SYMMETRY RESULTS FOR NONLINEAR ELLIPTIC OPERATORS WITH UNBOUNDED DRIFT

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Abstract. We prove the one-dimensional symmetry of solutions to elliptic equations of the form $-\mathrm{div}(e^{G(x)}a(|\nabla u|)\nabla u) = f(u)e^{G(x)}$, under suitable energy conditions. Our results holds without any restriction on the dimension of the ambient space.

Contents

1. Introduction

In this paper we study the one-dimensional symmetry of solutions to nonlinear equations of the following type:

$$\mathrm{div}(a(|\nabla u|)\nabla u) + a(|\nabla u|)\langle \nabla G(x), \nabla u\rangle + f(u) = 0, \tag{1}$$

or in a more compact form

$$-\mathrm{div}(e^{G(x)}a(|\nabla u|)\nabla u) = f(u)e^{G(x)}, \tag{2}$$

where $f \in C^1(\mathbb{R})$[1], $G \in C^2(\mathbb{R}^n)$ and $a \in C^{1,1}_{loc}((0, +\infty))$. We also require that the function a satisfies the following structural conditions:

$$a(t) > 0 \quad \text{for any } t \in (0, +\infty), \tag{3}$$

$$a(t) + a'(t)t > 0 \quad \text{for any } t \in (0, +\infty). \tag{4}$$

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[1] One could consider functions f which are only locally lipschitz continuous, as in [9]. To avoid inessential technicalities, we do not treat this case here.

Observe that the general form of (2) encompasses, as very special cases, many elliptic singular and degenerate equations. Indeed, if $G \equiv 0$ and $a(t) = t^{p-2}$, $1 < p < +\infty$, or $a(t) = 1/\sqrt{1+t^2}$ then we obtain the p-Laplacian and the mean curvature equations respectively. Moreover, if $a(t) \equiv 1$ and $G(x) = -|x|^2/2$ equation (1) boils down to the classical Ornstein-Uhlenbeck operator for which we refer to [1] and the references therein.

To prove the one-dimensional symmetry of solutions we follow the approach introduced in [5] and further developed in [9].

Following [5, 9, 3], we define $A : \mathbb{R}^n \to Mat(n \times n), \lambda_1 \in C^0((0, +\infty)), \lambda_G \in C^0(\mathbb{R}^{2n})$ as follow

$$A_{hk}(\xi) := \frac{a'(|\xi|)}{|\xi|}\xi_h\xi_k + a(|\xi|)\delta_{hk} \quad \text{for any } 1 \leq h, k \leq n, \tag{5}$$

$$\lambda_1(t) := a(t) + a'(t)t \quad \text{for any } t > 0 \tag{6}$$

and

$$\lambda_G(x) := \text{maximal eigenvalue of } \nabla^2 G(x). \tag{7}$$

Definition 1.1. *We say that u is a weak solution to (1) if $u \in C^1(\mathbb{R}^n)$ and denoted by $\mathrm{d}\mu = e^{G(x)}\mathrm{d}x$*

$$\int_{\mathbb{R}^n} \langle a(|\nabla u|)\nabla u, \nabla\varphi\rangle - f(u)\varphi \, \mathrm{d}\mu = 0 \qquad \forall \varphi \in C_c^1(\mathbb{R}^n) \tag{8}$$

and either (A1) or (A2) is satisfied, where :

(A1) $\{\nabla u = 0\} = \emptyset$.

(A2) $a \in C^0([0, +\infty))$ *and*

$$\text{the map} \quad t \to ta(t) \quad \text{belongs to} \quad C^1([0, +\infty)).$$

Notice that (8) is well-defined, thanks to (A1) or (A2).

Notice also that weak solutions to (1) are critical points of the functional

$$I(u) := \int_{\mathbb{R}^n} \Big(\Lambda(|\nabla u|) + F(u)\Big)\mathrm{d}\mu \tag{9}$$

where $F'(t) = -f(t)$, $\mathrm{d}\mu = e^{G(x)}\mathrm{d}x$ and

$$\Lambda(t) := \int_0^t a(|\tau|)\tau \mathrm{d}\tau.$$

The regularity assumption $u \in C^1(\mathbb{R}^n)$ is always fulfilled in many important cases, like those involving the p-Laplacian operator or the mean curvature operator. For instance, when $a(t) = t^{p-2}$, $1 < p < +\infty$, any distribution solution $u \in W^{1,p}_{loc}(\mathbb{R}^n) \cap L^{\infty}_{loc}(\mathbb{R}^n)$ is of class C^1, by the well-known results in [16, 22]). In light of this, and in view of the great generality of the function a, it is natural to work in the above setting.

Definition 1.2. *Let $h \in L^1_{loc}(\mathbb{R}^n)$ and let u be a weak solution to (1). We say that u is $h-$stable if*

$$\int_{\mathbb{R}^n} \langle A(\nabla u)\nabla\varphi, \nabla\varphi\rangle - f'(u)\varphi^2 \,\mathrm{d}\mu \geq \int_{\mathbb{R}^n} a(|\nabla u|)h\varphi^2 \,\mathrm{d}\mu \quad \forall\varphi \in C^1_c(\mathbb{R}^n). \tag{10}$$

Remark 1.3. When $a(t) \equiv 1$, Definition 1.2 boils down to the $h-$stability condition introduced in [2, 3].

When $h \equiv 0$, then u satisfies the classical stability condition [5, 9, 11, 10], and we simply say that u is stable. In particular, every minimum point of the functional (9) is a stable solution to (1).

Let us also point out that, in view of (A1) or (A2), the integral

$$\int_{\mathbb{R}^n} \langle A(\nabla u)\nabla\varphi, \nabla\varphi\rangle - f'(u)\varphi^2 - a(|\nabla u|)h\varphi^2 \,\mathrm{d}\mu \tag{11}$$

is well defined.[2] In particular, under the condition (A2) the function A can be extended by continuity at the origin, by setting $A_{hk}(0) := a(0)\delta_{hk}$.

We can now state our main symmetry results:

Theorem 1. *Assume $G \in C^2(\mathbb{R}^n)$ and $h \in L^1_{loc}(\mathbb{R}^n)$ with $h \geq \lambda_G$. Let $u \in C^1(\mathbb{R}^n) \cap C^2(\{\nabla u \neq 0\})$ with $\nabla u \in H^1_{loc}(\mathbb{R}^n)$ be a $h-$stable weak solution to (1).*

Assume that there exists $C > 0$ such that

$$\lambda_1(t) \leq Ca(t) \quad \forall t > 0, \tag{12}$$

and one of the following conditions holds

(a) *there exists $C_0 \geq 1$ such that $\int_{B_R} a(|\nabla u|)|\nabla u|^2 \mathrm{d}\mu \leq C_0 R^2$ for any $R \geq C_0$,*
(b) *$n = 2$ and u satisfies $a(|\nabla u|)|\nabla u|^2 e^G \in L^\infty(\mathbb{R}^2)$.*

Then u is one-dimensional, i.e. there exists $\omega \in \mathbb{S}^{n-1}$ and $u_0 : \mathbb{R} \to \mathbb{R}$ such that

$$u(x) = u_0(\langle\omega, x\rangle) \quad \forall x \in \mathbb{R}^n. \tag{13}$$

Moreover,

$$\big\langle (h(x)\mathrm{I}_n - \nabla^2 G(x))\nabla u, \nabla u \big\rangle = 0 \quad \forall x \in \mathbb{R}^n. \tag{14}$$

In particular, if u_0 is not constant, there are C and g of class C^2 such that

$$G(x) = C(\langle\omega, x\rangle) + g(x'), \tag{15}$$

where $x' := x - \langle\omega, x\rangle\,\omega$ and $\lambda_G(x) = h(x) = C''(\langle\omega, x\rangle)$ for all $x \in \mathbb{R}^n$.

[2] cfr. also [9, footnote 1 at p. 742 and footnote 2 at page 743].

Remark 1.4. Paradigmatic examples satisfying the assumption (12) are the p-Laplacian operator, for any $p \in (1, +\infty)$, and the generalized mean curvature operator obtained by setting $a(t) := (1 + t^q)^{-\frac{1}{q}}$, with $q > 1$.

Theorem 2. *Let $G(x) := -|x|^2/2$, $a(t) := t^{p-2}$ with $p > 1$ and let $u \in C^1(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n)$ be a monotone weak solution to (1), i.e., such that*

$$\partial_i u(x) > 0 \quad \forall x \in \mathbb{R}^n, \tag{16}$$

for some $i \in \{1, \dots, n\}$. Suppose that u satisfies either (a) or (b) in Theorem 1. Then u is one-dimensional. Moreover, if either $p = 2$ or $a(t) := (1 + t^q)^{-\frac{1}{q}}$ with $q > 1$, then the same conclusion holds for every monotone weak solution $u \in C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Theorem 3. *Let u be a bounded weak solution to*

$$\Delta u - \langle x, \nabla u \rangle + f(u) = 0 \tag{17}$$

with Morse index k. Then,

(i) *if $k \leq 2$ then u is one-dimensional;*
(ii) *if $3 \leq k \leq n$ then u is a function of at most $k - 1$ variables, i.e. there exists $C \in Mat((k-1) \times n)$ and $u_0 : \mathbb{R}^{k-1} \to \mathbb{R}$ such that*

$$u(x) = u_0(Cx) \quad \forall x \in \mathbb{R}^n. \tag{18}$$

The result in Theorem 3 should be compared with the analysis in [14], where the author shows that a minimal surface in the Gauss space, with Morse index less than or equal to n, is necessarily a hyperplane through the origin. These minimal surfaces are important geometric objects as they correspond to self-shrinkers for the mean curvature flow, which are the model of generic singularities. Since the minimal surface equation in the Gauss space arises as singular limit, as $\epsilon \to 0$, of the equations

$$\Delta u - \langle x, \nabla u \rangle - \frac{W'(u)}{\epsilon} = 0,$$

where W is a double-well potential (see for instance [23]), it is natural to ask if there exist bounded solutions to (17), with Morse index less than or equal to n, which are not one-dimensional.

2. A geometric Poincaré inequality

We recall the following result which has been proved in [9].

Lemma 2.1. *For any $\xi \in \mathbb{R}^n \setminus \{0\}$, the matrix $A(\xi)$ is symmetric and positive definite and its eigenvalues are $\lambda_1(|\xi|), \cdots, \lambda_n(|\xi|)$, where λ_1 is as in (6) and $\lambda_i(t) = a(t)$ for every $i = 2, \dots, n$. Moreover,*

$$\langle A(\xi)\xi, \xi \rangle = |\xi|^2 \lambda_1(|\xi|), \tag{19}$$

and

$$0 \leq \langle A(\xi)(V - W), (V - W) \rangle = \langle A(\xi)V, V \rangle + \langle A(\xi)W, W \rangle - 2 \langle A(\xi)V, W \rangle, \tag{20}$$

for any $V, W \in \mathbb{R}^n$ and any $\xi \in \mathbb{R}^n \setminus \{0\}$.

Lemma 2.2. *Let $u \in C^1(\mathbb{R}^n) \cap C^2(\{\nabla u \neq 0\})$ with $\nabla u \in H^1_{loc}(\mathbb{R}^n)$ be a weak solution to (1). Then for any $i = 1, \ldots, n$, and any $\varphi \in C^1_c(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n} \langle A(\nabla u)\nabla u_i, \nabla\varphi\rangle - a(|\nabla u|)\langle \nabla u, \nabla(G_i)\rangle \varphi - f'(u)u_i\varphi \, \mathrm{d}\mu = 0. \tag{21}$$

Proof. By Lemma 2.2 in [9] we have

$$\text{the map} \quad x \to W(x) := a(|\nabla u(x)|)\nabla u(x) \quad \text{belongs to } W^{1,1}_{loc}(\mathbb{R}^n, \mathbb{R}^n), \tag{22}$$

therefore, since $e^{G(x)} \in C^2(\mathbb{R}^n)$ we get

$$We^G \in W^{1,1}_{loc}(\mathbb{R}^n, \mathbb{R}^n). \tag{23}$$

By Stampacchia's Theorem (see, e.g. [18, Theorem 6:19]), we get $\partial_i(We^G) = 0$ for almost any $x \in \{We^G = 0\} = \{W = 0\}$, that is

$$\partial_i(We^G) = 0$$

for almost any $x \in \{\nabla u = 0\}$. In the same way, by Stampacchia's Theorem and (A2), it can be proven that $\nabla u_i(x) = 0$, and hence $A(\nabla u(x))\nabla u_i(x) = 0$, for almost any $x \in \{\nabla u = 0\}$. Moreover, the following relation holds (see [9] for the proof)

$$\partial_i(We^G) = (A(\nabla u)\nabla u_i + a(|\nabla u|)\nabla u G_i)e^G \quad \text{on } \{\nabla u \neq 0\}, \tag{24}$$

and thanks to the previous observations

$$\partial_i(We^G) = (A(\nabla u)\nabla u_i + a(|\nabla u|)\nabla u G_i)e^G \quad \textit{a.e.} \text{ in } \mathbb{R}^n. \tag{25}$$

Applying (8) with φ replaced by φ_i and making use of (23) and (25), we obtain

$$\begin{aligned}
0 &= \int_{\mathbb{R}^n} a(|\nabla u|)\langle \nabla u, \nabla \varphi_i\rangle - f(u)\varphi_i \, \mathrm{d}\mu \\
&= \int_{\mathbb{R}^n} -\langle A(\nabla u)\nabla u_i, \nabla\varphi\rangle - a(|\nabla u|)\langle \nabla u, \nabla\varphi\rangle G_i \, \mathrm{d}\mu \\
&+ \int_{\mathbb{R}^n} f'(u)u_i\varphi + f(u)\varphi G_i \, \mathrm{d}\mu \\
&= \int_{\mathbb{R}^n} -\langle A(\nabla u)\nabla u_i, \nabla\varphi\rangle - a(|\nabla u|)\langle \nabla u, \nabla(\varphi G_i)\rangle \, \mathrm{d}\mu \\
&+ \int_{\mathbb{R}^n} a(|\nabla u|)\langle \nabla u, \nabla G_i\rangle \varphi + f'(u)u_i\varphi + f(u)\varphi G_i \, \mathrm{d}\mu.
\end{aligned}$$

Recalling (8), applied with φ replaced by φG_i, we obtain the thesis. □

From now on, we use A and a, as a short-hand notation for $A(\nabla u)$ and $a := a(|\nabla u|)$ respectively. In the following result we prove that every monotone solution to (1) is indeed $h-$stable.

Lemma 2.3. *Assume that u is a weak solution to* (1) *and that there exists $i \in \{1, \ldots, n\}$ such that*

$$u_i := \partial_i u(x) > 0 \quad \forall x \in \mathbb{R}^n \tag{26}$$

then u is $h-$stable, with

$$h(x) := \frac{\langle \nabla u(x), \nabla G_i(x)\rangle}{u_i(x)}.$$

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $\psi := \varphi^2/u_i$. We use (20) with $V := \varphi \nabla u_i/u_i$ and $W := \nabla\varphi$ to obtain that

$$\frac{2\varphi}{u_i}\langle A\nabla u_i, \nabla\varphi\rangle - \frac{\varphi^2}{u_i^2}\langle A\nabla u_i, \nabla u_i\rangle \le \langle A\nabla\varphi, \nabla\varphi\rangle.$$

From this and Lemma 2.2 we get

$$\begin{aligned}0 &= \int \langle A\nabla u_i, \nabla\psi\rangle - a\langle \nabla u, \nabla G_i\rangle \psi - f'(u)u_i\psi \,\mathrm{d}\mu \\ &= \int 2\frac{\varphi}{u_i}\langle A\nabla u_i, \nabla\varphi\rangle - \frac{\varphi^2}{u_i^2}\langle A\nabla u_i, \nabla u_i\rangle - a\frac{\varphi^2}{u_i}\langle \nabla u, \nabla G_i\rangle - f'(u)\varphi^2 \,\mathrm{d}\mu \\ &\le \int \langle A\nabla\varphi, \nabla\varphi\rangle - a\frac{\varphi^2}{u_i}\langle \nabla u, \nabla G_i\rangle - f'(u)\varphi^2 \,\mathrm{d}\mu.\end{aligned} \tag{27}$$

Notice that we can apply Lemma 2.2 since, in view of (26), u has no critical points and thus it is of class C^2, by the classical regularity results. □

The following Lemma can be proved using the same tecniques implemented in [9, Lemma 2.4],

Lemma 2.4. *Let $h \in L^1_{loc}(\mathbb{R}^n)$. Let $u \in C^1(\mathbb{R}^n) \cap C^2(\{\nabla u \neq 0\})$ with $\nabla u \in H^1_{loc}(\mathbb{R}^n)$ be a $h-$stable weak solution to (1). Then, (10) holds for any $\varphi \in H^1_0(B)$ and for any ball $B \subset \mathbb{R}^n$. Moreover, under the assumptions of Lemma 2.2,*

$$\int_{\mathbb{R}^n} \langle A(\nabla u)\nabla u_i, \nabla\varphi\rangle - a(|\nabla u|)\langle \nabla u, \nabla(G_i)\rangle \varphi - f'(u)u_i\varphi \,\mathrm{d}\mu = 0. \tag{28}$$

for any $i = 1, \dots, n$, any $\varphi \in H^1_0(B)$ and any ball $B \subset \mathbb{R}^n$.

Proposition 2.5. *Let $h \in L^1_{loc}(\mathbb{R}^n)$ and $u \in C^1(\mathbb{R}^n) \cap C^2(\{\nabla u \neq 0\})$ with $\nabla u \in H^1_{loc}(\mathbb{R}^n)$ be a $h-$stable weak solution to (1). Then, for every $\varphi \in C^1_c(\mathbb{R}^n)$ it holds*

$$\begin{aligned}\int_{\mathbb{R}^n} a(|\nabla u|)h(x)|\nabla u|^2\varphi^2 \,\mathrm{d}\mu \le \int_{\mathbb{R}^n} &|\nabla u|^2\langle A\nabla\varphi, \nabla\varphi\rangle + a(|\nabla u|)\left\langle \nabla^2 G\nabla u, \nabla u\right\rangle \varphi^2 \\ &+ \varphi^2\Big[\langle A\nabla|\nabla u|, \nabla|\nabla u|\rangle - \sum_{i=1}^n \langle A(\nabla u)\nabla u_i, \nabla u_i\rangle\Big]\mathrm{d}\mu.\end{aligned} \tag{29}$$

Proof. We start observing that by Stampacchia's Theorem, since $\mu << \mathcal{L}^n$, we get

$$\nabla|\nabla u|(x) = 0 \quad \mu-\text{a.e. } x \in \{|\nabla u| = 0\}, \tag{30}$$

$$\nabla u_j(x) = 0 \quad \mu-\text{a.e. } x \in \{|\nabla u| = 0\} \subseteq \{u_j = 0\}, \tag{31}$$

for any $j = 1, \dots, n$. Let $\varphi \in C^1_c(\mathbb{R}^n)$ and $i = 1, \dots, n$. Using (21) with test function $u_i\varphi^2$ and summing over $i = 1, \dots, n$ we get

(32)

$$\int_{\mathbb{R}^n} \sum_{i=1}^n \left\langle A(\nabla u)\nabla u_i, \nabla(u_i\varphi^2)\right\rangle - f'(u)|\nabla u|^2\varphi^2 \,\mathrm{d}\mu = \int_{\mathbb{R}^n} a(|\nabla u|)\left\langle \nabla^2 G\nabla u, \nabla u\right\rangle \varphi^2 \,\mathrm{d}\mu$$

Using (10) with test function $|\nabla u|\varphi$ (note that this choice is possible thanks to Lemma 2.4) we then get

(33)

$$\begin{aligned}\int_{\mathbb{R}^n} a(|\nabla u|)h(x)|\nabla u|^2\varphi^2 \,\mathrm{d}\mu &\le \int_{\mathbb{R}^n} \Big\langle \Big(A(\nabla u(x))\nabla(|\nabla u|\varphi)\Big), \nabla(|\nabla u|\varphi)\Big\rangle - f'(u)|\nabla u|^2\varphi^2 \,\mathrm{d}\mu \\ &= \int_{\mathbb{R}^n} |\nabla u|^2 \langle A\nabla\varphi, \nabla\varphi\rangle \,\mathrm{d}\mu + \int_{\{\nabla u\neq 0\}} \varphi^2 \langle A\nabla|\nabla u|, \nabla|\nabla u|\rangle \\ &\quad + 2\varphi|\nabla u| \langle A\nabla\varphi, \nabla|\nabla u|\rangle - f'(u)|\nabla u|^2\varphi^2 \,\mathrm{d}\mu\end{aligned}$$

and by (32) we conclude that

(34)

$$\begin{aligned}\int_{\mathbb{R}^n} a(|\nabla u|)h(x)|\nabla u|^2\varphi^2 \,\mathrm{d}\mu &\le \int_{\mathbb{R}^n} |\nabla u|^2 \langle A\nabla\varphi, \nabla\varphi\rangle \,\mathrm{d}\mu + \int_{\{\nabla u\neq 0\}} a(|\nabla u|) \left\langle \nabla^2 G\nabla u, \nabla u\right\rangle \varphi^2 \mathrm{d}\mu \\ &\quad + \int_{\{\nabla u\neq 0\}} \varphi^2 \Big[\langle A\nabla|\nabla u|, \nabla|\nabla u|\rangle - \sum_{i=1}^n \langle A(\nabla u)\nabla u_i, \nabla u_i\rangle \Big] \mathrm{d}\mu.\end{aligned}$$

which is the thesis. □

Remark 2.6. Letting

$$L_{u,x} := \{y \in \mathbb{R}^n \mid u(y) = u(x)\},$$

we denote by $\nabla_T u$ the tangential gradient of u along $L_{u,x} \cap \{\nabla u \neq 0\}$, and by $k_1, \ldots, k_{n-1}$ the principal curvatures of $L_{u,x} \cap \{\nabla u \neq 0\}$.

(35)

$$\langle A\nabla|\nabla u|, \nabla|\nabla u|\rangle - \sum_{i=1}^n \langle A(\nabla u)\nabla u_i, \nabla u_i\rangle = a\Big[|\nabla|\nabla u||^2 - \sum_{i=1}^n |\nabla u_i|^2\Big] - a'|\nabla u||\nabla_T|\nabla u||^2$$

and using (6) we get

$$\begin{aligned}\langle A\nabla|\nabla u|, \nabla|\nabla u|\rangle &- \sum_{i=1}^n \langle A(\nabla u)\nabla u_i, \nabla u_i\rangle \\ &= -\lambda_1|\nabla_T|\nabla u||^2 - a(|\nabla u|)\Big(\sum_{i=1}^n |\nabla u_i|^2 - |\nabla_T|\nabla u||^2 - |\nabla|\nabla u||^2\Big)\end{aligned} \tag{36}$$

Notice that the quantity

$$\sum_{i=1}^n |\nabla u_i|^2 - |\nabla|\nabla u||^2 - |\nabla_T|\nabla u||^2$$

has a geometric interpretation, in the sense that it can be expressed in terms of the principal curvatures of level sets of u.

More precisely, the following formula holds (see [9, 20, 21])

$$\sum_{i=1}^{n}|\nabla u_i|^2-|\nabla|\nabla u||^2-|\nabla_T|\nabla u||^2=|\nabla u|^2\sum_{j=1}^{n-1}k_j^2 \qquad \text{on } L_{u,x}\cap\{\nabla u\neq 0\}\,, \tag{37}$$

so that (34) becomes

$$\begin{aligned}&\int_{\{\nabla u\neq 0\}} a(|\nabla u|)h(x)|\nabla u|^2\varphi^2+\Big[\lambda_1|\nabla_T|\nabla u||^2+a(|\nabla u|)|\nabla u|^2\sum_{j=1}^{n-1}k_j^2\Big]\varphi^2\\&\quad -a(|\nabla u|)\left\langle\nabla^2G\nabla u,\nabla u\right\rangle\varphi^2\ \mathrm{d}\mu\\&\leq\int_{\mathbb{R}^n}\langle A\nabla\varphi,\nabla\varphi\rangle\,|\nabla u|^2\mathrm{d}\mu.\end{aligned}$$

Rearranging the terms, we obtain

$$\begin{aligned}&\int_{\{\nabla u\neq 0\}} a(|\nabla u|)\left\langle(h(x)I-\nabla^2G)\nabla u,\nabla u\right\rangle\varphi^2+\Big[\lambda_1|\nabla_T|\nabla u||^2+a(|\nabla u|)|\nabla u|^2\sum_{j=1}^{n-1}k_j^2\Big]\varphi^2\ \mathrm{d}\mu\\&\leq\int_{\mathbb{R}^n}\langle A\nabla\varphi,\nabla\varphi\rangle\,|\nabla u|^2\mathrm{d}\mu,\end{aligned}\tag{38}$$

where $I\in Mat(n\times n)$ denotes the identity matrix.
Notice that from (38) we also obtain

$$\int_{\{\nabla u\neq 0\}} a(|\nabla u|)\left\langle(h(x)I-\nabla^2G)\nabla u,\nabla u\right\rangle\varphi^2\mathrm{d}\mu\leq\int_{\mathbb{R}^n}\langle A\nabla\varphi,\nabla\varphi\rangle\,|\nabla u|^2\mathrm{d}\mu. \tag{39}$$

3. One-dimensional symmetry of solutions

In this section we will use (38) to prove several one-dimensional results for solutions to (1), following the approach introduced in [5] and then developed in [9]. Notice that, more recently, a similar approach has also been used to handle semilinear equations in Riemannian and subriemannian spaces (see [6, 7, 8, 12, 13, 19]) and also to study problems involving the Ornstein-Uhlenbeck operator [2], as well as semilinear equations with unbounded drift [3].

The following Lemma is proved in [9, 13].

Lemma 3.1. *Let $g\in L^\infty_{loc}(\mathbb{R}^n,[0,+\infty))$ and let $q>0$. Let also, for any $\tau>0$,*

$$\eta(\tau):=\int_{B_\tau} g(x)\mathrm{d}x. \tag{40}$$

Then, for any $0<r<R$,

$$\int_{B_R\setminus B_r}\frac{g(x)}{|x|^q}\mathrm{d}x\leq q\int_r^R\frac{\eta(\tau)}{|\tau|^{q+1}}\mathrm{d}\tau+\frac{1}{R^q}\eta(R) \tag{41}$$

Proof of Theorem 1.

Let us fix $R>0$ (to be taken appropriately large in what follows) and $x\in\mathbb{R}^n$ and let us define

$$\varphi(x) := \begin{cases} 1 & \text{if} \quad x\in B_{\sqrt{R}} \\ 2\frac{\log(R/|x|)}{\log(R)} & \text{if} \quad x\in B_R\setminus B_{\sqrt{R}} \\ 0 & \text{if} \quad x\in\mathbb{R}^n\setminus B_R, \end{cases} \tag{42}$$

where $B_R := \{y\in\mathbb{R}^n \mid |y|<R\}$. Obviously $\varphi\in Lip(\mathbb{R}^n)$ and

$$|\nabla\varphi(x)| \le C_2\frac{\chi_{\sqrt{R},R}(x)}{\log(R)|x|}$$

for suitable $C_2>0$. Hence for every $R>e$, (38) together with $h\ge\lambda_G$ yields

(43)

$$\int_{\{\nabla u\neq 0\}\cap\overline{B}_R}\Big[\lambda_1|\nabla_T|\nabla u||^2 + a(|\nabla u|)|\nabla u|^2\sum_{j=1}^{n-1}k_j^2\Big]\varphi^2\,\mathrm{d}\mu \le \int_{\mathbb{R}^n}\langle A(\nabla u)\nabla\varphi,\nabla\varphi\rangle\,|\nabla u|^2\mathrm{d}\mu$$

therefore, by (12)

(44)

$$\begin{aligned}\int_{\{\nabla u\neq 0\}\cap\overline{B}_R}\Big[\lambda_1|\nabla_T|\nabla u||^2 + a(|\nabla u|)|\nabla u|^2\sum_{j=1}^{n-1}k_j^2\Big]\varphi^2\,\mathrm{d}\mu &\le (1+C)\int_{\mathbb{R}^n}a(|\nabla u|)|\nabla\varphi|^2|\nabla u|^2\mathrm{d}\mu \\ &\le \frac{(1+C)C_2^2}{\log(R)^2}\int_{B_R\setminus B_{\sqrt{R}}}\frac{a(|\nabla u|)|\nabla u|^2}{|x|^2}\mathrm{d}\mu.\end{aligned}$$

Applying Lemma 3.1 with $g=a(|\nabla u|)|\nabla u|^2e^G$ and $q=2$, and recalling that

$$\int_{B_R}a(|\nabla u|)|\nabla u|^2\mathrm{d}\mu\le C_0R^2$$

for R large, we obtain

(45)

$$\begin{aligned}\int_{\{\nabla u\neq 0\}\cap\overline{B}_R}\Big[\lambda_1|\nabla_T|\nabla u||^2 + a(|\nabla u|)|\nabla u|^2\sum_{j=1}^{n-1}k_j^2\Big]\varphi^2\,\mathrm{d}\mu &\le \frac{(1+C)C_0C_2^2}{\log(R)^2}\Big[2\int_{\sqrt{R}}^{R}\frac{1}{|\tau|}\mathrm{d}\tau+1\Big] \\ &\le 2\frac{(1+C)C_0C_2^2}{\log(R)}.\end{aligned}$$

Therefore, sending $R\to+\infty$ in (45) we get

$$k_j(x)=0 \quad \text{and} \quad |\nabla_T|\nabla u||(x)=0 \tag{46}$$

for every $j=1,\dots,n-1$ and every $x\in\{\nabla u\neq 0\}$. From this and Lemma 2.11 in [9] we get the one-dimensional symmetry of u.

Let us now suppose $n=2$ and $a(|\nabla u|)|\nabla u|^2 e^G \in L^\infty(\mathbb{R}^2)$. Taking in (38) the following test function

$$\varphi(x) = \max\left[0, \min\left(1, \frac{\ln R^2 - \ln|x|}{\ln R}\right)\right], \tag{47}$$

recalling that $h \geq \lambda_G$ and following [9, Cor. 2.6], we then obtain

$$\int_{\{\nabla u\neq 0\}\cap \overline{B}_R} \Big[\lambda_1|\nabla_T|\nabla u||^2 + a(|\nabla u|)|\nabla u|^2 \sum_{j=1}^{n-1} k_j^2\Big]\varphi^2 \,\mathrm{d}\mu \leq C' \int_{B_{R^2}\setminus B_R} \frac{a(|\nabla u|(x))}{|x|^2\,(\ln R)^2}|\nabla u|^2 e^{G(x)}\mathrm{d}x$$

for some constant $C' > 0$. When $R \to +\infty$, since $a(|\nabla u|)|\nabla u|^2 e^{G(x)}$ is bounded, the r.h.s. term of the previous inequality goes to zero, and we conclude again that u is one-dimensional.

Assume now that u is not constant. If we take in (39) the same test functions as above, we get

$$\int_{\mathbb{R}^n} a(|\nabla u|)\,\big\langle (h(x)\mathrm{I}_n - \nabla^2 G(x))\nabla u, \nabla u\big\rangle \,\mathrm{d}\mu(x) = 0\,.$$

Using the fact that $u(x) = u_0(\langle \omega, x\rangle)$ and $a(t) > 0$ we obtain that $\big\langle (h(x)\mathrm{I}_n - \nabla^2 G(x))\omega, \omega\big\rangle = 0$ for all x such that $u_0'(\langle\omega, x\rangle) \neq 0$. Since u is not constant and is a solution to the elliptic equation (1), the set of points such that $u_0'(\langle\omega, x\rangle) = 0$ has zero measure, so, by the regularity of G we conclude that

$$\big\langle (h(x)\mathrm{I}_n - \nabla^2 G(x))\omega, \omega\big\rangle = 0 \qquad \forall\, x \in \mathbb{R}^n\,,$$

which gives (14) and (15).

□

As pointed out in [3], a Liouville type result follows from Theorem 1.

Corollary 3.2. *Let G, h, u satisfy the assumptions in Theorem 1. Assume further that $h \in C^0(\mathbb{R}^n)$ and $h(x) > \lambda_G(x)$ for some $x \in \mathbb{R}^n$. Then u is constant. In particular, if u is a stable solution, that is $h \equiv 0$, and $\lambda_G(x) < 0$ for some $x \in \mathbb{R}^n$, then u is constant.*

In the following lemma we give a sufficient condition for a solution u to satisfy condition (a) in Theorem 1.

Lemma 3.3. *Let u be a weak solution to (1). Then, for each $\varphi \in C_c^1(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} a(|\nabla u|)|\nabla u|^2\varphi \mathrm{d}\mu = -\int_{\mathbb{R}^n} a(|\nabla u|)\,\langle \nabla u, \nabla\varphi\rangle\, u\mathrm{d}\mu + \int_{\mathbb{R}^n} f(u)u\varphi\mathrm{d}\mu. \tag{48}$$

In particular, if $t \to ta(t) \in L^\infty((0,+\infty))$, $u \in L^\infty(\mathbb{R}^n)$ and $\mu(\mathbb{R}^n) < +\infty$ then there exists $C > 0$ such that

$$\int_{\mathbb{R}^n} a(|\nabla u|)|\nabla u|^2 \mathrm{d}\mu \leq C\,. \tag{49}$$

Proof. Clearly (48) follows by taking $u\varphi$ as test function in (8). Let us show (49). For every $R > 1$ let $\Phi_R \in C^\infty(\mathbb{R})$ be such that $\Phi_R(t) = 1$ if $t \leq R$, $\Phi_R(t) = 0$ if $t \geq R+1$ and

$\Phi'_R(t) \le 3$ for $t \in [R, R+1]$, and define $\varphi(x) := \Phi_R(|x|)$. Then $|\nabla\varphi(x)| \le |\Phi'_R(|x|)| \le 3$, and (48) yields

$$\int_{B_R} a(|\nabla u|)|\nabla u|^2 \mathrm{d}\mu \le 3\int_{B_{R+1}\setminus B_R} a(|\nabla u|)|\nabla u||u|\mathrm{d}\mu + \int_{B_{R+1}} |f(u)||u|\mathrm{d}\mu \le C,$$

which gives (49) by letting $R \to +\infty$.

□

In the rest of the section we fix $G(x) = -|x|^2/2$. We start with a result which follows directly from Lemma 2.3.

Lemma 3.4. *Let $G(x) := -|x|^2/2$ and assume that u is a monotone weak solution to* (1)*, i.e. there exists $i \in \{1, \ldots, n\}$ such that*

$$\partial_i u(x) > 0 \quad \forall x \in \mathbb{R}^n, \tag{50}$$

then $u \in C^2(\mathbb{R}^n)$ and u is $(-1)-$stable.

Proof of Theorem 2. We start observing that u is $(-1)-$stable by Lemma 2.3. Since $\nabla^2 G(x) = -Id$ we have

$$-1 = h(x) = \lambda_G(x) = -1. \tag{51}$$

If $a(t) = t^{p-2}$ for some $p > 1$ then

$$\lambda_1(t) = (p-1)t^{p-2} = (p-1)a(t) \quad \forall t > 0 \tag{52}$$

and the conclusion follows by Theorem 1. If $a(t) = (1+t^q)^{-\frac{1}{q}}$ with $q > 1$ then

$$\lambda_1(t) = (1+t^q)^{-\frac{1}{q}} - (1+t^q)^{-\frac{q+1}{q}} t^q \le a(t) \quad \forall t > 0, \tag{53}$$

$$ta(t) \le 1 \quad \forall t > 0. \tag{54}$$

By Lemma 3.3 and (54) there exists $C > 0$ such that

$$\int_{\mathbb{R}^n} a(|\nabla u|)|\nabla u|^2 \mathrm{d}\mu \le C. \tag{55}$$

Notice that, if $a(t) = 1$ for every $t > 0$, by Theorem [17, Theorem 4.1] we have $u \in H^2(\mathbb{R}^n, \mu)$, so that (55) holds in this case, too. The conclusion follows by (53), (55) and Theorem 1. □

4. Solutions with Morse index bounded by the euclidean dimension

In this section we will focus on the Ornstein-Uhlenbeck operator. More precisely, we will consider weak solutions $u \in H^1(\mathbb{R}^n, \mu) \cap L^\infty(\mathbb{R}^n)$ to

$$\Delta u - \langle x, \nabla u\rangle + f(u) = 0 \tag{56}$$

where $f \in C^1(\mathbb{R})$, and we will prove some new symmetry results for solutions with Morse index $k \le n$. We recall that, by Theorem [17, Theorem 4.1], bounded weak solutions to (56) satisfy $u \in H^2(\mathbb{R}^n, \mu) \cap L^\infty(\mathbb{R}^n)$.

Definition 4.1. *A bounded weak solution u to the Ornstein-Uhlenbeck operator has Morse index $k \in \mathbb{N}$ if k is the maximal dimension of a subspace X of $H^1(\mathbb{R}^n, \mu)$ such that*

$$Q_u(\varphi) := \int_{\mathbb{R}^n} |\nabla\varphi|^2 - f'(u)\varphi^2 \mathrm{d}\mu < 0 \quad \forall\varphi \in X \setminus \{0\}. \tag{57}$$

Remark 4.2. Let u be a bounded solution to (56) and let $L : H^2(\mathbb{R}^n, \mu) \to L^2(\mathbb{R}^n, \mu)$ be the linear operator defined as

$$L(v) := -\Delta v + \langle \nabla v, x\rangle - f'(u)v. \tag{58}$$

Notice that L is self-adjoint in $L^2(\mathbb{R}^n, \mu)$ with compact inverse, so that by the Spectral Theorem [15] there exists an orthonormal basis of $L^2(\mathbb{R}^n, \mu)$ consisting of eigenvectors of L, and each eigenvalue of L is real. Then, u has Morse index k if and only if L has exactly k strictly negative eigenvalues, repeated according to their geometric multiplicity (see for instance [17, Theorem 4.1]).

The following Proposition is proved in [2, Lemma 3.2].

Proposition 4.3. *Let u be a bounded weak solution to* (56)*. If for some $i = 1, \dots, n$, u_i is not identically zero then it is an eigenfunction of L with eigenvalue -1, i.e.*

$$\int_{\mathbb{R}^n} \langle \nabla u_i, \nabla\varphi\rangle + u_i\varphi - f'(u)u_i\varphi \;\mathrm{d}\mu = 0, \quad \forall\varphi \in H^1(\mathbb{R}^n, \mu). \tag{59}$$

We are now in a position to prove Theorem 3.

Proof of Theorem 3.

By [17, Theorem 4.1] every bounded weak solution to (56) belongs to $H^2(\mathbb{R}^n, \mu)$, hence $u_i \in H^1(\mathbb{R}^n, \mu)$ for all $i = 1, \dots, n$. Therefore, using (59) with u_i as test function we obtain

$$Q_u(u_i) = \int_{\mathbb{R}^n} |\nabla u_i|^2 - f'(u)u_i^2 \mathrm{d}\mu = -\int_{\mathbb{R}^n} u_i^2 \le 0, \quad \forall i = 1, \dots, n. \tag{60}$$

In particular

$$Q_u(u_i) < 0 \tag{61}$$

for every $i = 1, \dots, n$ such that u_i is not identically zero. Let L be the operator defined in (58). If $k = 0$ then u is stable, hence it is constant by Corollary 3.2. If $k = 1$ then, by Remark 4.2 and Proposition 4.3, it follows that -1 is the smallest eigenvalue of L, that is

$$\inf_{\varphi\in H^1(\mathbb{R}^n,\mu), ||\varphi||_{L^2(\mathbb{R}^n,\mu)}=1} \Big(\int_{\mathbb{R}^n} |\nabla\varphi|^2 - f'(u)\varphi^2 \;\mathrm{d}\mu\Big) = -1. \tag{62}$$

Using (62) it follows that u is $(-1)-$stable and therefore, by Theorem 1, u is one-dimensional. Assume now $2 \le k \le n$ and define $S := \{i \in \{1, \dots, n\} \mid u_i(x) \ne 0, \text{ for some } x \in \mathbb{R}^n\}$ and $X := \mathrm{span}_{i\in S}\{u_i\} \subset H^1(\mathbb{R}^n, \mu)$. Clearly,

$$Q_u(v) < 0 \quad \forall v \in X \setminus \{0\} \tag{63}$$

therefore, by Definition 4.1, X has dimension less or equal than k, i.e. there exists $I \subset S$ with $|I| \ge |S| - k$ such that $\{u_i\}_{i\in I}$ are linearly dependent [15]. This means that, up to an orthogonal change of variables, u depends on at most k variables. Let us assume by contradiction that u is a function of exactly k variables. We claim that -1 is the smallest eigenvalue of L, as before. Indeed, if this is not the case, then there exist $\lambda < -1$ and $v \in H^1(\mathbb{R}^n, \mu)$, with $v \not\equiv 0$, such that $L(v) = \lambda v$. Therefore, by the linear independence of eigenvectors associated to different eigenvalues, it follows that $Y := \text{span}\{u_i, v\}$ has dimension equal to $k+1$ and $Q_u(w) < 0$ for every $w \in Y \setminus \{0\}$ which is in contradiction with the fact that u has Morse index k. This proves that u is a function of at most $(k-1)$ variables, as claimed.

□

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