \to F \) for some \( x \in \Omega \).

Problem 1 can be reformulated in this setting by letting \( F \) be the graph \((\omega_1, >)\) where \( \omega_1 \) is the first uncountable ordinal. This depends on the fact that a subgraph \( H \) of \((\mathbb{N}, \mathbb{N}^2)\) does not contain an infinite path if and only if it admits a rank function with values in \( \omega_1 \). Therefore, if a random subgraph \( \mathcal{X} \) of \((\mathbb{N}, \mathbb{N}^2)\) has no infinite paths, it is defined a \( \mu \)-measurable map \( \varphi : \Omega \to \omega_1^N \) where \( \varphi(x)(i) \) is the rank of the vertex \( i \in \mathbb{N} \) in the graph \( \mathcal{X}(x) \). It turns out that \( \varphi^\#(\mu) \) is a compactly supported Borel measure on \( \omega_1^N \), and that \( \varphi(\mathcal{X}_{i,j}) \leq A_{i,j} := \{ x \in \omega_1^N : x_i > x_j \} \). As a consequence, in the determination of the threshold for existence of infinite paths

\[
\lambda_c := \sup \left\{ \inf_{(i,j) \in \mathbb{N}^2} \mu(\mathcal{X}_{i,j}) : \mathcal{X} \text{ random graph without infinite paths} \right\},
\]

we can set \( \Omega = \omega_1^N \), \( \mathcal{X}_{i,j} = A_{i,j} \), and reduce to the variational problem on the convex set \( \mathcal{M}^N_{\mathcal{X}}(\omega_1^N) \) of compactly supported probability measures on \( \omega_1^N \):

\[
\lambda_c = \sup_{m \in \mathcal{M}^N_{\mathcal{X}}(\omega_1^N)} \inf_{(i,j) \in \mathbb{N}^2} m(A_{i,j}).
\]

As a next step, we show that in [1.2] we can equivalently take the supremum in the smaller class of all the compactly supported exchangeable measures on \( \omega_1^N \) (see Appendix B and references therein for a precise definition). Thanks
to this reduction, we can explicitly compute $\lambda_c = 1/2$ (Theorem 4.5). We note that the supremum in (1.2) is not attained, which implies that for $\mu(X_{i,j}) \geq 1/2$ infinite paths occurs with positive probability.

In Section 5 we consider again Problem 2 and we give a complete solution when $F$ is a finite graph, showing in particular that

$$
\lambda_c = c_0(F) := \sup_{\lambda \in \Sigma_F} \sum_{(a,b) \in E_F} \lambda_a \lambda_b,
$$

where $\Sigma_F$ is the set of all sequences $\{\lambda_a\}_{a \in V_F}$ with values in $[0,1]$ and such that $\sum_{a \in V_F} \lambda_a = 1$. By the appropriate choice of $F$ we can determine the thresholds for the existence of paths of a given finite length (Section 3 and Remark 5.2), or for the property of having chromatic number $\geq n$ (Section 6).

We can consider Problems 1 and 2 for a random subgraph $X$ of an arbitrary directed graph $G$, not necessarily equal to $(\mathbb{N}, \mathbb{N}^{(2)})$. However, it can be shown that, if we replace $(\mathbb{N}, \mathbb{N}^{(2)})$ with a finitely branching graph $G$ (such as a finite dimensional network), the probability that $X$ has an infinite path may be zero even if $\inf_{e \in E_G} \mu(X_e)$ is arbitrarily close to 1 (Proposition 4.7). Another variant is to consider subgraphs of $\mathbb{R}^{(2)}$ rather than $\mathbb{N}^{(2)}$ but it turns out that this makes no difference in terms of the threshold for having infinite paths in random subgraphs (Remark 4.8).

In Section 7 we fix again $G = (\mathbb{N}, \mathbb{N}^{(2)})$ and we ask if a random subgraph $X$ of $G$ contains an infinite clique, i.e. a copy of $G$ itself. More generally we consider the following problem.

**Problem 3.** Let $(\Omega, A, \mu)$ be a probability space. Let $\lambda > 0$ and for all $(i_1, \ldots, i_k) \in \mathbb{N}^{(k)}$, let $X_{i_1, \ldots, i_k}$ be a measurable subset of $X$ with $\mu(X_{i_1, \ldots, i_k}) \geq \lambda$. Is there an infinite set $J \subset \mathbb{N}$ such that $\bigcap_{(i_1, \ldots, i_k) \in J} X_{i_1, \ldots, i_k}$ is non-empty?

This problem is a random version of the classical Ramsey theorem [R:28] (we refer to [GP:73, PR:05], and references therein, for various generalization of Ramsey theorem). Clearly Ramsey theorem implies that the answer to Problem 3 is positive when $\Omega$ is finite. Moreover it can be shown that the answer remains positive when $\Omega$ is countable (Example 7.3). However when $\Omega = [0,1]$ (with the Lebesgue measure) the probability that $X$ contains an infinite clique may be zero even when $\inf_{e \in E_G} \mu(X_e)$ is arbitrarily close to 1 (see Example 7.2). We will show that Problem 3 has a positive answer if the indicator functions of the sets $X_{i_1, \ldots, i_k}$ all belong to a compact subset of $L^1(\Omega, \mu)$ (see Theorem 7.5).

Our original motivation for the above problems came from the following situation. Suppose we are given a space $E$ and a certain family $\Omega$ of sequences on $E$ (e.g., minimizing sequences of a functional, or orbits of a discrete dynamical system, etc). A typical, general problem asks for existence of a sequence in the family $\Omega$ that admits a subsequence with a prescribed property. One approach to it is by means of measure theory. The archetypal situation here come from recurrence theorems: one may ask if there exists a subsequence which belongs frequently to a given subset $C$ of the “phase” space $\Omega$ (we refer to such sequences as “$C$-recurrent orbits”). If we
consider the set \( X_i := \{ x \in \Omega : x_i \in C \} \), then a standard sufficient condition for existence of \( C \)-recurrent orbits is \( \mu(X_i) \geq \lambda > 0 \), for some probability measure \( \mu \) on \( \Omega \). In fact is easy to check that the set of \( C \)-recurrent orbits has measure at least \( \lambda \) by an elementary version of a Borel-Cantelli lemma (see Proposition 7.1). This is indeed the existence argument in the Poincaré Recurrence Theorem for measure preserving transformations. A more subtle question arises when one looks for a subsequence satisfying a given relation between two successive (or possibly more) terms: given a sub-
set \( R \) of \( E \times E \) we look for a subsequence \( x_{i_k} \) such that \( (x_{i_k}, x_{i_k+1}) \in R \) for all \( k \in \mathbb{N} \). As before, we may consider the subset of \( \Omega \), with double indices \( i < j \), \( X_{i,j} := \{ x \in \Omega : (x_i, x_j) \in R \} \) and we are then led to Problem 1.

2. Notations

We follow the set-theoretical convention of identifying a natural number \( p \) with the set \( \{0, 1, \ldots, p-1\} \) of its predecessors. More generally an ordinal number \( \alpha \) coincides with the set of its predecessors. With these conventions the set of natural numbers \( \mathbb{N} \) coincides with the least infinite ordinal \( \omega \). As usual \( \omega_1 \) denotes the first uncountable ordinal, namely the set of all countable ordinals.

Given two sets \( X, Y \) we denote by \( X^Y \) the set of all functions from \( Y \) to \( X \). If \( X, Y \) are linearly ordered we denote by \( X^{(Y)} \) the set of all increasing functions from \( Y \) to \( X \). In particular \( \mathbb{N}^{(p)} \) (with \( p \in \mathbb{N} \)) is the set of all increasing \( p \)-tuples from \( \mathbb{N} \), where a \( p \)-tuple \( i = (i_0, \ldots, i_{p-1}) \) is a function \( i : p \to \mathbb{N} \). The case \( p = 2 \), with the obvious identifications, takes the form \( \mathbb{N}^{(2)} = \{ (i, j) : i < j \} \).

Any function \( f : X \to X \) induces a function \( f_* : X^Y \to X^Y \) by \( f_*(u) = f \circ u \). On the other hand a function \( f : Y \to Z \) induces a function \( f^* : X^Z \to X^Y \) by \( f^*(u) = u \circ f \). In particular if \( S : \mathbb{N} \to \mathbb{N} \) is the successor function, \( S^* : X^{\mathbb{N}} \to X^{\mathbb{N}} \) is the shift map.

We let \( \mathcal{E}_r(\mathbb{N}), \text{Inj}(\mathbb{N}), \text{Incr}(\mathbb{N}) \subset \mathbb{N}^{\mathbb{N}} \) be the families of maps \( \sigma : \mathbb{N} \to \mathbb{N} \) which are compactly supported permutations, injective functions and strictly increasing functions, respectively. Note that with the above conventions \( \text{Incr}(\mathbb{N}) = \mathbb{N}^{(\omega)} \).

Given a measurable function \( \psi : X \to Y \) between two measurable spaces and given a measure \( m \) on \( X \), we denote as usual by \( \psi_#(m) \) the induced measure on \( Y \).

Given a compact metric space \( \Lambda \), the space \( \mathcal{M}(\Lambda^N) \) of Borel measures on \( \Lambda^N \) can be identified with \( C(\Lambda^N)^* \), i.e. the dual of the Banach space of all continuous functions on \( \Lambda^N \). By the Banach-Alaoglu theorem the subset \( \mathcal{M}^1(\Lambda^N) \subset \mathcal{M}(\Lambda^N) \) of probability measures is a compact (metrizable) subspace of \( C(\Lambda^N)^* \) endowed with the weak* topology.

Given \( \sigma : \mathbb{N} \to \mathbb{N} \) we have \( \sigma^* : \Lambda^N \to \Lambda^N \) and \( \sigma_#^* : \mathcal{M}^1(\Lambda^N) \to \mathcal{M}^1(\Lambda^N) \). To simplify notations we also write \( \sigma \cdot m \) for \( \sigma \cdot \mu \). Note the contravariance of this action:

\[
\theta \cdot \sigma \cdot m = (\sigma \circ \theta) \cdot m.
\]

Similarly given \( r \in \mathbb{N} \) and \( \iota \in \mathbb{N}^{(r)} \), we have \( \iota^*_# : \mathcal{M}^1(\Lambda^N) \to \mathcal{M}^1(\Lambda^r) \) and we define \( \iota \cdot m = \iota^*_#(m) \).
Given a family $F \subset \mathbb{N}^\mathbb{N}$, we say that $m$ is $F$-invariant if $\sigma \cdot m = m$ for all $\sigma \in F$.

3. Finite paths in random subgraphs

As a preparation for the study of infinite paths (see Problem 1) we consider the case of finite paths. The following example shows that there are random subgraphs $X$ of $(\mathbb{N}, \mathbb{N}(2))$ such that $\inf_{e \in \mathbb{N}(2)} \mu_e$ is arbitrarily close to $1/2$, and yet $X$ has probability zero of having infinite paths.

Example 3.1. Let $p \in \mathbb{N}$ and let $\Omega = p^\mathbb{N}$ with the Bernoulli probability measure $\mu = B(1/p, \ldots, 1/p)$. For $i < j$ in $\mathbb{N}$ let $X_{i,j} = \{ x \in p^\mathbb{N} : x_i > x_j \}$. Then $\mu(X_{i,j}) = \frac{1}{2} (1 - \frac{1}{p})$ for all $(i,j) \in \mathbb{N}^2$ and yet for each $x \in \Omega$ the graph $X(x) = \{(i,j) \in \mathbb{N}^2 : x_i > x_j \}$ has no paths of length $\geq p$ (where the length of a path is the number of its edges).

We will next show that the bounds in Example 3.1 are optimal. We need:

Lemma 3.2. Let $p \in \mathbb{N}$ and let $m \in \mathcal{M}^1(p^\mathbb{N})$. Let

\[ A_{i,j} := \{ x \in p^\mathbb{N} : x_i > x_j \} \]

Then

\[ \inf_{(i,j) \in \mathbb{N}^2} m(A_{i,j}) \leq \frac{1}{2} \left( 1 - \frac{1}{p} \right) \]

Proof. The proof is a reduction to the case of exchangeable measures (see Appendix B). Note that if $\sigma \in \text{Incr}(\mathbb{N})$, then $(\sigma \cdot m)(A_{i,j}) = m(A_{\sigma(i),\sigma(j)})$. Hence the infimum in (3.2) can only increase replacing $m$ with $\sigma \cdot m$. By Theorem B.8 we can then assume that $m$ is asymptotically exchangeable, so that in particular the sequence $m_k = S_k \cdot m$ converges, in the weak* topology, to an exchangeable measure $m' \in \mathcal{M}^1(p^\mathbb{N})$. Since $p$ is finite, the sets $A_{i,j}$ are clopen, and therefore $\lim_{k \to \infty} m_k(A_{i,j}) = m'(A_{i,j}) = m'(A_{0,1})$. Noting that $m_k(A_{i,j}) = m(A_{i+k,j+k})$, it follows that

\[ \inf_{(i,j) \in \mathbb{N}^2} m(A_{i,j}) \leq \lim_{k \to \infty} m_k(A_{0,1}) = m'(A_{0,1}) = \frac{1}{2} \left( 1 - \frac{1}{p} \right) \]

where the latter inequality follows from Corollary B.11.

Theorem 3.3. Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and let $X : \Omega \to 2^{\mathbb{E}_G}$ be a random subgraph of $G := (\mathbb{N}, \mathbb{N}(2))$. Consider the set

\[ P := \{ x \in \Omega : X(x) \text{ has a path of length } \geq p \} \]

Assume $\inf_{e \in \mathbb{N}(2)} \mu_e > \frac{1}{2} (1 - \frac{1}{p})$. Then $\mu(P) > 0$. 

Proof. Suppose for a contradiction that $\mu(P) = 0$. We can then assume $P = \emptyset$ (otherwise replace $\Omega$ with $\Omega - P$). For $x \in \Omega$ let $\varphi(x) : \mathbb{N} \to p$ assign to each $i \in \mathbb{N}$ the length of the longest path starting from $i$ in $X(x)$. We thus obtain a function $\varphi : \Omega \to \mathbb{N}$ which is easily seen to be measurable (this is a special case of Lemma 4.3). Let $m = \varphi_\#(\mu) \in \mathcal{M}_1(p^1)$. Since $\varphi(X_{i,j}) \subset A_{i,j}$, we have $m(A_{i,j}) \geq \mu(X_{i,j}) \geq 1/2$ for all $i, j$, contradicting Lemma 3.2.

Having determined the critical threshold $\lambda_p = \frac{1}{2}(1 - \frac{1}{p})$, it follows that if $\inf_{e \in \mathbb{N}(2)} \mu(X_e) \geq \lambda \geq \lambda_p$, the lower bound for $\mu(P)$ grows linearly with $\lambda$. More precisely we have:

**Corollary 3.4.** In the setting of Theorem 3.3 let $\lambda \in [0, 1]$ and suppose that $\inf_{e \in \mathbb{N}(2)} \mu(X_e) \geq \lambda$. Then $\mu(P) \geq \frac{\lambda - \lambda_p}{1 - \lambda_p}$ where $\lambda_p = \frac{1}{2}(1 - \frac{1}{p})$.

*Proof. Suppose $\inf_{e \in \mathbb{N}(2)} \mu(X_e) \geq \lambda$. Consider the conditional probability $\mu(\cdot | \Omega - P) \in \mathcal{M}_1(\Omega)$. We have

\[
\mu(X_e | \Omega - P) \geq \frac{\mu(X_e) - \mu(P)}{1 - \mu(P)} \geq \frac{\lambda - \mu(P)}{1 - \mu(P)}.
\]

Clearly $\mu(P | \Omega - P) = 0$. Applying Theorem 3.3 to $\mu(\cdot | \Omega - P)$ it then follows that $\frac{\lambda - \mu(P)}{1 - \mu(P)} \leq \lambda_p$, or equivalently $\mu(P) \geq \frac{\lambda - \lambda_p}{1 - \lambda_p}$.\)

4. Infinite paths

By Theorem 3.3, if $\inf_{e \in \mathbb{N}(2)} X_{i,j} \geq 1/2$, then the random subgraph $X$ of $(\mathbb{N}, \mathbb{N}(2))$ has arbitrarily long finite paths, namely for each $p$ there is $x \in \Omega$ (depending on $p$) such that $X(x)$ has a path of length $\geq p$. We want to show that for some $x \in \Omega$, $X(x)$ has an infinite path. To this aim it is not enough to find a single $x$ that works for all $p$. Indeed, $X(x)$ could have arbitrarily long finite paths without having an infinite path. The existence of infinite paths can be neatly expressed in terms of the following definition.

**Definition 4.1.** Let $G$ be a countable directed graph and let $\omega_1$ be the first uncountable ordinal. We recall that the rank function $\varphi_G : V_G \to \omega_1 \cup \{\infty\}$ of $G$ is defined as follows. For $i \in V_G$,

$$\varphi_G(i) = \sup_{j : (i,j) \in E_G} \varphi_G(j) + 1.$$  

This is a well defined countable ordinal if $G$ has no infinite paths starting at $i$. In the opposite case we set

$$\varphi_G(i) = \infty$$

where $\infty$ is a conventional value bigger than all the countable ordinals. For notational convenience we will take $\infty = \omega_1$ so that $\omega_1 \cup \{\infty\} = \omega_1 \cup \{\omega_1\} = \omega_1 + 1$. Note that if $i$ is a leaf, $\varphi_G(i) = 0$. Also note that $G$ has an infinite path if and only if $\varphi_G$ assumes the value $\infty$.

Given a random subgraph $X : \Omega \to 2^{E_G}$ of $G$, we let $\varphi_X(x) = \varphi_{X(x)}$, namely $\varphi_X(x)(i)$ is the rank of the vertex $i$ in the graph $X(x)$. So $\varphi_X$ is a...
map from $\Omega$ to $(\omega_1 + 1)^{V_G}$. It can also be considered as a map from $\Omega \times V_G$ to $\omega_1 + 1$ by writing $\varphi(x, i)$ instead of $\varphi(x)(i)$.

**Remark 4.2.** We have $\varphi_{\alpha}(x, i) = \varphi_{\omega_1}(x, i)$ where $\varphi_{\alpha} : \Omega \to (\omega_1 + 1)^{V_G}$ is defined by induction on $\alpha \leq \omega_1$ as follows.

$$
\begin{align*}
\varphi_0(x, i) &= 0 \\
\varphi_{\alpha}(x, i) &= \sup\{\varphi_{\beta}(x, j) + 1 : \beta < \alpha, (i, j) \in X(x)\}
\end{align*}
$$

Since taking the supremum over a countable set preserves measurability, from Remark 4.2 it follows that for all $k \in \mathbb{N}$ and $\alpha < \omega_1$ the sets $\{x : \varphi_{\alpha}(x, k) = \alpha\}$ are measurable. We will show that $\{x : \varphi_{\omega_1}(x, k) = \omega_1\}$ is $\mu$-measurable, namely it is the union of a measurable set and a $\mu$-null set.

**Lemma 4.3.** Let $G$ be a countable directed graph, let $(\Omega, \mathcal{A}, \mu)$ be a probability space and let $X : \Omega \to 2^{E_G}$ be a random subgraph of $G$.

1. The set $P := \{x \in \Omega \mid X(x) \text{ has an infinite path}\}$ is $\mu$-measurable.
2. For all $\alpha \leq \omega_1$, the set $\{x \in \Omega \mid \varphi_{\alpha}(x, i) = \alpha\}$ is $\mu$-measurable.
3. $\varphi_{\alpha} : \Omega \to (\omega_1 + 1)^{V_G}$ is $\mu$-measurable and its restriction to $\Omega - P$ is essentially bounded, namely for some $\alpha_0 < \omega_1$ it takes values in $\alpha_0^{V_G}$ outside of a $\mu$-null set.

**Proof.** Fix $k \in \mathbb{N}$. The sequence of values $\mu(\{x : \varphi_{\alpha}(x, k) \leq \beta\})$ is increasing with $\beta$ and uniformly bounded by $1 = \mu(\Omega)$. So there is $\alpha_0 < \omega_1$ such that

$$
\mu(\{x \in \Omega : \varphi_{\alpha}(x, k) = \beta\}) = 0 \quad \text{for } \alpha_0 \leq \beta < \infty.
$$

It follows that $\{x : \varphi_{\alpha}(x, k) = \omega_1\}$ is $\mu$-measurable and $\varphi_{\alpha}$ is $\mu$-measurable. Since $P = \bigcup_k \{x : \varphi_{\alpha}(x, k) = \omega_1\}$, we have that $P$ is $\mu$-measurable as well.

Given an ordinal $\alpha$, we put on $\alpha$ the topology generated by the open intervals. Note that a non-zero ordinal is compact if and only if it is a successor ordinal, and it is metrizable if and only if it is countable. Let $\mathcal{M}_c(\omega_1^N)$ be the set of compactly supported Borel measures on $\omega_1^N$, namely the measures with support in $\alpha_0^N$ for some $\alpha_0 < \omega_1$. The following Lemma reduces to Lemma 3.2 if $\alpha_0$ is finite.

**Lemma 4.4.** Let $m \in \mathcal{M}_c(\omega_1^N)$ be a non-zero measure with compact support. Let

$$
A_{i,j} := \{x \in \mathbb{N}^N : x_i > x_j\}.
$$

Then

$$
\inf_{(i,j)\in\mathbb{N}^2} m(A_{i,j}) < \frac{m(\omega_1^N)}{2}.
$$

**Proof.** With no loss of generality we can assume that $m \in \mathcal{M}^1(\omega_1^N)$, i.e. $m(\omega_1^N) = 1$. We divide the proof into four steps.

**Step 1.** Letting $\partial \omega_1$ be the derived set of $\omega_1$, that is the subset of all countable limit ordinals, we can assume that

$$
m(\{x : x_i \in \partial \omega_1\}) = 0 \quad \forall i \in \mathbb{N}.
$$

Indeed, it is enough to observe that the left-hand side of equation (4.2) can only increase if we replace $m$ with $s_\#(m)$, where $s : \omega_1 \to \omega_1 \setminus \partial \omega_1$ is
the successor map sending \( \alpha < \omega_1 \) to \( \alpha + 1 \), and \( s_\#(m) = (s_*)_\# \), namely \( s_\#(m)(X) := m(\{ x \in \omega_1^N : s \circ x \in X \}) \).

**Step 2.** Since the support of \( m \) is contained in \( \alpha_0^N \), for some ordinal \( \alpha_0 < \omega_1 \), thanks to Theorem 1.3.8 we can assume that \( m \) is asymptotically exchangeable, i.e. the sequence \( m_k = S^k \cdot \sigma \cdot m \) converges, in the weak* topology, to an exchangeable measure \( m' \in \mathcal{M}^1(\omega_1^N) \), with support in \( \alpha_0^N \), for all \( \sigma \in \omega^{(\omega)} \). Note however that, unless \( \alpha_0 \) is finite, we cannot conclude that \( \lim_{k \to \infty} m_k(A_{i,j}) = m'(A_{i,j}) \) since the sets \( A_{i,j} = \{ x \in \omega_1^N | x_i > x_j \} \) are not clopen.

**Step 3.** We shall prove by induction on \( \alpha < \omega_1 \) that

\[
\inf_{(i,j) \in \beta^{(2)}} m(\{ x : x_j < x_i \leq \alpha \}) \leq m'(\{ x : x_1 < x_0 \leq \alpha \}).
\]

Indeed, for \( \alpha = 0 \) we have \( \{ x : x_j < x_i \leq 0 \} = \emptyset \), and (4.3) holds.

As inductive step, let us assume that (4.3) holds for all \( \alpha < \beta < \omega_1 \), and we distinguish whether \( \beta \) is a successor or a limit ordinal.

In the former case let \( \beta = \alpha + 1 \). For \( (i,j) \to +\infty \) (with \( i < j \)) we have:

\[
m(\{ x_j < x_i \leq \beta \}) = m(\{ x_j < x_i \leq \alpha \}) + m(\{ x_j \leq \alpha, x_i = \beta \}) \\
\leq m'(\{ x_1 < x_0 \leq \alpha \}) + m'(\{ x_1 \leq \alpha, x_0 = \beta \}) + o(1) \\
= m'(\{ x_1 < x_0 \leq \beta \}) + o(1),
\]

where we used the induction hypothesis, and the fact that \( \{ x_j \leq \alpha, x_i = \beta \} \) is clopen.

Let us now assume that \( \beta \) is a limit ordinal and let \( i \in \mathbb{N} \). We have

\[
\bigcap_{\alpha < \beta} \{ x : \alpha < x_i \leq \beta \} = \emptyset,
\]

so for all \( \varepsilon > 0 \) there exists \( \alpha < \beta \) such that

\[
m'(\{ \alpha < x_i \leq \beta \}) < \varepsilon.
\]

Since \( m' \) is exchangeable, we can choose the same \( \alpha \) for every \( i \). Moreover by assumption \( m(\{ x_i = \beta \}) = 0 \) for every \( i \in \mathbb{N} \). Hence there exists \( \alpha \leq \alpha_i < \beta \) such that

\[
m(\{ \alpha_i < x_i \leq \beta \}) < \varepsilon.
\]

Given \( i < j \), distinguishing the relative positions of \( x_i, x_j \) with respect to \( \alpha \) and \( \alpha_i \) we have:

\[
\{ x_j < x_i \leq \beta \} \subseteq \{ x_j < x_i \leq \alpha \} \\
\cup \{ x_j \leq \alpha < x_i \leq \beta \} \\
\cup \{ \alpha < x_j \leq \alpha_i \} \\
\cup \{ \alpha_i < x_i \leq \beta \}.
\]

which gives

\[
m(\{ x_j < x_i \leq \beta \}) \leq m(\{ x_j < x_i \leq \alpha \}) \\
+ m(\{ x_j \leq \alpha < x_i \leq \beta \}) \\
+ m(\{ \alpha < x_j \leq \alpha_i \}) \\
+ m(\{ \alpha_i < x_i \leq \beta \}).
\]
Since \( \{x_j \leq \alpha < x_i \leq \beta\} \) and \( \{\alpha < x_j \leq \alpha_i\} \) are both clopen, we can approximate their \( m \)-measure by their \( m' \)-measure. So we have:

\[
m \{x_j \leq \alpha < x_i \leq \beta\} = m' \{\{x_1 \leq \alpha < x_0 \leq \beta\}\} + o(1)
\]

for \( (i, j) \rightarrow \infty \)

and

\[
m \{\alpha < x_j \leq \alpha_i\} = m' \{\alpha < x_1 \leq \alpha_i\} + o(1)
\]

for \( j \rightarrow \infty \),

where we used Remark B.7 to allow \( j \rightarrow \infty \) keeping \( i \) fixed. Now note that by the choice of \( \alpha \), we have \( m' \{\{\alpha < x_1 \leq \alpha_i\}\} < \varepsilon \), and by induction hypothesis \( \inf_{(i, j) \in \mathbb{N}^2} m \{\{x_j < x_i \leq \alpha\}\} < m'\{\{x_1 < x_0 \leq \beta\}\} \). Hence, from (4.4) we obtain:

\[
\inf_{(i, j) \in \mathbb{N}^2} m \{\{x_j < x_i \leq \beta\}\} \leq m'\{\{x_1 < x_0 \leq \alpha\}\}
\]

\[
+ m'\{\{x_1 \leq \alpha < x_0 \leq \beta\}\} + o(1)
\]

\[
+ \varepsilon + o(1) + \varepsilon.
\]

Therefore,

\[
\inf_{(i, j) \in \mathbb{N}^2} m \{\{x_j < x_i \leq \beta\}\} \leq m'\{\{x_1 < x_0 \leq \beta\}\} + 2\varepsilon + o(1)
\]

Inequality (4.3) is then proved for all \( \alpha < \omega_1 \).

**Step 4.** We now conclude the proof of the theorem. From (4.3) it follows (4.5)

\[
\inf_{(i, j) \in \mathbb{N}^2} m(A_{i,j}) \leq m'\{\{x : x_1 < x_0\}\} = \frac{1}{2} (1 - m'\{\{x : x_1 = x_0\}\}) < \frac{1}{2}
\]

where we used the fact the \( m' \) is exchangeable and Corollary B.10.

**Theorem 4.5.** Let \( (\Omega, A, \mu) \) be a probability space and let \( \mathcal{X}: \Omega \rightarrow 2^{\mathbb{E}_G} \) be a random subgraph of \( G := (\mathbb{N}, \mathbb{N}^2) \). Consider the set

\[
P := \{x \in \Omega : \mathcal{X}(x) \text{ has an infinite path}\}.
\]

Assume \( \inf_{e \in \mathbb{N}^2} \mu(\mathcal{X}_e) \geq \frac{1}{2} \). Then \( \mu(P) > 0 \).

Note that by Example 3.1 the bound 1/2 is optimal.

**Proof.** Suppose for a contradiction \( \mu(P) = 0 \). We can then assume \( P = \emptyset \) (replacing \( \Omega \) with \( \Omega - P \)). Hence the rank function \( \varphi := \varphi\mathcal{X} : \Omega \rightarrow (\omega_1 + 1)^\mathbb{N} \) takes values in \( \omega_1^\mathbb{N} \). Let \( m = \varphi_\#(\mu) \in \mathcal{M}(\omega_1^\mathbb{N}) \). Note that \( \varphi(\mathcal{X}_{i,j}) \subset A_{i,j} := \{x \in \mathbb{P}^\mathbb{N} : x_i > x_j\} \). Hence \( m(A_{i,j}) \geq \mu(\mathcal{X}_{i,j}) \geq 1/2 \) for all \( (i, j) \in \mathbb{N}^2 \). This contradicts Lemma 4.4.

Reasoning as in Corollary 3.4 we obtain:

**Corollary 4.6.** Let \( 0 \leq \lambda < 1 \). If \( \inf_{e \in \mathbb{N}^2} \mu(\mathcal{X}_e) \geq \lambda \), then \( \mu(P) > \frac{\lambda - 1/2}{1 - 1/2} \).

Note that if we replace \( (\mathbb{N}, \mathbb{N}^2) \) with a finitely branching countable graph \( G \), then the threshold for the existence of infinite paths becomes 1, namely we cannot ensure the existence of infinite paths even if each edge of \( G \)
belongs to the random subgraph $\mathcal{X}$ with probability very close to 1. In fact, the following more general result holds:

**Proposition 4.7.** Let $G = (V_G, E_G)$ be graph admitting a coloring function $c: E_G \to \mathbb{N}$ such that each infinite path in $G$ meets all but finitely many colors (it is easy to see that a finitely branching countable graph $G$ has this property). Then for every $\varepsilon > 0$ there is a probability space $(\Omega, A, \mu)$ and a random subgraph $\mathcal{X}: \Omega \to 2^{E_G}$ of $G$ such that for all $x \in \Omega$, $\mathcal{X}(x)$ has no infinite paths, and yet $\mu(\mathcal{X}_e) > 1 - \varepsilon$ for all $e \in E_G$.

**Proof.** Let $(Z_n)_{n \in \mathbb{N}}$ be a disjoint family of infinite subsets of $\mathbb{N}$. Let $\mu$ be a probability measure on $\Omega := \mathbb{N}$ with $\mu(\{n\}) < \varepsilon$ for every $n$. Given $n \in \Omega$ let $\mathcal{X}(n)$ be the subgraph of $G$ (with vertices $V_G$) containing all edges $e \in E_G$ of color $c(e) \notin Z_n$. Given $e \in E_G$ there is at most one $n$ such that $c(e) \in Z_n$. Hence clearly $\mu(\mathcal{X}_e) \geq 1 - \varepsilon$, and yet $\mathcal{X}(n)$ has no infinite paths for any $n \in \Omega$. 

**Remark 4.8.** It is natural to ask whether the answer to Problem [1] changes if we substitute $\mathbb{N}$ with the set of the real numbers. Since $\mathbb{N} \subset \mathbb{R}$, the probability threshold for the existence of infinite paths can only decrease, but the following example shows that it still equals $1/2$. Let $\Omega = [0, 1]^\mathbb{R}$ equipped with the product Lebesgue measure $\mathcal{L}$, let $\varepsilon > 0$, and let

$$\mathcal{X}_{i,j} := \{ x \in \Omega : x_i > x_j + \varepsilon \},$$

for all $i < j \in \mathbb{R}$. The assertion follows observing that $\mathcal{L}(\mathcal{X}_{i,j}) = (1 - \varepsilon)^2/2$ for all $i < j \in \mathbb{R}$, and

$$\bigcap_{i \in \{1, \ldots, N\}} \mathcal{X}_{n_i, n_{i+1}} = \emptyset$$

whenever $n_i$ is a strictly increasing sequence of reals numbers, and $N > 1/\varepsilon$.

5. **Threshold functions for graph morphisms**

**Definition 5.1.** Let $F$ and $G$ be directed graphs. A graph morphism $\varphi: G \to F$ is a map $\varphi : V_G \to V_F$ such that $(\varphi(a), \varphi(b)) \in E_F$ for all $(a, b) \in E_G$. We write $G \to F$ if there is a graph morphism from $G$ to $F$.

The results of the previous sections were implicitly based on following observation:

**Remark 5.2.** Let $G$ be a directed graph.

1. $G$ has a path of length $\geq p$ if and only if $G \not\rightarrow (p, p^{(2)})$.
2. $G$ has an infinite path if and only if $G \not\rightarrow (\omega_1, \omega_1^{(2)})$.

This suggests to generalize the above results considering other properties of graphs that can be expressed in terms of non-existence of graph morphisms. Let us give the relevant definitions.

**Definition 5.3.** Given two directed graphs $F, G$ and given $i, j \in V_G$ let

$$A_{i,j}(F, G) := \{ u \in V_F : (u(i), u(j)) \in E_F \}$$

and define the relative capacity of $F$ with respect to $G$ as

$$c(F, G) := \sup_{m \in M^1(V_F)} \inf_{(i,j) \in E_G} m(A_{i,j}(F, G)) \in [0, 1].$$
Proof. Suppose for a contradiction \( \phi \) is a graph morphism from \( \Omega \) to \( V_{\phi}^G \) and \( \mathcal{X}_{i,j} = A_{i,j}(F,G) \).

**Theorem 5.4.** Let \( F \) and \( G \) be directed countable graphs, let \((\Omega, \mathcal{A}, \mu)\) be a probability space and let \( \mathcal{X}: \Omega \rightarrow 2^{\mathcal{E}_G} \) be a random subgraph of \( G \). Let \( P := \{ x \in \Omega \mid \mathcal{X}(x) \neq F \} \). Assume \( \inf_{e \in \mathcal{E}_G} \mu(\mathcal{X}_e) > c(F,G) \). Then \( \mu(P) > 0 \). Moreover there are examples in which \( P \) is empty and \( \inf_{e \in \mathcal{E}_G} \mu(\mathcal{X}_e) \) as close as \( c(F,G) \) as desired. So \( c(F,G) \) is the threshold for non-existence of graph morphisms \( f: \mathcal{X}(x) \rightarrow F \). To prove the second part it suffices to take \( \Omega = V_{\phi}^G \) and \( \mathcal{X}_{i,j} = A_{i,j}(F,G) \).

Proof. Suppose for a contradiction \( \mu(P) = 0 \). We can then assume \( P = \emptyset \) (replacing \( \Omega \) with \( \Omega - P \)). Hence for each \( x \in \Omega \) there is a graph morphism \( \varphi(x): \mathcal{X}(x) \rightarrow F \), which can be seen as an element of \( V_{\phi}^G \). We thus obtain a map \( \varphi: \Omega \rightarrow V_{\phi}^G \). By Lemma 5.7 below, \( \varphi \) can be chosen to be \( \mu \)-measurable. Since \( x \in \mathcal{X}_{i,j} \) implies \( (\varphi(x)(i), \varphi(x)(j)) \in E_F \), we have \( \varphi(\mathcal{X}_{i,j}) \subset A_{i,j}(F,G) \) for all \((i,j) \in E_G \). Let \( m := \varphi(\mu) \in M^1(V_{\phi}^G) \). Then \( m(A_{i,j}(F,G)) \geq \mu(\mathcal{X}_{i,j}) > c(F,G) \). This is absurd by definition of \( c(F,G) \).

Reasoning as in Corollary 3.4 we obtain:

**Corollary 5.5.** Suppose \( c(F,G) < 1 \). If \( \inf_{\mathcal{E}_G} \mu(\mathcal{X}_e) \geq \lambda \), then \( \mu(P) \geq \frac{\lambda - c(F,G)}{1 - c(F,G)} \).

**Remark 5.6.** If the sup in the definition of \( c(F,G) \) is not reached, it suffices to have the weak inequality \( \inf_{\mathcal{E}_G} \mu(\mathcal{X}_e) \geq c(F,G) \) in order to have \( \mu(P) > 0 \) (this is indeed the case of Theorem 5.4).

It remains to show that the map \( \varphi: \Omega \rightarrow V_{\phi}^G \) in the proof of Theorem 5.4 can be taken to be \( \mu \)-measurable.

**Lemma 5.7.** Let \( F, G \) be countable directed graphs, let \((\Omega, \mathcal{A}, \mu)\) be a probability space, and let \( \mathcal{X}: \Omega \rightarrow 2^{\mathcal{E}_G} \) be a random subgraph of \( G \).

1. The set \( \Omega_0 := \{ x \in \Omega \mid \mathcal{X}(x) \neq F \} \) is \( \mu \)-measurable (i.e. measurable with respect to the \( \mu \)-completion of \( \mathcal{A} \)).

2. There is an \( \mu \)-measurable function \( \varphi: \Omega_0 \rightarrow V_{\phi}^G \) that selects, for each \( x \in \Omega_0 \), a graph morphism \( \varphi(x): \mathcal{X}(x) \rightarrow F \).

3. If \( F \) is finite, then \( \Omega_0 \) is measurable and \( \varphi \) can be chosen measurable.

**Proof.** Given a function \( f: V_G \rightarrow V_F \), we have \( f: \mathcal{X}(x) \rightarrow F \) (i.e., \( f \) is a graph morphism from \( \mathcal{X}(x) \) to \( F \)) if and only if \( x \in \bigcap_{(i,j) \in V_G} \bigcup_{a,b} B_{i,j,a,b} \bigcap_{(i,j) \in V_G} B_{i,j,a,b} \), where \( x \in B_{i,j,a,b} \) says that \( f(i) = a, f(j) = b \) and \( x \in \mathcal{X}_{i,j} \). This shows that \( B := \{ (x,f) \mid f: \mathcal{X}(x) \rightarrow F \} \) is a measurable subset of \( \Omega \times V_{\phi}^G \). We are looking for a \( (\mu) \)-measurable function \( \varphi: \pi_\mathcal{X}(B) \rightarrow V_{\phi}^G \) whose graph is contained in \( B \).

Special case: Let us first assume that \( \Omega \) is a Polish space (i.e., a complete separable metric space) with its algebra \( \mathcal{A} \) of Borel sets. By Jankov - von Neumann uniformization theorem (see [K:95, Thm. 29.9]), if \( X, Y \) are Polish spaces and \( Q \subset X \times Y \) is a Borel set, then the projection \( \pi_X(Q) \subset X \) is universally measurable (i.e. it is \( m \)-measurable for every \( \sigma \)-finite Borel measure \( m \) on \( X \)), and there is a universally measurable function \( f: \pi_X(Q) \rightarrow Y \).
whose graph is contained in \( Q \). We can apply this to \( X = \Omega, Y = V^G_F \) and \( Q = B \) to obtain (1) and (2). It remains to show that if \( F \) is finite \( \pi_X(Q) \) and \( f \) can be chosen to be Borel measurable. To this aim it suffices to use the following uniformization theorem of Arsenin - Kunugui (see \[ K:79 \]).

6. Chromatic Number

We will apply the results of the previous section to study the chromatic number of a random subgraph of \( (\mathbb{N}, \mathbb{N}^{(2)}) \).

We recall that the chromatic number \( \chi(G) \) of a directed graph \( G \) is the smallest \( n \) such that there is a colouring of the vertices of \( G \) with \( n \) colours in such a way that \( a, b \in V_G \) have different colours whenever \( (a, b) \in E_G \) (see \[ B:79 \]).

For \( p \in \mathbb{N} \), let \( K_p \) be the complete graph on \( p \) vertices, namely \( K_p \) has set of vertices \( p = \{0, 1, \ldots, p - 1\} \) and set of edges \( \{(x, y) \in p^2 : x \neq y\} \).

Clearly \( \chi(K_p) = p \). Note also that:

\[
(6.1) \quad G \rightarrow K_p \iff \chi(G) \leq p.
\]

Now let \( (\Omega, \mathcal{A}, m) \) be a probability space, and let \( \mathbb{X} : \Omega \rightarrow 2^{E_G} \) be a random subgraph of \( G = (\mathbb{N}, \mathbb{N}^{(2)}) \). Let \( P = \{x \in \Omega : \chi(\mathbb{X}(x)) \geq p\} \). By Equation (6.1) and the results of the previous section, if \( \inf_{\mathcal{E} \in \mu(\mathbb{X}_e)} c(K_p, (\mathbb{N}, \mathbb{N}^{(2)})) \), then \( \mu(P) > 0 \). This however does not say much unless we manage to determine \( c(K_p, (\mathbb{N}, \mathbb{N}^{(2)})) \). We will show that \( c(K_p, (\mathbb{N}, \mathbb{N}^{(2)})) = (1 - \frac{1}{p}) \), so we have:

**Theorem 6.1.** Let \( (\Omega, \mathcal{A}, m) \) be a probability space, and let \( \mathbb{X} : \Omega \rightarrow 2^{E_G} \) be a random subgraph of \( (\mathbb{N}, \mathbb{N}^{(2)}) \). If \( \inf_{\mathcal{E} \in \mu(\mathbb{X}_e)} c(F, (\mathbb{N}, \mathbb{N}^{(2)})) \) (see Definition 5.3) for any finite graph \( F \). The following invariant of directed graphs has been studied in \[ R:82 \] and \[ FT:85 \] Section 3 (where it is called value).

**Definition 6.2.** Given a directed graph \( F \), we define the capacity of \( F \) as

\[
(6.2) \quad c_0(F) := \sup_{\lambda \in \Sigma_F} \sum_{(a,b) \in E_F} \lambda_a \lambda_b \in [0, 1],
\]

where \( \Sigma_F \) is the simplex of all sequences \( \{\lambda_a\}_{a \in V_F} \) of real numbers such that \( \lambda_a \geq 0 \) and \( \sum_{a \in V_F} \lambda_a = 1 \).
**Proposition 6.3.** If $F$ is a finite directed graph, then

$$
(6.3) \quad c\left(F, (\mathbb{N}, \mathbb{N}^{(2)})\right) = c_0(F).
$$

**Proof.** Let $G = (\mathbb{N}, \mathbb{N}^{(2)})$. The proof is a series of reductions.

**Step 1.** Note that if $\sigma \in \text{Incr}(\mathbb{N})$, then $\sigma \cdot m(A_{i,j}(F, G)) = m(A_{\sigma(i), \sigma(j)}).$

Hence the infimum in (5.2) can only increase replacing $m$ with $\sigma_{\#}^* (m)$. By Theorem B.8 there is $\sigma \in \text{Incr}(\mathbb{N})$ such that $\sigma \cdot m$ is asymptotically exchangeable. It then follows that we can equivalently take the supremum in (5.2) among the measures $m \in \mathcal{M}^1(V_F^x)$ which are asymptotically exchangeable.

**Step 2.** By definition if $m$ is asymptotically exchangeable there is an exchangeable measure $m'$ such that $\lim_{k \to \infty} m_k = m'$, where $m_k = S^k \cdot m$. Clearly $\inf_{(i,j) \in E_G} m(A_{i,j}(F, G)) \leq \lim_{k \to \infty} m_k(A_{0,1}(F, G)) = m'(A_{0,1}(F, G))$.

So the supremum in (5.2) coincides with $\sup_m m(A_{0,1}(F, G))$, for $m$ ranging over the exchangeable measures.

**Step 3.** Recalling (B.13), every exchangeable measure is a convex integral combination of Bernoulli measures $B_\lambda$, with $\lambda \in \Sigma_F$. It follows that it is sufficient to compute the supremum on the Bernoulli measures $B_\lambda$. We have:

$$
B_\lambda \left( \left\{ x \in V_F^N : (x_0, x_1) \in E_F \right\} \right) = \sum_{(a,b) \in E_F} B_\lambda \left( \left\{ x : x_0 = a, x_1 = b \right\} \right) = \sum_{(a,b) \in E_F} \lambda_a \lambda_b
$$

so that (5.2) reduces to (6.2).

Notice that if there is a morphism of graphs from $G$ to $F$, then $c_0(G) \leq c_0(F)$. Also note that $c_0(F) = 1$ if there is some $a \in V_F$ with $(a,a) \in E_F$. Recall that $F$ is said to be: **irreflexive** if $(a,a) \not\in E_F$ for all $a \in V_F$; **symmetric** if $(a,b) \in E_F \iff (b,a) \in E_F$ for all $a,b \in V_F$; **anti-symmetric** if $(a,b) \in E_F \implies (b,a) \not\in E_F$ for all $a,b \in V_F$.

The **clique number** $\text{cl}(F)$ of $F$ is defined as the largest integer $n$ such that there is a subset $S \subseteq V_F$ of size $n$ which forms a clique, namely $(a,b) \in E_F$ or $(b,a) \in E_F$ for all $a,b \in S$.

**Proposition 6.4.** (See also [FT:85 Section 3]) Let $F$ be a finite irreflexive directed graph. If $F$ is anti-symmetric, then

$$
(6.4) \quad c_0(F) = \frac{1}{2} \left( 1 - \frac{1}{\text{cl}(F)} \right).
$$

If $F$ is symmetric, then

$$
(6.5) \quad c_0(F) = 1 - \frac{1}{\text{cl}(F)}.
$$

In particular $c_0(K_p) = 1 - \frac{1}{p}$.

**Proof.** The anti-symmetric case follows from the symmetric one taking the symmetric closure. So we can assume that $F$ is symmetric. Let $\lambda \in \Sigma_F$ be a maximizing distribution, meaning that $c_0(F) = \sum_{(a,b) \in E_F} \lambda_a \lambda_b$, and let $S_\lambda$ be the subgraph of $F$ spanned by the support of $\lambda$, that is $V_{S_\lambda} = \{ a \in V_F : \lambda_a > 0 \}$. Given $a \in S_\lambda$ note that $\frac{\partial}{\partial \lambda_a} \sum_{(u,v) \in E_F} \lambda_u \lambda_v = 2 \sum_{b \in V_F : (a,b) \in E_F} \lambda_b$. 

From Lagrange’s multiplier Theorem it then follows that $\sum_{b \in \mathcal{V}_F: (a,b) \in \mathcal{E}_F} \lambda_b$ is constant, namely it does not depend on the choice of $a \in S_\lambda$. Since $\sum_{a \in S_\lambda} (\sum_{b: (a,b) \in \mathcal{E}_F} \lambda_a) = c_0(F)$, it follows that for each $a \in S_\lambda$ we have:

$$\sum_{b \in \mathcal{V}_F: (a,b) \in \mathcal{E}_F} \lambda_b = c_0(F).$$  \hspace{1cm} (6.6)$$

If $c, c' \in V_{S_\lambda}$, we can consider the distribution $\lambda' \in \Sigma_F$ such that $\lambda'_c = 0$, $\lambda'_{c'} = \lambda_c + \lambda_{c'}$, and $\lambda'_b = \lambda_b$ for all $b \in \mathcal{V}_F \setminus \{c, c'\}$. From (6.6) it then follows that $\lambda'$ is also a maximizing distribution whenever $(c, c) \notin \mathcal{E}_F$. (In fact $\sum_{(a,b) \in \mathcal{E}_F} \lambda'_a \lambda'_b = \sum_{(a,b) \in \mathcal{E}_F} \lambda_a \lambda_b - \lambda_c \sum_{b: (c,b) \in \mathcal{E}_F} \lambda_b + \lambda_c \sum_{b: (c',b) \in \mathcal{E}_F} \lambda_b = c_0(F) - \lambda_c c_0(F) + \lambda_c c_0(F)$.)

As a first consequence, $S_\lambda$ is a clique whenever $\lambda$ is a maximizing distribution with minimal support. Indeed, let $K$ be a maximal clique contained in $S_\lambda$, and assume by contradiction that there exists $a \in V_{S_\lambda} \setminus V_K$. Letting $a' \in V_K$ be a vertex of $F$ independent of $a$ (such an element exists since $K$ is a maximal clique), and letting $\lambda' \in \Sigma_F$ as above, we have $c_0(F) = \sum_{(a,b) \in \mathcal{E}_F} \lambda'_a \lambda'_b$, contradicting the minimality of $V_{S_\lambda}$.

Once we know that $S_\lambda$ is a clique, again from (6.6) we get that $\lambda$ is a uniform distribution, that is $\lambda_a = \lambda_b$, for all $a, b \in V_{S_\lambda}$. It follows

$$c_0(F) = 1 - \frac{1}{|S_\lambda|} \leq 1 - \frac{1}{\text{cl}(F)},$$

which in turn implies (6.4), the opposite inequality being realized by a uniform distribution on a maximal clique. $\square$

Notice that the proof of Proposition 6.4 shows that there exists a maximizing $\lambda \in \Sigma_F$ whose support is a clique (not necessarily of maximal order).

7. Infinite cliques

We recall the following standard Borel-Cantelli type result, which shows that Problem 3 has a positive answer for $k = 1$.

**Proposition 7.1.** Let $(\Omega, \mathcal{A}, \mu)$ be a probability space. Let $\lambda > 0$ and for each $i \in \mathbb{N}$ let $X_i \subseteq \Omega$ be a measurable set such that $\mu(X_i) \geq \lambda$. Then there is an infinite set $J \subseteq \mathbb{N}$ such that

$$\bigcap_{i \in J} X_i \neq \emptyset.$$  

**Proof.** The set $Y := \bigcap_{n} \bigcup_{i>n} X_i$ is a decreasing intersection of sets of (finite) measure greater than $\lambda > 0$, hence $\mu(Y) \geq \lambda$ and, in particular, $Y$ is non-empty. Now it suffices to note that any element $x$ of $Y$ belongs to infinitely many $X_i$’s. $\square$

Proposition 7.1 has the following interpretation: if we choose each element of $\mathbb{N}$ with probability greater or equal to $\lambda$, we obtain an infinite subset with probability grater or equal to $\lambda$.

The following example shows that Problem 3 has in general a negative answer for $k > 1$. 
Example 7.2. Let $p \in \mathbb{N}$ and consider the Cantor space $\Omega = p^\mathbb{N}$, equipped with the Bernoulli measure $B_{(1/p, \ldots, 1/p)}$, and let $X_{i,j} := \{ x \in \Omega : x_i \neq x_j \}$. Then each $X_{i,j}$ has measure $\lambda = 1 - 1/p$, and for all $x \in X$ the graph $G(x) := \{(i,j) \in \mathbb{N}^2 : x \in X_{i,j}\}$ does not contain cliques (i.e. complete subgraphs) of cardinality $(p + 1)$.

In view of Example 7.2, we need further assumptions in order to get a positive answer to Problem 3.

Example 7.3. By Ramsey theorem, Problem 3 has a positive answer if there is a finite set $S \subset \Omega$ such that each $X_{i_1, \ldots, i_k}$ has a non-empty intersection with $S$. In particular, this is the case if $\Omega$ is countable.

Proposition 7.4. Let $r > 0$. Assume that $\Omega$ is a compact metric space and each set $X_{i_1, \ldots, i_k}$ contains a ball $B_{i_1, \ldots, i_k}$ of radius $r > 0$. Then Problem 3 has a positive answer.

Proof. Applying Lemma A.1 to the centers of the balls $B_{i_1, \ldots, i_k}$ it follows that for all $0 < r' < r$ there exists a positive integer $N$ and a ball $B$ of radius $r'$ such that $$ B \subset \bigcap_{(j_1, \ldots, j_k) \in J[k]} X_{j_1, \ldots, j_k}. $$

We now give a sufficient condition for a positive answer to Problem 3.

Theorem 7.5. Let $(\Omega, \mu)$ be a probability space. Let $\lambda > 0$ and assume that we have the sets $\mu(X_{i_1, \ldots, i_k}) \geq \lambda$ for each $(i_1, \ldots, i_k) \in \mathbb{N}^k$. Assume further that the indicator functions of $X_{i_1, \ldots, i_k}$ belong to a compact subset $K$ of $L^1(\Omega, \mu)$. Then, for any $\varepsilon > 0$ there exists an infinite set $J \subset \mathbb{N}$ such that

$$ \mu \left( \bigcap_{(i_1, \ldots, i_k) \in J[k]} X_{i_1, \ldots, i_k} \right) \geq \lambda - \varepsilon. $$

Proof. Consider first the case $k = 1$. By compactness of $K$, for all $\varepsilon > 0$ there exist an increasing sequence $\{i_n\}$ and a set $X_\infty \subset X$, with $\mu(X_\infty) \geq \lambda$, such that

$$ \mu(X_\infty \Delta X_{i_n}) \leq \frac{\varepsilon}{2^n} \quad \forall n \in \mathbb{N}. $$

As a consequence, letting $J := \{i_n : n \in \mathbb{N}\}$ we have

$$ \mu \left( \bigcap_{n \in \mathbb{N}} X_{i_n} \right) \geq \mu \left( X_\infty \cap \bigcap_{n \in \mathbb{N}} X_{i_n} \right) \geq \mu(X_\infty) - \sum_{n \in \mathbb{N}} \mu(X_\infty \Delta X_{i_n}) \geq \lambda - \varepsilon. $$

For $k > 1$, we apply Lemma A.1 with

$$ M = K \subset L^1(\Omega, \mu), $$

$$ f(i_1, \ldots, i_k) = \chi_{X_{i_1, \ldots, i_k}} \in L^1(\Omega, \mu). $$

In particular, recalling Remark A.4 for all $\varepsilon > 0$ there exist $J = \sigma(\mathbb{N})$, $X_\infty \subset \Omega$, and $X_{i_1, \ldots, i_m} \subset X$, for all $(i_1, \ldots, i_m) \in J[m]$ with $1 \leq m < k$, such
that $\mu(X_\infty) \geq \lambda$ and for all $(i_1, \ldots, i_k) \in J^k$ it holds 

$$
\mu(X_\infty \Delta X_{i_1}) \leq \frac{\varepsilon}{2^{\sigma^{-1}(i_1)}}
$$

$$
\mu(X_{i_1 \ldots i_m} \Delta X_{i_1 \ldots i_{m+1}}) \leq \frac{\varepsilon}{2^{\sigma^{-1}(i_{m+1})}}.
$$

Reasoning as above, it then follows 

$$
\mu\left(\bigcap_{(i_1, \ldots, i_k) \in J^k} X_{i_1 \ldots i_k}\right) \leq \\
\sum_{i_1 \in \mathbb{N}} \mu(X_\infty \Delta X_{i_1}) + \sum_{i_1 < i_2} \mu(X_{i_1} \Delta X_{i_1 i_2}) + \\
\cdots + \sum_{i_1 < \cdots < i_k} \mu(X_{i_1 \ldots i_{k-1}} \Delta X_{i_1 \ldots i_k}) \leq C(k) \varepsilon,
$$

where $C(k) > 0$ is a constant depending only on $k$. Therefore 

$$
\mu\left(\bigcap_{(i_1, \ldots, i_k) \in J^k} X_{i_1 \ldots i_k}\right) \geq \mu\left(\bigcap_{(i_1, \ldots, i_k) \in J^k} X_{i_1 \ldots i_k}\right) \geq \mu(X_\infty) - \mu\left(\bigcap_{(i_1, \ldots, i_k) \in J^k} X_{i_1 \ldots i_k}\right) \geq \lambda - C(k) \varepsilon.
$$

Notice that from Theorem 7.5 it follows that Problem 3 has a positive answer if there exist an infinite $J \subseteq \mathbb{N}$ and sets $\overline{X}_{i_1 \ldots i_k} \subseteq X_{i_1 \ldots i_k}$ with $(i_1, \ldots, i_k) \in J^k$, such that $\mu(\overline{X}_{i_1 \ldots i_k}) \geq \lambda$ for some $\lambda > 0$, and the indicator functions of $\overline{X}_{i_1 \ldots i_k}$ belong to a compact subset of $L^1(\Omega, \mu)$.

**Remark 7.6.** We recall that, when $\Omega$ is a compact subset of $\mathbb{R}^n$ and the perimeters of the sets $X_{i_1 \ldots i_k}$ are uniformly bounded, then the family $X_{i_1 \ldots i_k}$ has compact closure in $L^1(\Omega, \mu)$ (see for instance [AFP:00, Thm. 3.23]). In particular, if the sets $X_{i_1 \ldots i_k}$ have equibounded Cheeger constant, i.e. if there exists $C > 0$ such that 

$$
\min_{E \subset X_{i_1 \ldots i_k}} \frac{\text{Per}(E)}{|E|} \leq C \quad \forall (i_1, \ldots, i_k) \in \mathbb{N}^k,
$$

then Problem 3 has a positive answer.

**APPENDIX A. A TOPOLOGICAL RAMSEY THEOREM**

The following metric version of Ramsey theorem reduces to the classical Ramsey theorem when $M$ is finite.

**Lemma A.1.** Let $M$ be a compact metric space, let $k \in \mathbb{N}$, and let $f : \mathbb{N}^k \rightarrow M$. Then there exists an infinite set $J \subset \mathbb{N}$ such that the limit 

$$
\lim_{(i_1, \ldots, i_k) \rightarrow +\infty} f(i_1, \ldots, i_k)
$$

is well-defined for all $(i_1, \ldots, i_k) \in J^k$.
exists.

Proof. Notice first that the thesis is trivial for \( k = 1 \), since the space \( M \) is compact. Assuming that the thesis holds for some \( k \in \mathbb{N} \), we want to prove it for \( k + 1 \). So let \( f : \mathbb{N}^{(k+1)} \rightarrow M \). By inductive assumption, for all \( j \in \mathbb{N} \) there exist a infinite set \( J_j \subset \mathbb{N} \) and a point \( x_j \in M \) such that \( x_j = \lim_{i_1, \ldots, i_k \rightarrow \infty} f(j, i_1, \ldots, i_k) \), with \( (i_1, \ldots, i_k) \in [J_j]^k \). Possibly extracting further subsequences we can also assume that

\[
(A.1) \quad d(x_j, f(j, i_1, \ldots, i_k)) \leq 1/2^j
\]

for all \( (i_1, \ldots, i_k) \in J_j^k \). Moreover, by a recursive construction, we can assume that \( J_{j+1} \subseteq J_j \). Now define \( \tau \in \text{Incr}(\mathbb{N}) \) by choosing \( \tau(0) \in \mathbb{N} \) and inductively \( \tau(n+1) \in J_{\tau(n)} \). Since \( J_{j+1} \subseteq J_j \) for all \( j \), this implies \( \tau(m) \in J_{\tau(m)} \) for all \( m > n \). By compactness of \( M \), there exists \( \lambda \in \text{Incr}(\mathbb{N}) \) and a point \( x \in M \) such that \( x_{\tau(\lambda(n))} \rightarrow x \) for \( n \rightarrow \infty \). Take \( J = \text{Im}(\tau \circ \lambda) \).

The thesis follows the triangle inequality

\[
d(x, f(j, i_1, \ldots, i_k)) \leq d(x, x_j) + d(x_j, f(j, i_1, \ldots, i_k)),
\]

noting that if \( j < i_1 < \ldots < i_k \) are in \( J \), then \( i_1, \ldots, i_k \in J_j \) (so Equation \( A.1 \) applies).

Note that in Lemma \( A.1 \), the condition \( (i_1, \ldots, i_k) \rightarrow +\infty \) is equivalent to \( i_1 \rightarrow \infty \) (since \( i_1 < i_2 < \ldots < i_k \)). We would like to strengthen Lemma \( A.1 \) by requiring the existence of all the partial limits

\[
x = \lim_{i_{j(1)} \rightarrow \infty} \lim_{i_{j(2)} \rightarrow \infty} \ldots \lim_{i_{j(r)} \rightarrow \infty} x_{i_1 \ldots i_k}
\]

where \( 1 \leq r \leq k \) and \( (i_{j(1)}, \ldots, i_{j(r)}) \in J^{(r)} \) is a subsequence of \( (i_1, \ldots, i_k) \in J^{(k)} \). Note that the existence of all these \( 2^{k-1} \) partial limits does not follow from Lemma \( A.1 \). For instance \( \lim_{i_{j(1)} \rightarrow \infty} (-1)^{j_{(1)}} \) is 0 but \( \lim_{i_{j(1)} \rightarrow \infty} \lim_{j_{(1)} \rightarrow \infty} (-1)^{j_{(1)}} \)

does not exist.

To prove the desired strengthening it is convenient to introduce some terminology. Let \( \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\} \) be the one-point compactification of \( \mathbb{N} \). Given a distance \( \delta \) on \( \mathbb{N} \), we consider on \( \mathbb{N}^{(k)} \) the induced metric

\[
\delta_k((n_1, \ldots, n_k), (m_1, \ldots, m_k)) := \max_i \delta(n_i, m_i).
\]

Given \( \sigma \in \text{Incr}(\mathbb{N}) \), let \( \sigma_* : \mathbb{N}^{(k)} \rightarrow \mathbb{N}^{(k)} \) be the induced map defined by \( \sigma_* (n_1, \ldots, n_k) := (\sigma(n_1), \ldots, \sigma(n_k)) \). Given \( f : \mathbb{N}^{(k)} \), by the following theorem there is an infinite \( J \subset \mathbb{N} \) such that all the partial limits of \( f \mid J^{(k)} \) exist. Moreover the arbitrarity of \( \delta \) shows that we can impose an arbitrary modulus of convergence on all the partial limits of \( f \circ \sigma_* \), where \( \sigma \in \text{Incr}(\mathbb{N}) \) is an increasing enumeration of \( J \).

**Theorem A.2.** Let \( M \) be a compact metric space, let \( k \in \mathbb{N} \), and let \( f : \mathbb{N}^{(k)} \rightarrow M \). Then, for any distance \( \delta \) on \( \overline{\mathbb{N}} \) there exists \( \sigma \in \text{Incr}(\mathbb{N}) \) such that \( f \circ \sigma_* : \mathbb{N}^{(k)} \rightarrow M \) is 1-Lipschitz. As a consequence, it can be extended to a 1-Lipschitz function on the closure of \( \mathbb{N}^{(k)} \) in \( \mathbb{N}^k \).

We need:

**Lemma A.3.** Let \( \delta \) be a metric on \( \overline{\mathbb{N}} \). Then there is another metric \( \delta^* \) on \( \overline{\mathbb{N}} \) such that
(1) \( \delta^*(x, y) \leq \delta(x, y) \) for all \( x, y \).
(2) \( \delta^* \) is monotone in the following sense: \( \delta^*(x', y') \leq \delta^*(x, y) \) for all \( x' > x, y' > y \), provided \( x \neq y \).

Note that the monotonicity requirement is rather strong: for instance the metric \( \delta(n, m) := |1/n - 1/m| \) is not monotone.

**Proof.** The idea is to define \( \delta^*(x, y) = \delta(\psi(x), \psi(y)) \) for a suitable \( \psi \in \text{Incr}(\mathbb{N}) \). To this aim let

\[
\varepsilon(n) := \frac{1}{2} \inf_{m \geq n} \delta(n, m)
\]

and note that for \( x < y \) we have

\[
\varepsilon(x) + \varepsilon(y) \leq 2\varepsilon(x) \leq \delta(x, y).
\]

Choose \( \rho \in \text{Incr}(\mathbb{N}) \) such that \( \sup_{x' \geq \rho(x)} \delta(x', \infty) < \varepsilon(x) \). Given \( x \neq y \), for all \( x' \geq \rho(x), y' \geq \rho(y) \) we have

\[
\delta(x', y') \leq \delta(x', \infty) + \delta(y', \infty) \leq \varepsilon(x) + \varepsilon(y) \leq \delta(x, y).
\]

To finish the proof it is enough to choose \( \psi \in \text{Incr}(\mathbb{N}) \) so that for \( x' > x \) we have \( \psi(x') > \rho(\psi(x)) \). One way of doing this is to define \( \psi(0) = 0 \) and inductively \( \psi(n + 1) = \rho(\psi(n)) \).

**Proof of Theorem A.2.** By Lemma A.3 we can assume that \( \delta \) is monotone, namely \( \delta(x', y') \leq \delta(x, y) \) for all \( x' > x, y' > y \), provided \( x \neq y \).

To prove the theorem we proceed by induction on \( k \). When \( k = 1 \), consider the function \( \varepsilon(n) \) in (A.2). By compactness of \( M \) there exist \( x \in M \) and a subsequence \( f \circ \sigma \) of \( f \) converging to \( x \) with the property

\[
d_M(f(\sigma n), x) \leq \varepsilon(n).
\]

For \( n \neq m \) it follows from (A.3) that

\[
d_M(f(\sigma n), f(\sigma m)) \leq \delta(n, m).
\]

So \( f \circ \sigma \) is 1-Lipschitz.

Now assume inductively that the thesis holds for some \( k \in \mathbb{N} \), and let us prove it for \( k + 1 \). So let \( f : \mathbb{N}^{(k+1)} \to M \). We need to prove the existence of \( \sigma \in \text{Incr}(\mathbb{N}) \) such that

\[
d_M(f(\sigma(n, m)), f(\sigma(n', m'))) \leq \delta_k(1)((n, m), (n', m'))
\]

for all \( (n, m) \in \mathbb{N}^{(k+1)} \) and \( (n', m') \in \mathbb{N}^{(k+1)} \), where \( m = (m_1, \ldots, m_k) \) and \( m' = (m'_1, \ldots, m'_k) \).

Given \( n \in \mathbb{N} \) define \( f_n : \mathbb{N}^{(k)} \to M \) by

\[
f_n(m) := \begin{cases} f(n, m) & \text{if } n < m_1, \\ \bot & \text{if } n \geq m_1 \end{cases}
\]

where \( \bot \) is an arbitrary element of \( M \). Note that the condition \( n < m_1 \) is equivalent to \( (n, m) \in \mathbb{N}^{(k+1)} \).

By inductive assumption, for all \( n \in \mathbb{N} \) there exists \( \theta_n \in \text{Incr}(\mathbb{N}) \) such that \( f_n \circ \theta_{n*} : \mathbb{N}^{(k)} \to M \) is 1-Lipschitz. By a recursive construction, we can also assume that \( \theta_{n+1} \) is a subsequence of \( \theta_n \), namely \( \theta_{n+1} = \theta_n \circ \gamma_n \) for some \( \gamma_n \in \text{Incr}(\mathbb{N}) \). Indeed to obtain \( \theta_{n+1} \) as desired it suffices to apply the induction hypothesis to \( f_{n+1} \circ \theta_{n*} : \mathbb{N}^{(k)} \to M \) rather than directly to \( f_{n+1} \).
Since $f_n \circ \theta_n$ is 1-Lipshitz, there exist the limit 
\[
  g(n) := \lim_{\min(m) \to \infty} f(n, \theta_n(m)) \tag{A.10}
\]
Passing to a subsequence we can further assume that all the values of $f_n \circ \theta_n$ are within distance $\frac{1}{2} \varepsilon(n)$ from its limit, namely:
\[
d_M(g(n), f(n, \theta_n(m))) < \frac{1}{2} \varepsilon(n). \tag{A.8}
\]
Let $J_n := \theta_n(N) \subset N$ and let $\tau \in \text{Incr}(N)$ be such that:
\[
\tau(n + 1) \in J_{\tau(n)}. \tag{A.9}
\]
It then follows that
\[
\forall n, m \in \tau(N) \quad m > n \implies m \in J_n. \tag{A.10}
\]
For later purposes we need to define $\tau(n + 1)$ as an element of $J_{\tau(n)}$ bigger than its $n + 1$-th element, namely $\tau(n + 1) > \theta_{\tau(n)}(n + 1)$. So, for the sake of concreteness, we define inductively $\tau(0) := 0$ and $\tau(n + 1) := \theta_{\tau(n)}(n + 2)$. It then follows that:
\[
\forall i, j \in \tau(N) \quad \forall k \in N \quad j > i, j \geq k \implies \tau(j) > \theta_{\tau(i)}(k). \tag{A.11}
\]
Reasoning as in the case $k = 1$, there is $\lambda \in \text{Incr}(N)$ and $x_\infty \in M$ such that
\[
d_M(g(\lambda(n)), x_\infty) < \frac{1}{2} \varepsilon(n). \tag{A.12}
\]
Now define $\sigma := \tau \circ \lambda \in \text{Incr}(N)$. Note that $\sigma(N) \subset \tau(N)$ so \ref{A.10} and \ref{A.11} continue to hold with $\sigma$ instead of $\tau$. We claim that $f \circ \sigma_* : \mathbb{N}^{[k+1]} \to M$ is 1-Lipschitz.

As a first step we show that
\[
\exists k > m : (f \circ \sigma_*)(n, m) = (f_{\sigma(n)} \circ \theta_{\sigma(n)})(n, k) \tag{A.13}
\]
where $k > m$ means that $k_i > m_i$ for all respective components. To prove \ref{A.13} recall that $(f \circ \sigma_*)(n, m) = f(\sigma(n), \sigma(m_1), \ldots, \sigma(m_k))$. Since $n < \min(m)$, by \ref{A.10} the elements $\sigma(m_1), \ldots, \sigma(m_k)$ are in the image of $\theta_{\sigma(n)}$, namely for each $i$ we have $\sigma(m_i) = \theta_{\sigma(n)}(k_i)$ for some $k_i \in N$. Moreover applying \ref{A.11} we must have $k_i > m_i$. The proof of \ref{A.13} is thus complete.

It follows from \ref{A.13} and \ref{A.8} that $(f \circ \sigma_*)(n, m)$ is within distance $\frac{1}{2} \varepsilon(\sigma(n))$ from its limit $g(\sigma(n))$, which in turn is within distance $\frac{1}{2} \varepsilon(n)$ from its limit $x_\infty$ by \ref{A.12}. We have thus proved:
\[
d_M(f(\sigma_*(n, m)), x_\infty) < \frac{1}{2} \varepsilon(\sigma(n)) + \frac{1}{2} \varepsilon(n). \tag{A.14}
\]
Recalling that for $x \neq y$ we have $\varepsilon(x) + \varepsilon(y) \leq \delta(x, y)$, it follows that for $n \neq n'$ the left-hand side of \ref{A.6} is bounded by $\frac{1}{2} \delta(\sigma(n), \sigma(n')) + \frac{1}{2} \delta(n, n')$, which in turn is $\leq \delta(n, n')$ by monotonicity of $\delta$.

If remains to prove \ref{A.6} in the case $n = n'$. Given $m, m'$ as in \ref{A.6}, we apply \ref{A.13} to get $k > m, k' > m'$ with $(f \circ \sigma_*)(n, m) = (f_{\sigma(n)} \circ \theta_{\sigma(n)})(n, k)$ and $(f \circ \sigma_*)(n, m') = (f_{\sigma(n)} \circ \theta_{\sigma(n)})(n, k')$.

Using the monotonicity of $\delta$ and the fact that $f_{\sigma(n)} \circ \theta_{\sigma(n)}$ is 1-Lipschitz, it follows that:
\[
d_M(f(\sigma_*(n, m)), f(\sigma_*(n, m'))) \leq \delta_k(k, k') \leq \delta_k(m, m'). \tag{A.15}
\]
Remark A.4. Theorem A.2 implies that there exists an infinite set \( J = \sigma(N) \subset \mathbb{N} \) such that, for all \( 0 \leq m < k \) and \( (i_1, \ldots, i_m) \in J^m \), there are limit points \( x_{i_1 \ldots i_m} \in M \) with the property
\[
x_{i_1 \ldots i_m} = \lim_{(i_{m+1}, \ldots, i_k) \to \infty} x_{i_1 \ldots i_k},
\]
where we set \( x_{i_1 \ldots i_k} := f(i_1, \ldots, i_k) \). Moreover, by choosing the distance
\[
\delta(n, m) = \varepsilon |2^{-n} - 2^{-m}|,
\]
we may also require
\[
d_M (x_{i_1 \ldots i_m}, x_{i_1 \ldots i_k}) \leq \frac{\varepsilon}{2^{\sigma^{-1}(i_{m+1})}} \quad \forall (i_1, \ldots, i_k) \in J^k.
\]

Appendix B. Exchangeable measures

Let \( \Lambda \) be a compact metric space. We recall a classical notion of exchangeable measure due to De Finetti [DF:74], showing some equivalent conditions.

**Proposition B.1.** Given \( m \in M^1(\Lambda^N) \), the following conditions are equivalent:

a) \( m \) is \( S_c(N) \)-invariant;

b) \( m \) is \( \text{Inj}(N) \)-invariant;

c) \( m \) is \( \text{Incr}(N) \)-invariant.

**Definition B.2.** If \( m \) satisfies one of these equivalent conditions we say that \( m \) is exchangeable.

Notice that an exchangeable measure is always shift-invariant, while there are shift-invariant measures which are not exchangeable. To prove Proposition B.1 we need some preliminary results concerning measures satisfying condition (c).

**Definition B.3.** Given \( m \in M(\Lambda^N) \) and \( f \in L^p(\Lambda^N) \), with \( p \in [1, +\infty] \), we let
\[
\tilde{f} = E(f|\mathcal{A}_s) \in L^p(\Lambda^N)
\]
be the conditional probability of \( f \) with respect to the \( \sigma \)-algebra \( \mathcal{A}_s \) of the shift-invariant Borel subsets of \( \Lambda^N \). In particular, \( \tilde{f} \) is shift-invariant, and by Birkhoff’s theorem (see for instance [P:82]) we have
\[
\tilde{f} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ S^k,
\]
where the limit holds almost everywhere and in the strong topology of \( L^1(\Lambda^N) \).

**Lemma B.4.** If \( m \in M^1(\Lambda^N) \) is \( \text{Incr}(N) \)-invariant, then for all \( f \in L^\infty(\Lambda^N, m) \) we have
\[
\tilde{f} = \lim_{n \to \infty} f \circ S^n,
\]
where the limit is taken in the weak* topology of \( L^\infty(\Lambda^N) \), namely for every \( g \in L^1(\Lambda^N, m) \) we have
\[
\lim_{n \to \infty} \int_{\Lambda^N} g(f \circ S^n) \, dm = \int_{\Lambda^N} g\tilde{f} \, dm.
\]
Proof. It suffices to prove that \( \lim_{n \to \infty} f \circ S^n \) exists, since in that case it is necessarily equal to the (weak *) limit of the arithmetic means \( \frac{1}{n} \sum_{k=0}^{n-1} f \circ S^k \), and therefore to \( \tilde{f} \) (since \( \tilde{f} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ S^k \) in an even stronger topology). Since the sequence \( f \circ S^n \) is equibounded in \( L^\infty(\Lambda^N, m) \), it is enough to prove \( \text{(B.2)} \) for all \( g \) in a dense subset \( D \) of \( L^1(\Lambda^N) \). We can take \( D \) to be the set of those functions \( g \in L^1(\Lambda^N, m) \) depending on finitely many coordinates (namely \( g(x) = h(x_1, \ldots, x_r) \) for some \( r \in \mathbb{N} \) and some \( h \in L^1(\Lambda^r, m) \)). The convergence of \( \text{(B.2)} \) for \( g(x) = h(x_1, \ldots, x_r) \) follows from the fact that \( \sigma \cdot m = m \) for all \( \sigma \in \text{Incr}(\mathbb{N}) \), which implies that the quantity in \( \text{(B.2)} \) is constant for all \( n > r \). Indeed to prove that 

\[
\int_{\Lambda^N} g(f \circ S^n) \, dm = \int_{\Lambda^N} g(f \circ S^{n+l}) \, dm
\]

it suffices to consider the function \( \sigma \in \text{Incr}(\mathbb{N}) \) which fixes \( 0, \ldots, r-1 \) and sends \( i \) to \( i + l \) for \( i \geq r \). \( \square \)

We are now ready to prove the equivalence of the conditions in the definition of exchangeable measure.

Proof of Proposition \([B.7]\). Since \( \mathcal{G}_c(\mathbb{N}) \subset \text{Inj}(\mathbb{N}) \) and \( \text{Incr}(\mathbb{N}) \subset \text{Inj}(\mathbb{N}) \), the implications \( b) \Rightarrow a) \) and \( b) \Rightarrow c) \) are obvious. The implication \( a) \Rightarrow b) \) is also obvious since it is true on the Borel subsets of \( \Lambda^N \) of the form \( x \in \Lambda^N : x_{i_1} \in A_1, \ldots, x_{i_r} \in A_r \), which generate the whole Borel \( \sigma \)-algebra of \( \Lambda^N \).

Let \( m \in \mathcal{M}^1(\Lambda^N) \) be \( \text{Incr}(\mathbb{N}) \)-invariant, and let us prove that \( m \) is \( \text{Inj}(\mathbb{N}) \)-invariant. So let \( \sigma \in \text{Inj}(\mathbb{N}) \). We must show that

\[
\int_{\Lambda^N} g \, dm = \int_{\Lambda^N} g \circ \sigma^* \, dm,
\]

for all \( g \in C(\Lambda^N) \). It suffices to prove \( \text{(B.3)} \) for \( g \) in a dense subset \( D \) of \( C(\Lambda^N) \). So we can assume that \( g(x) \) has the form \( g_0(x_0) \cdot \ldots \cdot g_r(x_r) \) for some \( r \in \mathbb{N} \) and \( g_1, \ldots, g_r \in C(\Lambda) \). Note that \( g_i(x_i) = (g_i \circ P_i)(x) \) where \( P_i : \Lambda^N \to \Lambda \) is the projection on the \( i \)-th coordinate. Since \( P_i = P_0 \circ S^i \) where \( S^i \) is the shift, we can apply Lemma \( \text{(B.4)} \) to obtain

\[
\int_{\Lambda^N} g \, dm = \int_{\Lambda^N} g_1 \circ P_1 \cdot \ldots \cdot g_r \circ P_1 \, dm.
\]

Reasoning in the same way for the function \( g \circ \sigma^* \), we finally get

\[
\int_{\Lambda^N} g \circ \sigma^* \, dm = \int_{\Lambda^N} g_1 \circ P_1 \cdot \ldots \cdot g_r \circ P_1 \, dm = \int_{\Lambda^N} g \, dm.
\]

\( \square \)

Definition \( B.5 \). We say that \( m \in \mathcal{M}^1(\Lambda^N) \) is asymptotically exchangeable if the limit \( m' = \lim_{\theta \to \infty} \text{Incr}(\mathbb{N}) \cdot m \) exists in \( \mathcal{M}^1(\Lambda^N) \) and is an exchangeable measure.

Remark \( B.6 \). Note that if \( m \) is asymptotically exchangeable, then:

\[
m' := \lim_{\text{min}_{\theta \to \infty} \text{Incr}(\mathbb{N})} \theta \cdot m
\]

\[
= \lim_{k \to \infty} S^k \cdot m.
\]

However it is possible that \( \lim_{k \to \infty} S^k \cdot m \) exists and is exchangeable, and yet \( m \) is not asymptotically exchangeable. As an example one may start with the Bernoulli probability measure \( \mu \) on \( 2^\mathbb{N} \) with \( \mu(\{x_i = 0\}) = 1/2 \) and then
consider the conditional probability \( m(\cdot) = \mu(\cdot|A) \) where \( A \subset 2^\mathbb{N} \) is the set of those sequences \( x \in 2^\mathbb{N} \) satisfying \( x_{(n+1)^2} = 1 - x_n^2 \) for all \( n \).

**Remark B.7.** If \( m \) is asymptotically exchangeable and \( m' = \lim_{k \to \infty} S^k \cdot m \), then for all \( r \in \mathbb{N} \) and \( g_1, \ldots, g_r \in C(\Lambda) \) we have

\[
(B.6) \quad \lim_{i_1, \ldots, i_r \to \infty} \int_{\Lambda^N} g_1(x_{i_1}) \cdots g_r(x_{i_r}) \, dm = \int_{\Lambda^N} g_1(x_1) \cdots g_r(x_r) \, dm'.
\]

**Theorem B.8.** Given \( m \in \mathcal{M}^1(\Lambda^\mathbb{N}) \) there is \( \sigma \in \omega(\omega) \) such that \( \sigma \cdot m \) is asymptotically exchangeable.

**Proof.** Fix \( m \in \mathcal{M}^1(\Lambda^\mathbb{N}) \). Given \( r \in \omega \) consider the function \( f : \omega^r \to \mathcal{M}^1(\Lambda^r) \) sending \( \iota \) to \( \iota \cdot m \in \mathcal{M}^1(\Lambda^r) \). By Lemma A.1 there is an infinite set \( J_r \subset \omega \) such that

\[
(B.7) \quad \lim_{\min(\iota) \to \infty, \iota \in J^r} \iota \cdot m
\]

exists in \( \mathcal{M}^1(\Lambda^r) \). By a diagonal argument we choose the same set \( J = J_r \) for all \( r \). Let \( \sigma \in \text{Incr}(\mathbb{N}) \) be such that \( \sigma(\mathbb{N}) = J \). We claim that \( \sigma \cdot m \) is asymptotically exchangeable. To this aim consider \( m_k := S^k \cdot \sigma \cdot m \in \mathcal{M}^1(\Lambda^\mathbb{N}) \). By compactness there is an accumulation point \( m' \in \mathcal{M}^1(\Lambda^\mathbb{N}) \) of \( \{m_k\}_{k \in \mathbb{N}} \). We claim that

\[
(B.8) \quad \lim_{\min(\theta) \to \infty, \theta \in J^\omega} \theta \cdot \sigma \cdot m = m',
\]

hence in particular \( m_k \to m' \) (taking \( \theta = S^k \)). Note that the claim also implies that \( m' \) is exchangeable. Indeed, given an increasing function \( \gamma : \mathbb{N} \to \mathbb{N} \), to show \( \gamma \cdot m' = m' \) it suffices to replace \( \theta \) with \( \theta \circ \gamma \) in equation (B.8). Since the subset of \( C(\Lambda^\mathbb{N}) \) consisting of the functions depending on finitely many coordinates is dense, it suffices to prove that for all \( r \in \mathbb{N} \) and \( \iota \in \mathbb{N}^r \) the limit

\[
(B.9) \quad \lim_{\min(\iota) \to \infty, \iota \in J^\omega} \iota \cdot \theta \cdot \sigma \cdot m
\]

exists in \( \mathcal{M}^1(\Lambda^r) \) (the limit being necessarily \( \iota \cdot m' \)). This is however just a special case of equation (B.7). \( \square \)

We give below some representation results for exchangeable measures. First note that if \( \Lambda \) is countable, a measure \( m \in \mathcal{M}^1(\Lambda^\mathbb{N}) \) is determined by the values it takes on the sets of the form \( \{x : x_{i_1} = a_1, \ldots, x_{i_r} = a_r\} \).

**Lemma B.9.** If \( \Lambda \) is countable, a measure \( m \in \mathcal{M}(\Lambda^\mathbb{N}) \) is exchangeable if and only if it admits a representation of the following form. There is a probability space \((\Omega, \mu)\) (which in fact can be taken to be \((\Lambda^\mathbb{N}, m)\)) and a family \( \{\psi_a\}_{a \in \Lambda} \) in \( L^\infty(\Omega, \mu) \) such that for all \( i_1 < \ldots < i_r \in \mathbb{N} \) we have

\[
(B.10) \quad m(\{x : x_{i_1} = a_1, \ldots, x_{i_r} = a_r\}) = \int_{\Omega} \psi_{a_1} \cdot \cdots \cdot \psi_{a_r} \, d\mu.
\]
Proof. Since the right-hand side of the equation does not depend on \(i_1, \ldots, i_r\) a measure \(m \in \mathcal{M}^1(\Lambda^N)\) admitting the above representation is clearly exchangeable. Conversely if \(m\) is exchangeable it suffices to take \(\psi_a = \overline{\chi_a}\) where \(\chi_a\) is the characteristic function of the set \(\{x : x_0 = a\}\). We can in fact obtain the desired result by a repeated application of Equation \((\text{B.2})\) after observing that the characteristic function \(\chi_{\{x : x_i = a_1, \ldots, x_r = a_r\}}\) is the product \(\chi_{\{x : x_i = a_1\}} \cdot \ldots \chi_{\{x : x_r = a_r\}}\) and \(\chi_{\{x : x_i = a\}} = \chi_a \circ (S^*)^i\). \(\square\

Corollary B.10. If \(\Lambda\) is countable and \(m \in \mathcal{M}^1(\Lambda^N)\) is exchangeable, then \(m(\{x \in \Lambda^N : x_0 = x_1\}) \neq 0\).

Proof. By \((\text{B.10})\) \(m(\{x \in \Lambda^N : x_0 = x_1\}) = \sum_{a \in \Lambda} \int \psi_a \, d\mu \neq 0\). \(\square\)

Corollary B.11. If \(p \in \mathbb{N}\) and \(m \in \mathcal{M}^1(\Lambda^N)\) is exchangeable, then \(m(\{x \in \Lambda^N : x_0 = x_1\}) \geq \frac{1}{p}\).

Proof. Write \(m(\{x \in \Lambda^N : x_0 = x_1\}) = \sum_{a \in \Lambda} \int_{\Omega} \psi_a \, d\mu\) and apply the Cauchy-Schwarz inequality to the linear operator \(\sum \int_{\Omega} \psi_a \, d\mu\) to obtain

\[
(B.11) \quad \left(\sum_{a < p} \int_{\Omega} \psi_a \, d\mu\right) \cdot \left(\sum_{a < p} \int_{\Omega} 1 \, d\mu\right) \geq \left(\sum_{a < p} \int_{\Omega} \psi_a \, d\mu\right)^2
\]

which gives the desired result. \(\square\)

Thanks to a theorem of De Finetti, suitably extended in [HS:55] there is an integral representation à la Choquet for the exchangeable measures on \(\Lambda^N\), where \(\Lambda\) is a compact metric space. More precisely, in [HS:55] it is shown that the extremal points of the (compact) convex set of all exchangeable measures are given by the product measures \(\sigma^N\), with \(\sigma \in \mathcal{M}^1(\Lambda)\). As a consequence, Choquet theorem [C:69] provides an integral representation for any exchangeable measure \(m\) on \(\Lambda^N\), i.e. there is a probability measure \(\mu \in \mathcal{M}^1(\Lambda)\) such that

\[
(B.12) \quad m = \int_{\mathcal{M}^1(\Lambda)} \sigma^N \, d\mu(\sigma).
\]

When \(\Lambda\) is finite, i.e. \(\Lambda = p = \{0, \ldots, p-1\}\) for some \(p \in \mathbb{N}\), we can identify \(\mathcal{M}^1(\Lambda)\) with the symplex \(\Sigma_p\) of all \(\lambda \in [0,1]^p\) such that \(\sum_{i=0}^{p-1} \lambda_i = 1\). Given \(\lambda \in \Sigma_p\), we denote by \(B_\lambda\) the product measure on \(\Lambda^N\), namely the unique measure making all the events \(\{x : x_i = a\}\) independent with measure \(B_\lambda(\{x : x_i = a\}) = \lambda_a\). In this case, \((\text{B.12})\) becomes

\[
(B.13) \quad m = \int_{\Sigma_p} B_\lambda \, d\mu(\lambda),
\]

where \(\mu\) is a probability measure on \(\Sigma_p\).

We finish this excursionus on exchangeable measures with the following result:

**Proposition B.12.** Let \(m \in \mathcal{M}^1(\Lambda^N)\) be exchangeable, then for all \(f \in L^1(\Lambda^N)\) the following conditions are equivalent:

a) \(f\) is \(\Theta_c(\mathbb{N})\)-invariant;

b) \(f\) is \(\text{Inj}(\mathbb{N})\)-invariant;
c) \( f \) is shift-invariant.

**Proof.** Since \( \Theta_c(\mathbb{N}) \subset \text{Inj}(\mathbb{N}) \) and \( s \in \text{Inj}(\mathbb{N}) \), the implications b) \( \Rightarrow \) a) and b) \( \Rightarrow \) c) are obvious.

In order to prove that a) \( \Rightarrow \) b), we let \( F = \{ \sigma \in \text{Inj}(\mathbb{N}) : f = f \circ \sigma^* \} \), which is a closed subset of \( \text{Inj}(\mathbb{N}) \) containing \( \Theta_c(\mathbb{N}) \). Then, it is enough to observe that \( \Theta_c(\mathbb{N}) \) is a dense subset of \( \text{Inj}(\mathbb{N}) \subset \mathbb{N}^\mathbb{N} \), with respect to the product topology of \( \mathbb{N}^\mathbb{N} \), so that \( F = \Theta_c(\mathbb{N}) = \text{Inj}(\mathbb{N}) \).

Let us prove that c) \( \Rightarrow \) a). Let \( \sigma \in \Theta_c(\mathbb{N}) \) and let \( n \) be such that \( \sigma(i) = i \) for all \( i \geq n \). It follows that \( S^k \circ \sigma^* = S^k \), for all \( k \geq n \). As a consequence, for \( m \)-almost every \( x \in \Lambda^\mathbb{N} \) it holds
\[
f \circ \sigma^*(x) = f \circ S^n \circ \sigma^*(x) = f \circ S^n(x) = f(x),
\]
where the first equality holds since the measure \( m \) is \( \Theta_c(\mathbb{N}) \)-invariant. \( \square \)

Notice that from Proposition B.12 it follows that \( \tilde{f} \) is \( \text{Inj}(\mathbb{N}) \)-invariant for all \( f \in L^1(\Lambda^\mathbb{N}) \). In particular, for an exchangeable measure, the \( \sigma \)-algebra of the shift-invariant sets coincides with the (a priori smaller) \( \sigma \)-algebra of the \( \text{Inj}(\mathbb{N}) \)-invariant sets.

**References**


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