Regularity results for boundaries in $\mathbb{R}^2$ with prescribed anisotropic curvature

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Abstract

In this paper we consider the anisotropic perimeter

$$P_\varphi(E) = \int_{\partial E} \varphi(\nu_E)\, \mathrm{d}\mathcal{H}^1$$

defined on subsets $E \subset \mathbb{R}^2$, where the anisotropy φ is a (possibly non symmetric) norm on $\mathbb{R}^2$ and ν_E is the exterior unit normal vector to ∂E. We consider quasi-minimal sets E (which include sets with prescribed curvature) and we prove that $\partial E \setminus \Sigma(E)$ is locally a bi-lipschitz curve and the singular set $\Sigma(E)$ is closed and discrete.
We then classify the global P_φ-minimal sets. In particular we find that global minimal sets may have a singular point if and only if $\{\varphi \leq 1\}$ is a triangle or a quadrilateral and that sets with two singularities exist if and only if $\{\varphi \leq 1\}$ is a triangle.
We finally show that the boundary of a subset of $\mathbb{R}^2$ which locally minimizes the anisotropic perimeter plus a volume term (prescribed constant curvature) is contained, up to a translation and a rescaling, in the boundary of the Wulff shape determined by the anisotropy.

1 Introduction

The aim of this paper is to characterize the subsets of a given open set $\Omega \subset \mathbb{R}^2$ which minimize (locally in Ω) a general anisotropic functional of the form

$$E \mapsto \int_{\partial^* E \cap \Omega} \varphi(\nu_E(x))\, \mathrm{d}\mathcal{H}^1(x) - \lambda |E|, \tag{1}$$

where φ is a generic positive one-homogeneous convex function expressing the anisotropy, ν_E is the exterior unit normal to the reduced boundary $\partial^* E$ and $\lambda \in \mathbb{R}$ is a given constant. When $\lambda = 0$ the functional in (1) provides a notion of anisotropic perimeter in $\mathbb{R}^2$.

In the isotropic case, i.e. when $\varphi(x) = |x|$, this problem has been studied by several authors in any dimension [11], [10]. In particular, it is well–known that, given a set $E \subset \mathbb{R}^2$ minimizing (1), the part of ∂E lying inside Ω is smooth and has constant mean curvature equal to λ; this implies (because we are in two dimensions) that ∂E coincides, locally in Ω, with the boundary of a circle of radius $1/\lambda$.

As it is shown in [7], the isotropic version of (1) is strictly related to equilibrium surfaces in capillarity phenomena, like for example the problem of identifying the shape of a liquid rising in a narrow tube.

When the function φ^2 is smooth and uniformly convex, it is well–known [6], [1] that problem (1) always admits minimizers which are hyper–surfaces of class $\mathcal{C}^{1,\alpha}$ out of a closed singular set of zero $\mathcal{H}^{n-1}$–measure. We point out that, in our case, the function φ^2 is not necessarily differentiable nor uniformly convex, hence such regularity results cannot be applied. However, in two dimensions one can still get some regularity results. Indeed, in [4] a class of functionals more general than (1) is studied and it is proved that a minimizer has boundary which is locally a lipschitz curve out of a closed singular set Σ of zero $\mathcal{H}^1$–measure. In this paper we improve this result by showing that Σ is a discrete set (see Theorem 3.4).

When φ is piecewise linear the anisotropy is called *crystalline*. In this respect, another motivation for considering the functional (1) comes from the theory of evolutions by crystalline mean curvature. Indeed, as it is shown in [14], [5], when the starting set is an "admissible" polyhedron, during such evolutions some facets may break or bend. If we let Ω be one of these facets then the appearing fracture corresponds to the boundary of a set $E \subset \Omega$ which is a minimum of (1) (see [15], [14]).

We point out that in [12] a problem strictly related to ours is considered, that is to analyze the structure and regularity properties of two–dimensional clusters minimizing the anisotropic perimeter, under variations which preserve the volume.

The plan of the paper is the following: in Section 2 we introduce some notation that we shall use throughout the paper. In Section 3 we introduce the class of ω-minimal sets, which include the minima of (1). In Theorem 3.4 we prove that the singular points of an ω-minimizer are isolated. In Theorem 3.11 we completely describe the minima of (1) for $\lambda = 0$ and $\Omega = \mathbb{R}^2$. In Section 4 we prove that, given a minimizer E of (1), the connected components of $\partial E \cap \Omega$ are contained, up to a translation, in $\frac{1}{\lambda}\partial W_\varphi$, where the Wulff shape W_φ is the analogous of the unit ball in the anisotropic setting (Theorem 4.5).

2 Notations

Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ be a function such that

1. $\varphi(x) = 0 \Leftrightarrow x = 0$;
2. $\varphi(tx) = t\varphi(x) \qquad \forall t > 0$;
3. $\varphi(x - y) \leq \varphi(x - z) + \varphi(z - y)$.

A function with these properties will be called a *general norm* on $\mathbb{R}^2$.

We define $\varphi^o\colon \mathbb{R}^2 \to \mathbb{R}$ as

$$\varphi^o(v) = \sup_{\xi \neq 0} \frac{\langle \xi, v \rangle}{\varphi(\xi)}$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product of $\mathbb{R}^2$. It is not difficult to check that φ^o is also a general norm on $\mathbb{R}^2$ and that

$$\varphi(\xi) = \sup_{v \neq 0} \frac{\langle \xi, v \rangle}{\varphi^o(v)}.$$

We will call *Wulff shape* the set $W_\varphi := \{x \in \mathbb{R}^2 \colon \varphi^o(x) < 1\}$ and *Frank diagram* the set $F_\varphi := \{\xi \in \mathbb{R}^2 \colon \varphi(\xi) < 1\}$. We define the duality (multivalued) maps T and T^o by

$$\begin{aligned} T(v) &:= \{\xi \in \mathbb{R}^2 \colon \varphi(\xi) = \varphi^o(v),\ \langle \xi, v \rangle = \varphi(\xi)\varphi^o(v)\} \\ T^o(\xi) &:= \{v \in \mathbb{R}^2 \colon \varphi^o(v) = \varphi(\xi),\ \langle \xi, v \rangle = \varphi(\xi)\varphi^o(v)\}. \end{aligned}$$

Notice that if H is the half-space $H = \{x \colon \langle \nu, x \rangle < 0\}$ with exterior normal vector $\nu_H = \nu$ then, given $x \in \partial W_\varphi$, $W_\varphi - x \subset H$ if and only if $x \in T^o(\nu)$ (or equivalently $\nu \in T(x)$).

We will denote with $\mathcal{H}^k$ the k-dimensional Hausdorff measure and we let $|A| := \mathcal{H}^2(A)$ for $A \subset \mathbb{R}^2$ be the Lebesgue measure.

Given $v \in \mathbb{R}^2 \setminus \{0\}$ we say that a set S is a *graph along* v, if it is not possible to find two different points $x, y \in S$ such that $x - y = \lambda v$ for any $\lambda \in \mathbb{R}$.

The anisotropic perimeter of a set E in the open set $A \subset \mathbb{R}^2$ is defined by

$$P_\varphi(E, A) := \sup \left\{ \frac{1}{|W_\varphi|} \int_E \operatorname{div} \psi(x) \, \mathrm{d}x \colon \psi \in \mathcal{C}^1_c(A; \mathbb{R}^2), \varphi^o(\psi(y)) \leq 1 \quad \forall y \in A \right\}.$$

We let $B_\rho(x) := \{y \colon ||x - y|| < \rho\}$ be the usual euclidean ball of $\mathbb{R}^2$, $B_\rho := B_\rho(0)$ and we define

$$\begin{aligned} \partial E &:= \big\{x \in \mathbb{R}^2 \colon \forall \rho > 0 \quad |E \cap B_\rho(x)| \in \,]0, |B_\rho(x)|[\,\big\}, \\ \overline{E} &:= \big\{x \in \mathbb{R}^2 \colon \forall \rho > 0 \quad |E \cap B_\rho(x)| \neq 0\big\}. \end{aligned}$$

It holds, as usual, that $\overline{E}, \partial E$ are closed sets, $\mathring{E}$ is open and $\overline{E} = \partial E \cup \mathring{E}$. Notice that if $|E \triangle F| = 0$ then $\partial E = \partial F$ ($E \triangle F := (E \setminus F) \cup (F \setminus E)$).

3 ω-minimal sets

Let $\omega\colon [0, \infty[\,\to [0, \infty[$ be an increasing function such that $\lim_{\rho \to 0} \omega(\rho) = 0$ and let $\Omega \subset \mathbb{R}^2$ be an open set.

We say that a set $E \subset \mathbb{R}^2$ is ω-minimal in Ω, if

$$P_\varphi(E, B_\rho(x)) \leq P_\varphi(F, B_\rho(x)) + \omega(\rho)\sqrt{|E \triangle F|}$$

whenever $x \in \partial E$, $B_\rho(x) \Subset \Omega$ and $E \triangle F \Subset B_\rho(x)$ (the notation $A \Subset B$ means that $\overline{A}$ is a compact subset of B).

We denote by $\mathcal{M}_\omega(\Omega)$ the family of all ω-minimal sets in Ω. Notice that this class of ω-minimal sets is contained in the one considered in [4]. In particular we have a lower and upper density estimates $\theta\rho \leq \mathcal{H}^1(\partial E \cap B_\rho(x)) \leq \Theta\rho$ for every $x \in \partial E$ and for all ρ smaller than a constant ρ_ω depending only on ω. We also have $\mathcal{H}^1(\partial E \setminus \partial^* E) = 0$ where $\partial^* E$ is the usual *reduced boundary*, i.e. the set of points $x \in \partial E$ where the euclidean external normal vector $\nu_E(x)$ can be defined. Moreover a representation formula is available:

$$P_\varphi(E, A) = \int_{\partial E \cap A} \varphi(\nu_E(x)) \, \mathrm{d}\mathcal{H}^1(x).$$

3.1 Regularity results for ω-minimal sets

For $E \in \mathcal{M}_\omega(\Omega)$ we define the singular set $\Sigma(E)$ as the set of all points $x \in \partial E \cap \Omega$ such that it is not possible to find a neighborhood U of x and a bilipschitz curve $\gamma\colon]0,1[\to \partial E \cap U$. Clearly $\Sigma(E)$ is relatively closed in Ω. In [4] it has been proven[1] that for $E \in \mathcal{M}_\omega(\Omega)$ there holds $\mathcal{H}^1(\Sigma(E)) = 0$ Given a set $E \in \mathcal{M}_\omega(\Omega)$ it is not difficult to show that if $\Sigma(E)$ has an accumulation point then (by a blow-up argument) there exists a minimal set $E_0 \in \mathcal{M}_0$ with at least two singular points. Unfortunately we will see that a minimal set with two singular points does exist when W_φ is a triangle. To include also this case we are going to use a slighty different tecnique to get a regularity result for $E \in \mathcal{M}_\omega(\Omega)$ (Theorem 3.4). Instead of proving a "decay of excess" result, we rely on [2] where the structure of general sets with finite perimeter is investigated.

We will be interested in the study of connected components of $\Omega \setminus E$ for $E \in M_\omega(\Omega)$. First of all notice that if $E \in \mathcal{M}_\omega(\Omega)$ then, clearly, $-(\mathbb{R}^2 \setminus E) \in \mathcal{M}_\omega(-\Omega)$. Moreover if A is a connected component of $\Omega \cap \mathring{E}$ by [2, Theorem 2] we know that $P_\varphi(E, B) = P_\varphi(A, B) + P_\varphi(E \setminus A, B)$ for all open sets $B \subset \Omega$. We want to prove that $A \in \mathcal{M}_\omega(\Omega)$. Given A' such that $A' \triangle A \Subset B_\rho(x)$ we let $E' := A' \cup (E \setminus A)$. Then we have $E \triangle E' \subset A \triangle A'$ so, as needed,

$$\begin{aligned}
P_\varphi(A, B_\rho(x)) &= P_\varphi(E, B_\rho(x)) - P_\varphi(E \setminus A, B_\rho(x)) \\
&\leq P_\varphi(E', B_\rho(x)) - P_\varphi(E \setminus A, B_\rho(x)) + \omega(\rho)\sqrt{|E \triangle E'|} \\
&\leq P_\varphi(A', B_\rho(x)) + \omega(\rho)\sqrt{|E \triangle E'|} \\
&\leq P_\varphi(A', B_\rho(x)) + \omega(\rho)\sqrt{|A \triangle A'|}.
\end{aligned}$$

So every connected component of $\Omega \setminus \partial E$, which is either a component of $\Omega \cap \mathring{E}$ or a component of $\Omega \setminus \overline{E}$ has the property $\mathcal{H}^1 \llcorner (\Omega \cap \partial A) = \mathcal{H}^1 \llcorner (\Omega \cap \partial^* A)$.

For a Jordan curve γ we denote by $\mathrm{int}(\gamma)$ the open set bounded by γ and by $\mathrm{ext}(\gamma) = \mathbb{R}^2 \setminus \overline{\mathrm{int}(\gamma)}$.

Lemma 3.1 *Let $\gamma\colon [-1,1] \to \mathbb{R}^2$ be a lipschitz parameterization of a closed Jordan curve. Suppose that $E = \mathrm{int}(\gamma) \in \mathcal{M}_\omega(\Omega)$ where $\Omega \subset \mathbb{R}^2$ is an open set, $\gamma(0) \in \Omega$. Then there exists $\varepsilon > 0$ such that $\gamma_{|]-\varepsilon,\varepsilon[}\colon]-\varepsilon, \varepsilon[\to \gamma(]-\varepsilon, \varepsilon[)$ is bi-lipschitz.*

[1] Actually in [4] a different notion of $\Sigma(E)$ is given (let us call it $\Sigma'(E)$). Since a regularity result was obtained for $\partial E \setminus \Sigma'(E)$ we get that $\Sigma(E) \subset \Sigma'(E)$. On the other hand in the following section we will classify all singular global minima, and it is easily seen that for such sets Σ and Σ' coincide. The identification can be then extended to all ω-minimal sets by a blow-up argument.

Proof:
Let L be the lipschitz constant for γ and let $M > 0$ be such that $M^{-1}||\xi|| \leq \varphi(\xi) \leq M||\xi||$ for all $\xi \in \mathbb{R}^2$. Choose $r > 0$ so that $B_{4r}(\gamma(0)) \Subset \Omega$, $\omega(r) \leq (4\sqrt{\pi}M)^{-1}$ and choose $\varepsilon > 0$ small enough so that $\varepsilon \leq r/(5L)$ and $\gamma(]-\varepsilon,\varepsilon[) \subset B_r(\gamma(0))$. For simplicity we also suppose that $\gamma(-1) = \gamma(1) \notin B_r(\gamma(0))$. Let $s, t \in\,]-\varepsilon,\varepsilon[$ be given, $S := [\gamma(s),\gamma(t)]$ be the segment between $\gamma(s)$ and $\gamma(t)$, $\rho := 2\sup_{\tau\in[s,t]} ||\gamma(\tau) - \gamma(s)||$ and $B := B_\rho(\gamma(s))$. Notice that $\rho \leq 2L|s-t| \leq 4L\varepsilon$.

Define $V = \gamma^{-1}(B)$ and let $\mathcal{F}$ be the family of the connected components of V. Clearly every $I \in \mathcal{F}$ is an open interval $I =]s_I, t_I[\subset]-\varepsilon,\varepsilon[$ and $\gamma(s_I), \gamma(t_I) \in \partial B$. If $\gamma(]s_I,t_I[) \cap S \neq \emptyset$ we may define $s'_I := \inf\{\tau \in [s_I,t_I] : \gamma(\tau) \in S\}$ and $t'_I := \sup\{\tau \in [s_I,t_I] : \gamma(\tau) \in S\}$. Hence we define a new curve $\alpha: [-1,1] \to \mathbb{R}^2$ by

$$\alpha(t) = \begin{cases} \frac{t-s'_I}{t'_I-s'_I}\gamma(s'_I) + \frac{t'_I-t}{t'_I-s'_I}\gamma(t'_I) & \text{if } \exists I \in \mathcal{F}: t \in]s_I,t_I[\\ & \text{and } \gamma(]s_I,t_I[) \cap S \neq \emptyset \\ \gamma(t) & \text{otherwise} \end{cases}$$

Notice that $\gamma([-1,1])\triangle\alpha([-1,1]) \Subset B$ hence there exists $\eta < \rho$ such that $\gamma([-1,1])\triangle\alpha([-1,1]) \Subset B' = B_\eta(\gamma(s))$. Consider now the family $\mathcal{G}$ of all connected components of $B \setminus \alpha([-1,1])$ and define $E' = E \setminus B \cup \bigcup\{A \in \mathcal{G}: A \setminus B' \subset E\}$. Clearly $E\triangle E' \subset B' \Subset B$. Moreover $\partial E' \cap B \subset \alpha(V)$ and $\alpha(t) \in B \setminus S \Rightarrow \nu_{E'}(\alpha(t)) = \nu_E(\gamma(t))$.

Suppose now that the curve γ (and hence α) is counter-clockwise oriented and define $\xi^\perp$ to be the $\pi/2$ clockwise rotation of ξ so that

$$P_\varphi(E,B) = \sum_{I\in\mathcal{F}} \int_I \varphi(\gamma'(\tau)^\perp)\,\mathrm{d}\tau.$$

Let now $\bar{I}$ be the component of $\mathcal{F}$ which contains s and t. By the minimality of E, we get

$$\begin{aligned} P_\varphi(E,B) &\leq P_\varphi(E',B) + \omega(\rho)\sqrt{\pi}\rho \leq \sum_{I\in\mathcal{F}\setminus\{\bar{I}\}} \int_I \varphi(\alpha'(\tau)^\perp)\,\mathrm{d}\tau \\ &\quad + \int_{\bar{I}\setminus[s'_{\bar{I}},t'_{\bar{I}}]} \varphi(\gamma'(\tau)^\perp)\,\mathrm{d}\tau + \varphi(\gamma(t'_{\bar{I}}) - \gamma(s'_{\bar{I}})) + \omega(\rho)\sqrt{\pi}\rho \\ &\leq P_\varphi(E,B) - \int_{[s'_{\bar{I}},t'_{\bar{I}}]} \varphi(\gamma'(\tau)^\perp)\,\mathrm{d}\tau + \varphi(\gamma(t'_{\bar{I}}) - \gamma(s'_{\bar{I}})) + \omega(\rho)\sqrt{\pi}\rho. \end{aligned}$$

Let now L be the lipschitz constant of γ and $M > 0$ be such that

$$M^{-1}||\xi|| \leq \varphi(\xi) \leq M||\xi|| \qquad \forall\xi \in \mathbb{R}^2.$$

Notice also that $s'_{\bar{I}} \leq s < t \leq t'_{\bar{I}}$ and $[\gamma(s'_{\bar{I}}),\gamma(t'_{\bar{I}})] \subset [\gamma(s),\gamma(t)]$ so that

$$\begin{aligned} 0 &\leq -\int_{[s'_{\bar{I}},t'_{\bar{I}}]} \varphi(\gamma'(\tau)^\perp)\,\mathrm{d}\tau + \varphi(\gamma(t'_{\bar{I}}) - \gamma(s'_{\bar{I}})) + \omega(\rho)\sqrt{\pi}\rho \\ &\leq -\frac{L}{M}|s-t| + M||\gamma(t)-\gamma(s)|| + 2\omega(r)\sqrt{\pi}L|t-s| \end{aligned}$$

whence we obtain the lipschitz inequality for γ^{-1}:

$$|t-s| \leq \frac{2M^2}{L}||\gamma(t)-\gamma(s)||.$$

□

We recall the following theorem which can be found in [13, Theorem 2.1].

Theorem 3.2 *Let $A \subset \mathbb{R}^2$ be a bounded open and simply connected set such that there exists a continuous surjective curve $\gamma: \partial B_1 \to \partial A$. Then there exists a continuous surjective mapping $\Phi: \overline{B_1} \to \overline{A}$ such that $\Phi_{|B_1}$ is bijective.*

The next theorem is, as a matter of fact, a kind of regularity result for the boundary of a connected component A of our minimizer E. Roughly speaking it says that every point on ∂A can be joined with some (and hence all) internal points of A. The result is achieved showing that A locally contains a continuous image of a disk.

Lemma 3.3 *Let $x_0 \in \mathbb{R}^2$, $\rho > 0$, $E \in \mathcal{M}_\omega(B_{2\rho}(x_0))$ and let A be a connected component of $B_{2\rho}(x_0) \setminus \partial E$ such that $A \cap B_\rho(x_0) \neq \emptyset$. Then, given $x \in \partial A \cap B_\rho(x_0)$, $y \in A$ there exists a continuous curve $\gamma: [0,1] \to \mathbb{R}^2$ such that $\gamma(0) = x$, $\gamma(1) = y$ and $\gamma(]0,1]) \subset A$.*

Proof:
By [2, Theorem 4] we know that there exists a family $\{\Gamma_k : k \in I \subset \mathbb{Z}\}$ of Jordan curves such that $\Gamma_k \subset \partial A$, $\mathcal{H}^1(\Gamma_i \cap \Gamma_j) = 0$, $\mathrm{int}(\Gamma_i)$, $\mathrm{int}(\Gamma_j)$ are either disjoint or one is contained in the other ($i \neq j$) and $\mathcal{H}^1(\partial A \setminus \bigcup_k \Gamma_k) = 0$ (recall that ∂A is closed and $\partial A \supset \partial^M A \supset \partial^* A$ where $\partial^M A$ is the boundary considered in [2]).

Notice that $\Gamma_i \cap \Gamma_j$ (with $i \neq j$) contains at most one point. Otherwise $\mathbb{R}^2 \setminus (\Gamma_i \cup \Gamma_j)$ would have at least 4 connected components of which only one cointains A. This is a contradiction since $\Gamma_1 \cup \Gamma_2 \subset \partial A$.

Let now J be the set of $k \in I$ such that $\Gamma_k \cap B_\rho(x_0) \neq \emptyset$. If $\Gamma_k \cap \partial B_{2\rho}(x_0) \neq \emptyset$ then $\mathcal{H}^1(\Gamma_k) \geq \rho$. Otherwise $\Gamma_k \Subset B_{2\rho}(x_0)$ so that $\mathrm{int}(\Gamma_k)$ contains a connected component F of $\mathbb{R}^2 \setminus \partial E$. By [4, Lemma 6.11] there exist $\varepsilon \in]0, \rho[$ (which does not depend on k) such that $\mathrm{diam}(F) \geq \varepsilon$ and hence $\mathcal{H}^1(\Gamma_k) \geq \varepsilon$.

So J is a finite set since $\varepsilon \# J \leq \sum_{k \in J} \mathcal{H}^1(\Gamma_k) \leq \mathcal{H}^1(\partial E \cap B_{2\rho}(x_0)) < \infty$.

Finally we consider a point $x \in \partial E \cap B_\rho(x_0)$. Suppose $\Gamma_1, \dots, \Gamma_N$ are the Jordan curves containing x and suppose they are ordered in such a way that (for some $1 \leq M \leq N$)

$$\mathrm{int}(\Gamma_1) \subset \ldots \subset \mathrm{int}(\Gamma_M) \subset \mathrm{ext}(\Gamma_{M+1}) \subset \ldots \subset \mathrm{ext}(\Gamma_N).$$

Let U be a neighborhood of x such that $(\Gamma_1 \cup \ldots \cup \Gamma_N) \cap U \subset \partial A \cap U$. Then either $A' = \mathrm{int}(\Gamma_1)$ or $A' = \mathrm{int}(\Gamma_2) \setminus \overline{\mathrm{int}(\Gamma_1)}$ is such that $A' \cap U \subset A$. In any case (by Jordan theorem or by Theorem 3.2) there exists a continuous surjective mapping $\Phi: \overline{B_1} \to \overline{A'}$.

Clearly there exists a continuous curve $\gamma_1: [0,1] \to \overline{B_1}$ such that $\gamma(0) = x$ and $\gamma(]0,1]) \subset \Phi^{-1}(A' \cap U)$ (notice that $\Phi^{-1}(A' \cap U)$ is a neighborhood of $\Phi^{-1}(x)$ in $\overline{B_1}$). On the other hand, since A is open and connected, given $y \in A$, there exists a coninuous curve γ_2 joining y with $\Phi(\gamma_1(1))$. Joining the two curves $\Phi(\gamma_1)$ and γ_2 we get the curve γ we were looking for. □

Theorem 3.4 *Let $E \in \mathcal{M}_\omega(\Omega)$. Then $\Sigma(E) \cap \Omega$ is a discrete set. Moreover, if W_φ is neither a triangle nor a quadrilateral then $\Sigma(E) = \emptyset$.*

Proof:
Let $B_{2\rho}(x_0) \subset \Omega$ and let $\mathcal{F}$ be the family of the connected components of $B_{2\rho}(x_0) \setminus \partial E$ which meet $B_\rho(x_0)$. As in the previous lemma it is easily seen that $\mathcal{F}$ is finite, in fact, for every component $A \in \mathcal{F}$ either $A \cap \partial B_{2\rho}(x_0) \neq \emptyset$ or $A \Subset B_{2\rho}(x_0)$ and in any case $P_\varphi(A, B_{2\rho})(x_0) \geq \varepsilon$.

Consider now the map $\lambda: \Sigma(E) \cap B_\rho(x_0) \to \mathcal{P}(\mathcal{F})$ ($\mathcal{P}(\mathcal{F})$ is the family of subsets of $\mathcal{F}$) defined by $\lambda(x) = \{A \in \mathcal{F}: x \in \partial A\}$. We are going to prove that given $\alpha \in \mathcal{P}(\mathcal{F})$ then $\lambda^{-1}(\alpha)$ contains at most two points. Since $\mathcal{F}$ is a finite set, this is enough to conclude that $\Sigma(E) \cap B_\rho(x_0)$ is finite.

Choose $\alpha \in \mathcal{P}(\mathcal{F})$. First of all we claim that if α has less than three elements then $\lambda^{-1}(\alpha)$ is empty. In fact suppose $x \in \Sigma(E) \cap B_\rho(x_0)$ and $\lambda(x) = \alpha$. Since $x \in \partial E$ it is clear that $\alpha = \lambda(x)$ contains at least one component of $\mathring{E} \cap B_\rho(x_0)$ and one component of $B_\rho(x_0) \setminus \overline{E}$. If α contains exactly two components then reasoning as in the proof of Lemma 3.3 we can prove that there exists a neighborhood U of x and a jordan curve Γ such that $\partial E \cap U \supset \Gamma \cap U$. Actually we have $\partial E \cap U = \Gamma \cap U$ since Γ is such that $\mathcal{H}^1(\partial E \cap U) = \mathcal{H}^1(\Gamma \cap U)$ and given a point $y \in \partial E \setminus \Gamma$ we could find $\varepsilon > 0$ such that $B_\varepsilon(y) \cap \Gamma = \emptyset$ and hence, by the lower density estimate for ω-minimal sets, $\mathcal{H}^1(\partial E \cap B_\varepsilon(y)) > 0$. So there exists a (possibly smaller) neighborhood U of x such that $\Gamma \cap U = \partial E \cap U$. By means of an chord-arc reparameterization we can suppose that γ is lipschitz and by Lemma 3.1 we get that $\gamma_{|\gamma^{-1}(\overline{U})}$ is bi-lipschitz and the claim is proved.

So we may suppose that α has at least three elements. Suppose by contradiction that $\lambda^{-1}(\alpha)$ has three or more elements. Then there exists $A_1, A_2, A_3 \in \mathcal{F}$ such that $\bigcap_{i=1}^3 \partial A_i \supset \{x_1, x_2, x_3\}$. Choose a point $a_i \in A_i$ ($i = 1, 2, 3$). Given $i, j \in \{1, 2, 3\}$, by Lemma 3.3 it is possible to find a continuous curve $\gamma_{ij}: [0, 1] \to \mathbb{R}^2$ such that $\gamma_{ij}(0) = x_i$, $\gamma_{ij}(1) = a_j$ and $\gamma_{ij}(]0, 1]) \subset A_j$. Notice that the supports $\gamma_{ij}(]0, 1[)$ are disjoint, and this leads to a contradiction since the graph having vertices $x_1, x_2, x_3, a_1, a_2, a_3$ and edges γ_{ij} cannot be planar.

If W_φ is neither a triangle nor a quadrilateral, then $\Sigma(E) = \emptyset$ by [4, Theorem 6.18]. This result can also be obtained by a blow-up argument from Theorem 3.8 below. □

3.2 Classification of global minimal sets

In this section we aim at the classification of all global minimal sets i.e. we are going to characterize all sets $E \in \mathcal{M}_0 = \mathcal{M}_0(\mathbb{R}^2)$.

We recall the following elementary lemma.

Lemma 3.5 *Let $C \subset \mathbb{R}^2$ be closed and convex. For $k \geq 2$, let $x_1, \ldots, x_k$ be extremal points of C with $x_i \neq x_j$ for $i \neq j$. Let also $0 \leq \alpha_i \leq \pi$ be the angle defined by C in x_i. Then there holds*

$$\sum_{i=1}^{k} \alpha_i \geq (k-2)\,\pi,$$

and the equality holds if and only if C is the "k-agon" (possibly degenerate) spanned by $x_1, \ldots, x_k$.

Proposition 3.6 (calibration) *Let $I \subset \mathbb{N}$, $A_i \subset \mathbb{R}^k$ $(i \in I)$ be sets of locally finite perimeter, $\bigcup_{i\in I} A_i = \mathbb{R}^k$, $A_i \cap A_j = \emptyset$ for $i \neq j$. Let $n_i \subset \mathbb{R}^k$ and let $n\colon \mathbb{R}^k \to \mathbb{R}^k$ be defined by $n(x) = \sum_{i\in I} n_i \chi_{A_i}(x)$. Let also $E \subset \mathbb{R}^k$ be a set of finite perimeter.*

Suppose moreover that

(i) for $\mathcal{H}^{k-1}$-a.e $x \in \partial^* A_i \cap \partial^* A_j$ it holds $\langle n_i, \nu_{A_i}(x)\rangle + \langle n_j, \nu_{A_j}(x)\rangle = 0$;

(ii) $\varphi^o(n_i) \leq 1$ for all $i \in I$;

(iii) $\lim_{y\to x}\langle \nu_E(x), n(y)\rangle = \varphi(\nu_E(x))$ for $\mathcal{H}^{k-1}$-a.e. $x \in \partial^* E$.

Then $E \in \mathcal{M}_0(\mathbb{R}^k)$.

Proof:
First of all we prove that $\operatorname{div} n = 0$ in the sense of distributions. We recall [3, Ch. 4.4] that

$$\mathcal{H}^{k-1}\left(\left\{x \in \bigcup_{i\in I} \partial^* A_i \colon \#\{i \in I \colon x \in \partial^* A_i\} \neq 2\right\}\right) = 0.$$

In fact, given $\psi \in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned}
\langle \operatorname{div} n, \psi\rangle &= -\int \langle n(x), D\psi(x)\rangle \,\mathrm{d}x = -\sum_{i\in I}\Big\langle n_i, \int \chi_{A_i}(x) D\psi(x)\,\mathrm{d}x\Big\rangle \\
&= \sum_{i\in I}\int_{\partial^* A_i} \psi(x)\langle n_i, \nu_{A_i}(x)\rangle \,\mathrm{d}\mathcal{H}^{k-1}(x) \\
&= \sum_{i<j}\int_{\partial^* A_i \cap \partial^* A_j} \psi(x)(\langle n_i, \nu_{A_i}(x)\rangle + \langle n_j, \nu_{A_j}(x)\rangle)\,\mathrm{d}\mathcal{H}^{k-1}(x) = 0.
\end{aligned}$$

Consider now a ball B and a set of finite perimeter F such that $E \triangle F \Subset B$; consider also a family of mollifiers $\rho_\varepsilon \in C_c^\infty(\mathbb{R}^n)$. We recall that $\operatorname{div}(n * \rho_\varepsilon) = (\operatorname{div} n) * \rho_\varepsilon = 0$ so that

$$\begin{aligned}
0 &= \int \operatorname{div}(n * \rho_\varepsilon)(x)(\chi_F(x) - \chi_E(x))\,\mathrm{d}x = \langle (D\chi_E - D\chi_F), n * \rho_\varepsilon\rangle \\
&= \langle (D\chi_E - D\chi_F) \llcorner B, n * \rho_\varepsilon\rangle = \langle D\chi_E \llcorner B, n * \rho_\varepsilon\rangle - \langle D\chi_F \llcorner B, n * \rho_\varepsilon\rangle.
\end{aligned}$$

Since by hypothesis, for $\mathcal{H}^{k-1}$-a.e. $x \in \partial^* E$,

$$\lim_{y\to 0} |\langle \nu_E(x), n(x+y)\rangle - \varphi(\nu_E(x))| = 0$$

we get

$$\lim_{\varepsilon\to 0^+} \int |\langle \nu_E(x), n(x+y)\rangle - \varphi(\nu_E(x))|\, \rho_\varepsilon(y)\,\mathrm{d}y = 0$$

that is $\lim_\varepsilon \langle \nu_E(x), (n * \rho_\varepsilon)(x)\rangle = \varphi(\nu_E(x))$. Hence we have $\lim_\varepsilon \langle D\chi_E \llcorner B, n * \rho_\varepsilon\rangle = \int_B \varphi(\nu_E(x))\,\mathrm{d}\mathcal{H}^{k-1}(x) = P_\varphi(E, B)$. On the other hand, notice that $\varphi^o(n * \rho_\varepsilon(x)) \leq \sup_{y\in\mathbb{R}^k} \varphi^o(n(y)) \leq 1$ hence

$$\langle D\chi_F \llcorner B, n * \rho_\varepsilon\rangle = \int_{\partial^* F \cap B} \langle \nu_F(x), n * \rho_\varepsilon(x)\rangle \,\mathrm{d}\mathcal{H}^{k-1}(x) \leq P_\varphi(F, B)$$

and we obtain, as desired

$$0 \leq P_\varphi(E,B) - P_\varphi(F,B).$$

$\square$

Definition 3.7 *Given a set $E \subset \mathbb{R}^2$, which is a cone over the origin, we say that E is φ-*flat *if for any connected component A of $\mathring{E}$ (resp. $\mathbb{R}^2 \setminus \overline{E}$) there exists $x \in \partial W_\varphi$ such that $W_\varphi - x$ (resp. $x - W_\varphi$) is contained in A.*

Notice that, if E is a φ-flat cone, both $\mathring{E}$ and $\mathbb{R}^2 \setminus \overline{E}$ have a finite number of connected components.

In the following theorem we characterize the minimal cones $E \in \mathcal{M}_0(\mathbb{R}^2)$.

Theorem 3.8 *Let $E \subset \mathbb{R}^2$ be a cone centered in the origin. Then $E \in \mathcal{M}_0(\mathbb{R}^2)$ if and only if E is φ-flat. In particular, one of the following conditions hold (see Figures 1,2,3):*

(i) ∂E is the union of two half-lines and there exists $x \in \partial W_\varphi$ such that $W_\varphi - x \subset \mathring{E}$ and $x - W_\varphi \subset \mathbb{R}^2 \setminus \overline{E}$;

(ii) W_φ is a quadrilateral and ∂E is the union of four half-lines parallel to the edges of W_φ and having the same exterior normal vector;

(iii) W_φ is a triangle and ∂E is the union of six half-lines parallel to the edges of W_φ and having the same exterior normal vector;

(iv) W_φ is a triangle and ∂E is the union of four half-lines, three of which are parallel to an edge of W_φ and have the same exterior normal vector.

Proof:
Let $E \in \mathcal{M}_0$ be a cone on the origin. Every connected component A of $\mathring{E}$ is minimal. Notice that ∂A is simply the union of two half-lines. By cutting from A a small triangle with vertex in the origin it is easily seen that for A to be minimal there must exist $x \in \partial W_\varphi$ such that $W_\varphi - x \subset A$. This is true for every connected component A of $\mathring{E}$ and the analogous result is obtained for $-A$ when A is a connected component of $\mathbb{R}^2 \setminus \overline{E}$. This implies that E is φ-flat.

We now prove the other implication. Let $E \subset \mathbb{R}^2$ be a φ-flat cone and let $A_1, \ldots, A_{2N}$ be the connected components of $\mathbb{R}^2 \setminus \partial E$, enumerated in such a way that A_i is consecutive to A_{i+1}. Letting $I = \{1, 3, \ldots, 2N-1\}$, $J = \{2, 4, \ldots, 2N\}$ suppose also that $A_i \subset \mathring{E}$ for $i \in I$ and $A_i \subset \mathbb{R}^2 \setminus \overline{E}$ for $i \in J$.

Let $x_i \in \partial W_\varphi$ be such that $W_\varphi - x_i \subset A_i$ if $i \in I$, and $x_i - W_\varphi \subset A_i$ if $i \in J$; let α_i be the angle of A_i in 0 and β_i be the angle of W_φ in x_i.

We are going to construct a function $n\colon \mathbb{R}^2 \to \mathbb{R}^2$ which satisfies the hypothesis of Proposition 3.6.

First of all we claim that $N \leq 3$. Suppose that $\sum_{i\in I} \alpha_i \leq \sum_{i \in J} \alpha_i$ (the other case being similar) and notice that the points x_i for $i \in I$ are all different. So, by Lemma 3.5 we get

$$\pi \geq \sum_{i\in I} \alpha_i \geq \sum_{i \in I} \beta_i \geq (N-2)\pi$$

that is $N \leq 3$.

So we consider the three cases.

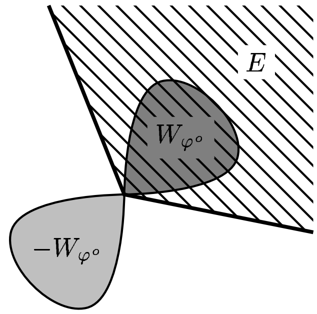

Figure 1: There may be infinite many minimal cones with one component. If W_φ has no vertices then these cones are half-planes. In this and in the following pictures translations of W_φ are filled with dark gray while translations of $-W_\varphi$ are filled with light gray.

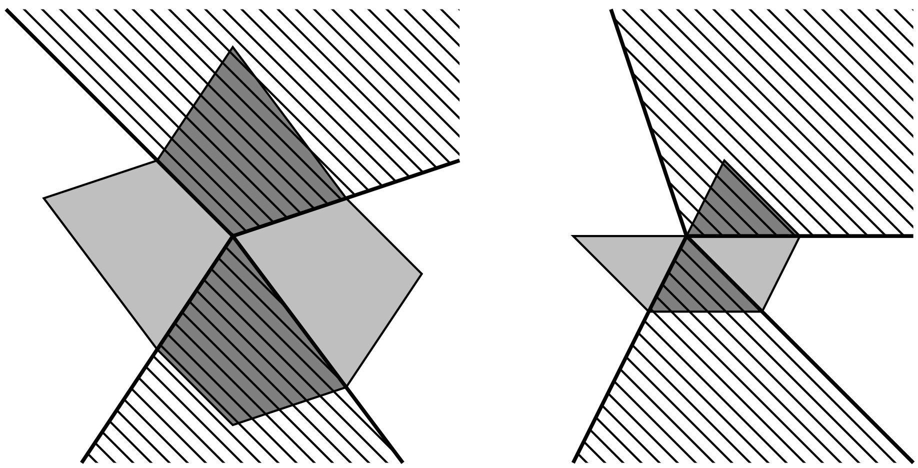

Figure 2: On the left a minimal cone with two components when W_φ is a quadrilateral. The only other minimal cone is its complementary set. On the right one of infinitely many minimal cones with two components when W_φ is a triangle.

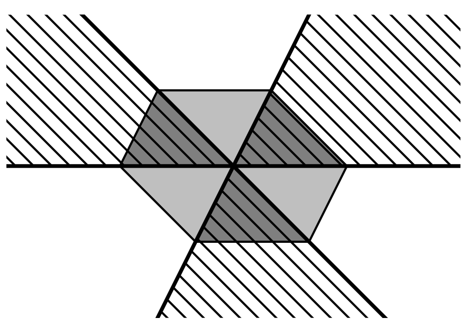

Figure 3: A minimal cone with three components when W_φ is a triangle. The only other minimal cone is the complementary set.

(i) Assume that $N = 1$. This implies that $A_1 = \mathring{E}$ and $A_2 = \mathbb{R}^2 \setminus \overline{E}$ are connected, and one of them is convex. Assuming that A_1 is convex we have $x_1 - W_\varphi \subset A_2$. Then it is easy to prove that $n(x) = x_1$ satisfies the hypothesis of Proposition 3.6. Notice that if W_φ has angles there are infinitely many cones E of this kind otherwise E is an half-plane.

(ii) Assume that $N = 2$. If the points $x_1 \dots x_4$ are all distinct, then we have (by Lemma 3.5)

$$2\pi = \alpha_1 + \dots + \alpha_4 \geq \beta_1 + \dots + \beta_4 \geq 2\pi$$

which implies that W_φ is the convex hull of $x_1, \dots, x_4$ (i.e. a quadrilateral or a triangle) and that $\alpha_i = \beta_i$ ($i = 1, \dots, 4$). Again it is not difficult to show that $n(x) = x_i$ for $x \in A_i$ is a calibration of E.

Suppose now that $x_i = x_j$ for some $i \neq j$. Since i, j cannot have the same parity, we can safely assume that $i = 1, j = 2$. So we have $x_1 = x_2$, $\beta_1 = \beta_2$. Since $W_\varphi - x_1 \subset A_1$ and $x_1 - W_\varphi \subset A_2$ we get $\alpha_1 + \alpha_2 \geq \pi + \beta_1$ so that (again by Lemma 3.5)

$$2\pi = \alpha_1 + \dots + \alpha_4 \geq \pi + \beta_1 + \beta_3 + \beta_4 \geq \pi + \pi$$

which means that W_φ is the triangle spanned by x_1, x_3, x_4 and that $\alpha_1 + \alpha_2 = \beta_1 + \pi$, $\alpha_3 = \beta_3$, $\alpha_4 = \beta_4$. Again we define $n(x) = x_i$ for $x \in A_i$. This is a calibration since it is constant on $A_1 \cup A_2$ and $\alpha_3 = \beta_3$, $\alpha_4 = \beta_4$. Notice moreover that there are infinitely many cones E of this kind.

(iii) Assume $N = 3$. In this case, by Lemma 3.5 we find that $\alpha_i = \beta_i$ for $i \in I$ and W_φ is the triangle spanned by x_1, x_3, x_5. So $\alpha_2 + \alpha_4 + \alpha_6 = \pi$, $\alpha_i = \beta_i$ for $i \in J$ and $x_1 = x_4$, $x_2 = x_5$, $x_3 = x_6$. So $n(x) = x_i$ for $x \in A_i$ is again a calibration for E.

□

If W_φ is a triangle we are going to show that there exist sets $E, E' \in \mathcal{M}_0(\mathbb{R}^2)$ such that E, E' are not cones, $\Sigma(E)$ contains two distinct points and $\Sigma(E')$ contains one point.

Suppose W_φ is the triangle spanned by the points $x_1, x_2, x_3 \in \mathbb{R}^2$. Define $C_i = \bigcup_{\rho>0} \rho(W_\varphi - x_i)$, $C_i' = \bigcup_{\rho>0} \rho(x_i - W_\varphi)$ and choose two points $y_1, y_2 \in \mathbb{R}^2$ such that $y_2 \in \mathbb{R}^2 \setminus (y_1 + (C_3 \cup C_2'))$.

We choose E with three components: $E = E_1 \cup E_2 \cup E_3$. In particular, $E_2 = y_2 + C_2$, $E_3 = y_1 + C_3$ and E_1 can be any set such that $E_1 \supset (y_1 + C_1) \cup (y_2 + C_1)$, $\mathbb{R}^2 \setminus E_1 \subset (y_1 + (C_2' \cup C_3 \cup C_1'))) \cup (y_2 + (C_3' \cup C_2 \cup C_1'))$ and such that $\nu_{E_1}(x) \in T(x_1)$ for every $x \in \partial E_1$ (see Figure 4).

We define also E' by $E' = E_1 \cup E_2'$ where E_2' is any set such that $E_1' \supset (y_1 + C_1)$, $\mathbb{R}^2 \setminus E_1' \supset y_1 + C_1'$ and such that $\nu_{E_1'}(x) \in T(x_1)$ for every $x \in \partial E_1'$.

Proposition 3.9 *The sets E, E' described before are global minima of P_φ (i.e. $E, E' \in \mathcal{M}_0(\mathbb{R}^2)$).*

Proof:
Consider $n\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $n(x) = x_2$ for $x \in (y_1 + C_2') \cup (y_2 + C_2)$, $n(x) = x_3$ for $x \in (y_1 + C_3) \cup (y_2 + C_3')$ and $n(x) = x_1$ otherwise. The construction of E

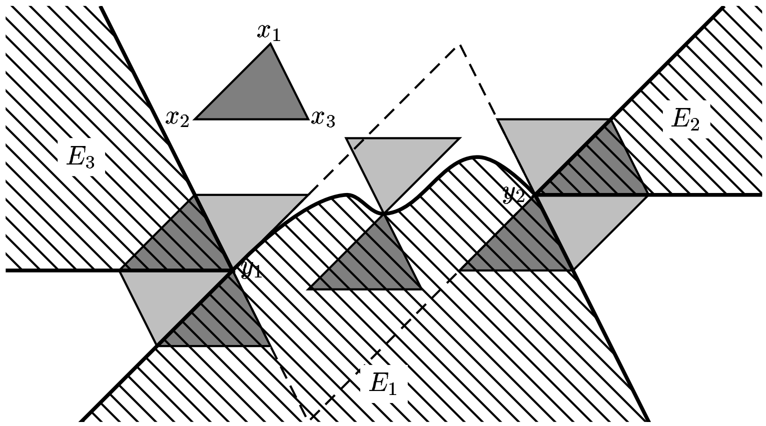

Figure 4: A global minimum with two singular points.

guarantees that Proposition 3.6 can be applied. The same proof applies to E' with a similar calibration. □

From Proposition 3.6 we know that any set $E \subset \mathbb{R}^2$ such that there exists $n \in \mathbb{R}^2$ with $\nu_E(x) \in T(n)$ for almost every $x \in \partial E$ is a global minimum. In the following theorem we show that the converse is also true, when the Wulff Shape is neither a triangle nor a quadrilateral. We also characterize all singular global minima of the anisotropic perimeter.

We say that a Caccioppoli set E is a *φ-flat global lipschitz subgraph* if there exists $n \in \mathbb{R}^2$ such that $\nu_E(x) \in T(n)$ for almost every $x \in \partial E$ and given $y \in \mathbb{R}^2$ there exist exactly one $x \in \partial E$ such that $y - x = \lambda n$ for some $\lambda \in \mathbb{R}$.

Lemma 3.10 *Let $E \in \mathcal{M}_\omega(\mathbb{R}^2)$ and let S be a connected component of ∂E. Then there exist a set $G \in \mathcal{M}_\omega(\mathbb{R}^2)$ such that $\partial G = S$ and $\nu_E(x) = \nu_G(x)$ for $\mathcal{H}^1$-a.e. $x \in S$.*

Proof:
Let F be the union of E and all the connected components C of $\mathbb{R}^2 \setminus \overline{E}$ such that $\overline{C} \cap S = \emptyset$. Clearly S is a connected component of ∂F too. Let now G be the set F minus the union of all the connected components C of $\mathring{F}$ such that $\overline{C} \cap S = \emptyset$. As noted in Section 3.1, by cutting components from an ω-minimal set the ω-minimality is preserved hence $G \in \mathcal{M}_\omega(\mathbb{R}^2)$.

Moreover it is easily seen that every connected component C of $\mathbb{R}^2 \setminus \partial G$ has the property $\overline{C} \cap S \neq \emptyset$. We now prove that $\partial G = S$. Indeed, if T is a connected component of ∂G, by the regularity results for G we know that T splits the plane $\mathbb{R}^2$ in, at least, two connected components. If T were different from S, then any connected component of $\mathbb{R}^2 \setminus T$ which does not contain S would contain a component of $\mathring{E}$ (or of $\mathbb{R}^2 \setminus \overline{E}$) which does not meet S.

To complete the proof notice that the normal vector to E in a point $x \in S$ is preserved in each operation. □

Theorem 3.11 (singular global minima) *Let $E \in \mathcal{M}_0(\mathbb{R}^2)$ (see Figure 5).*

1. *If W_φ is neither a triangle nor a quadrilateral then $\Sigma(E) = \emptyset$ and E is a φ-flat global lipschitz subgraph.*

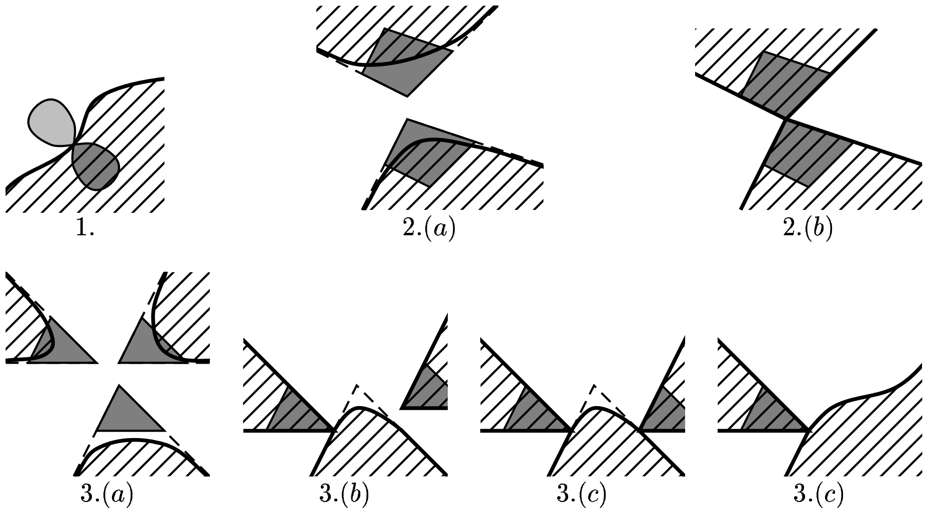

Figure 5: The global minimal sets as classified in Theorem 3.11

2. If W_φ is a quadrilateral then one of the following is true:

(a) $\Sigma(E) = \emptyset$ and ∂E has at most two connected components, each of which is the boundary of a φ-flat global lipschitz subgraph;

(b) $\Sigma(E)$ is composed by a single point, say 0, and E is one of the cones described in Theorem 3.8.

3. If W_φ is a triangle then one of the following is true:

(a) $\Sigma(E) = \emptyset$ and ∂E has at most three connected components each of which is the boundary of a φ-flat global lipschitz subgraph;

(b) $\Sigma(E)$ is composed by a single point and ∂E has two connected components, one of which is the boundary of a φ-flat global lipschitz subgraph whereas the other coincides with the boundary of one of the sets described in Proposition 3.9;

(c) E is one of the sets described in Proposition 3.9.

Proof:
We divide the proof into three steps.

Step 1. Assume that ∂E is connected and $\Sigma(E) = \emptyset$. By [4, Proposition 6.2] we know that the excess of E is zero on every ball. This means that there exists $n \in \mathbb{R}^2$ such that $\nu_E(x) \in T(n)$ for $\mathcal{H}^1$-a.e. $x \in \partial E$, which implies that E is a φ-flat global lipschitz subgraph.

Step 2. Assume that ∂E is connected and $0 \in \Sigma(E)$. In the proof of [4, Theorem 6.17] it has been proved that it is possible to find a neighbourhood $U = \rho W_\varphi$ of 0 and a set F such that $E \triangle F \subset U$, $F \cap U$ is a cone centred in 0 and $P_\varphi(E, U) = P_\varphi(F, U)$. Clearly (by a blow-up argument) the cone $F \cap U$ is the restriction in U of a global singular minimal cone.

Suppose that this minimal cone is of type (ii) or (iii) in the classification given in Theorem 3.8. We claim that, in this case, E is itself a global minimal cone. Notice that $\nu_F(x)$ is an extremal point of $\mathcal{F}_\varphi$ for all $x \in \partial F \cap \rho W_\varphi$ so

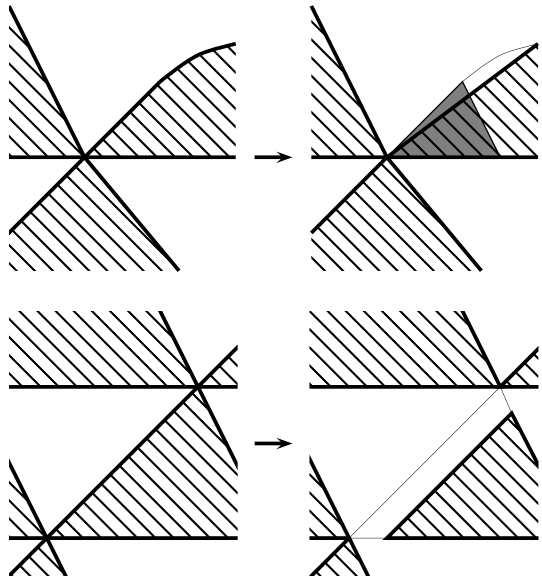

Figure 6: How to lessen the perimeter of E in the proof of Theorem 3.11

equality must hold in [4, (14)] and by Jensens's inequality we get $E \cap U = F \cap U$. By the regularity results of the previous section we know that the connected components of $\partial E \setminus \Sigma(E)$ are lipschitz curves and that $\Sigma(E)$ is discrete. Consider one of the curves starting from 0. If it were not straight, it would be possible to replace it with a segment and we would obtain a minimal cone (in U) which has not the right normal vectors, and this is impossible by Theorem 3.8. Therefore, all the curves starting from 0 are straight lines. If one of these curves met another singular point then it could be possible to decrease perimeter with a suitable variation (see Figure 6). So there is only one singular point and the set is a minimal cone.

Suppose now that the cone $F \cap U$ is of type (iv). Reasoning as before we obtain that all but one of the curves of ∂E starting from 0 are straight lines and do not meet other singular points. Consider the curve that does not need to be straight. This curve may either go to infinity or meet another singular point y_1. In any case, the normal vector to the curve has to lie in $T(n)$ for some fixed $n \in \mathbb{R}^2$. These properties guarantee that E is one of the sets considered in Proposition 3.9.

Step 3. Assume that ∂E is not connected. By Lemma 3.10 and the previous steps each connected component C of ∂E is either the boundary of a φ-flat global lipschitz subgraph or one of the sets described in Proposition 3.9. In any case C can be split into one, two or three global lipschitz graphs, two of which can only meet (without crossing) in one singular point. Let $\{S^i\}_{i\in I}$, $I \subseteq \mathbb{Z}$, be such global lipschitz graphs. We have that $\partial E = \bigcup_{i\in I} S^i$ and $\bigcup_{i\neq j} S^i \cap S^j = \Sigma(E)$. We can also suppose that the curves S^i are oriented so that the set E lies on the left of such curves.

Let $E_k := E/k$. By the volume density estimates and the compactness result for ω-minimizers [4, Proposition 3.3, Proposition 3.4] we know that a subsequence of E_k converges in $L^1_{\mathrm{loc}}(\mathbb{R}^2)$, to a set $E_\infty \in \mathcal{M}_0(\mathbb{R}^2)$ and if $x_k \in$

∂E_k, $x_k \to x$ for some $x \in \mathbb{R}^2$, then $x \in \partial E_\infty$ (in particular $0 \in \partial E_\infty$). This means that S^i/k converges[2] to a global lipschitz graph S^i_∞ containing 0 and $\partial E_\infty = \bigcup_{i\in I} S^i_\infty$.

We now claim that $S^i_\infty \cap S^j_\infty = \{0\}$ for all $i \neq j$. Suppose by contradiction that there exists $x \in S^i_\infty \cap S^j_\infty$, $x \neq 0$ and let $x^i_k \in S^i$ and $x^j_k \in S^j$ be two sequences such that $x^i_k/k \to x$ and $x^j_k/k \to x$. We can assume that the curves S^i and S^j lie in the boundary of the same connected component of $\mathring{E}$ or of $\mathbb{R}^2 \setminus \overline{E}$ because if not there would exist another curve S^l between S^i and S^j such that $x \in S^l$ (so that we could replace S^i or S^j with S^l).

Choose now two points $\bar{x}^i \in S^i$, $\bar{x}^j \in S^j$. For each k we can construct a set F_k which has the same boundary of E but with the curves between $\bar{x}^i$ and x^i and between $\bar{x}^j_k$ and x^j_k replaced with the segments $[\bar{x}^i, \bar{x}^j]$ and $[x^i_k, x^j_k]$.

The condition $P_\varphi(E, B_R) \leq P_\varphi(F_k, B_R)$ (for R large enought) then reads as

$$C(\|x^i_k - x^j_k\| + \|\bar{x}^i - \bar{x}^j\|) \geq \|x^i_k - \bar{x}^i\| + \|x^j_k - \bar{x}^j\|$$

where $C > 0$ is a suitable constant depending only on φ. Thus we get

$$C\frac{\|x^i_k - x^j_k\|}{k} \geq \frac{\|x^i_k\|}{k} + \frac{\|x^j_k\|}{k} - \frac{\|\bar{x}^i\|}{k} - \frac{\|\bar{x}^j\|}{k} - C\frac{\|\bar{x}^i - \bar{x}^j\|}{k}$$

and letting $k \to \infty$ we obtain

$$\liminf_{k\to\infty} \left\| \frac{x^i_k}{k} - \frac{x^i_k}{k} \right\| \geq \frac{2}{C}\|x\| > 0,$$

which contradicts the hypothesis $\|x^i_k/k - x^j_k/k\| \to 0$. We have proved that E_∞ is connected and $\Sigma(E_\infty) = \{0\}$. From Theorem 3.4 it follows that W_φ is a triangle or a quadrilateral.

If W_φ is a quadrilateral, by Step 2 we conclude that E_∞ is a minimal singular cone. Then necessarily $\#I = 2$, S^i $(i \in I)$ are regular[3] (otherwise ∂E would not be connected) and $\Sigma(E) = \emptyset$.

If W_φ is a triangle, by the previous step we conclude that E_∞ is either a singular minimal cone with three components or one of the sets E' described in Proposition 3.9. In the first case $\#I = 3$, $I = \{i_1, i_2, i_3\}$ and either $\Sigma(E) = \emptyset$ and S^i $(i \in I)$ are all disjoint or S^{i_1} is regular while $S^{i_2} \cup S^{i_3}$ is the boundary of one of the sets E' described in Proposition 3.9. In the second case (E_∞ is one of the sets E' described in Proposition 3.9) we have $\#I = 2$, which implies $\Sigma(E) = \emptyset$ and S^{i_1}, S^{i_2} are both regular. $\square$

4 Sets with prescribed constant curvature

Given $H \in L^1(\mathbb{R}^2)$ and $\Omega \subset \mathbb{R}^2$ we consider the functional $F_H(E)$, defined on sets E with locally finite perimeter:

$$F_H(E) := P_\varphi(E, \Omega) - \int_{E\cap\Omega} H. \tag{2}$$

[2] here and in the following the convergences are always meant to be "up to a subsequence"
[3] in this proof by "S is regular" we mean $\overline{S} \cap \Sigma(E) = \emptyset$

If E is a local minimizer of (2), that is, $E \subset \mathbb{R}^2$ is a set such that for every set E' with $E \triangle E' \Subset \Omega$ we have $F_H(E) \leq F_H(E')$, then, we say that E *is a set with curvature* H *in* Ω. The family of all these sets will be denoted by $\mathcal{N}_H(\Omega)$. As a matter of fact, when $\varphi(\xi) = ||\xi||$, then $H(x)$ is the curvature of the curve ∂E in x.

In the following we are going to consider the case $H \equiv \lambda$ where λ is a given constant.

We recall the following fundamental result (see [9], [8]).

Theorem 4.1 (isoperimetric inequality) *Let* $E \subset \mathbb{R}^2$ *be such that* $|E| < \infty$ *and* $P_\varphi(E, \mathbb{R}^2) = P_\varphi(W_\varphi, \mathbb{R}^2)$. *Then* $|E| \leq |W_\varphi|$ *and equality holds if and only if* $E = W_\varphi + h$ *for some* $h \in \mathbb{R}^2$.

Notice that if $\Omega =]a, b[\times \mathbb{R} \subset \mathbb{R}^2$ and $E \cap \Omega$ is the subgraph of a function $u \in H_0^1([a, b])$, then

$$F_1(E) = \int_a^b \varphi(-u'(y), 1) - u(y)\, \mathrm{d}y.$$

For $u \in H_0^1([a, b])$ we define as $\Gamma_u := \{(x, y) : x \in [a, b], y = u(x)\}$ the graph of u.

Lemma 4.2 *Let* $a, b \in \mathbb{R}$ *and let* $F : H_0^1([a, b]) \to \mathbb{R}$ *be defined as*

$$F(u) := \int_a^b \varphi(-u'(y), 1) - u(y)\, \mathrm{d}y.$$

Let I *be the set of all positive numbers* r *for which there exists a function* $u_r \in H_0^1([a, b])$ *such that* Γ_{u_r} *is contained, up to a translation, in the boundary of* rW_φ. *Then* $I = \emptyset$ *or* $I = [\bar{r}, +\infty[$ *for some* $\bar{r} > 0$*; moreover, if* $\bar{r} < 1$, u_r *is a minimum of* F *if and only if* $u_r = u_1$.

Proof:
Assume $I \neq \emptyset$. Notice that the function $u_r \in H_0^1([a, b])$ is uniquely determined by $r \in I$. Moreover, if $r \in I$, then $r' \in I$ for all $r' > r$. Indeed, let $\bar{x} \in [a, b]$ such that u_r attains its maximum at $\bar{x}$. For all $\lambda > 1$, there exist $c_\lambda \in \mathbb{R}$ and $x_\lambda \in [a, b]$ such that, defining

$$v_\lambda(x) := c_\lambda + \lambda u_r \left(x_\lambda + \frac{x - \bar{x}}{\lambda} \right) \qquad \lambda > 1,$$

we have $v_\lambda \in H_0^1([a, b])$, which implies $\lambda r \in I$ with $u_{\lambda r} = v_\lambda$. Reasoning in a similar way, it is easy to prove that the set I is closed, hence $I = [\bar{r}, +\infty[$ for some $\bar{r} > 0$.

Assume now $\bar{r} < 1$. First of all, let us suppose that $\mathcal{F}_\varphi$ is smooth and uniformly convex and let $\varphi = \varphi(\xi_1, \xi_2)$, $(\xi_1, \xi_2) \in \mathbb{R}^2$. In this case we can compute the Euler equation associated to the functional F:

$$\frac{\partial}{\partial y} \left(\frac{\partial \varphi}{\partial \xi_1}(-u'(y), 1) \right) = 1, \tag{3}$$

which is equivalent to $\frac{\partial \varphi}{\partial \xi_1}(-u'(y), 1)) = y + c$ for some $c \in \mathbb{R}$. Since the functional F is strictly convex in $H_0^1([a, b])$, if we prove that u_1 is a solution of (3), then u_1 is the unique minimizer of F in $H_0^1([a, b])$.

By assumption, there exists a point $\overline{x} = (\bar{y}, \bar{z}) \in \mathbb{R}^2$ such that $\Gamma_{u_1} \subset \partial W_\varphi + \overline{x}$. Letting $\nu_r(y) := (-u_r'(y), 1)/\varphi(-u_r'(y), 1)$, we have

$$\nabla\varphi(-u_1'(y), 1) = T^o(\nu_r(y)) = (y, u_1(y)) - \overline{x} = (y - \bar{y}, u_1(y) - \bar{z}).$$

which implies (3) with $c = -\bar{y}$.

Let us consider the general case. We can find a sequence of general norms φ_k which converge uniformly on compact subsets of $\mathbb{R}^2$ to φ and such that $\mathcal{F}_{\varphi_k}$ are smooth and uniformly convex. Let F_k and $\bar{r}_k$ be defined as F and $\bar{r}$ with φ replaced by φ_k. Notice that the sequence of functionals F_k converges uniformly to F on bounded subsets of $H_0^1([a, b])$, for $k \to \infty$. Moreover, since $\bar{r}_k \to \bar{r}$ we can assume $\bar{r}_k < 1$ for any k. Therefore, letting u_1^k be the unique minimizer of F_k, since $u_1^k \to u_1$ in $H_0^1([a, b])$ for $k \to \infty$, it follows that u_1 is a minimum of F.

Suppose now that u_r is another minimum of F for some $r \neq 1$, and assume for simplicity $r > 1$ (the case $r < 1$ can be treated in a similar way). Let us consider the function $v(y) := (u_1(y) + u_r(y))/2$. By the convexity of F it follows that v is also a minimum of F. Let $s \in [1, r]$ be such that

$$\int_a^b u_s(y)\,\mathrm{d}y = \int_a^b v(y)\,\mathrm{d}y.$$

We claim that $u_s = v$. Indeed, since $\Gamma_{u_s} \subset x_s + s\partial W_\varphi$ for some $x_s \in \mathbb{R}^2$, we can compare sW_φ with the set

$$B := \Big((x_s + sW_\varphi) \cap \{(y, z) : z \le 0\}\Big) \cup \Big\{(y, z) : y \in [a, b], 0 \le z \le v(y)\Big\}.$$

Then, since sW_φ and B have same volume and (anisotropic) perimeter, by Theorem 4.1 we get $B = sW_\varphi$, which implies $u_s = v$.

On the other hand, from the convexity of F and the minimality of u_1, u_r and u_s we get

$$\int_a^b \varphi\left(-\frac{u_1'(y) + u_r'(y)}{2}, 1\right) - \frac{u_1(y) + u_r(y)}{2}\,\mathrm{d}y = F(u_s) \tag{4}$$
$$= \frac{F(u_1) + F(u_r)}{2} = \int_a^b \frac{\varphi(-u_1'(y), 1) + \varphi(-u_r'(y), 1)}{2} - \frac{u_1(y) + u_r(y)}{2}\,\mathrm{d}y.$$

Since φ is convex, for all $y \in [a, b]$ in which u_r and u_1 (hence u_s) are differentiable, we have

$$\varphi\left(-\frac{u_1'(y) + u_r'(y)}{2}, 1\right) \le \frac{\varphi(-u_1'(y), 1) + \varphi(-u_r'(y), 1)}{2}, \tag{5}$$

and by (4) the equality must hold in (5). This implies that $\nu_r(y)$ and $\nu_1(y)$ belong to the same edge of $\mathcal{F}_\varphi$. It follows that either $\nu_1(y) = \nu_r(y)$ or $\nu_s(y)$ is an interior point of an edge of $\mathcal{F}_\varphi$, but the last possibility cannot hold since $\nu_s(y)$ is a vertex of $\mathcal{F}_\varphi$ for a.e. $y \in [a, b]$, hence $u_r = u_1$. □

Lemma 4.3 *Let A be an open cone centered in 0. Then*

$$P_\varphi(W_\varphi, A) = 2|W_\varphi \cap A|.$$

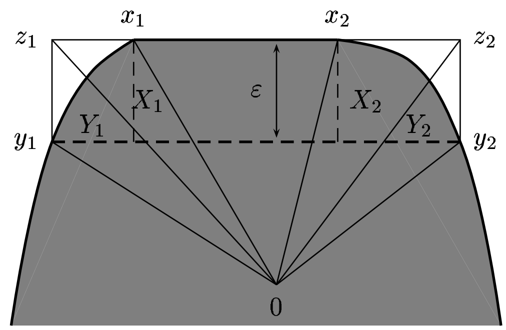

Figure 7: The construction used in Lemma 4.4.

Proof:
Notice that for $x \in \partial W_\varphi$ we have $\varphi(\nu_{W_\varphi}(x)) = \langle \nu_{W_\varphi}(x), x\rangle$, while for $x \in \partial A$ we have $\langle \nu_A(x), x\rangle = 0$. So we get

$$\begin{aligned} P_\varphi(W_\varphi, A) &= \int_{\partial W_\varphi \cap A} \varphi(\nu_{W_\varphi}(x))\, \mathrm{d}\mathcal{H}^1(x) \\ &= \int_{\partial(W_\varphi \cap A)} \langle x, \nu_{W_\varphi}(x)\rangle\, \mathrm{d}\mathcal{H}^1(x) \\ &= \int_{W_\varphi \cap A} \operatorname{div} x\, \mathrm{d}x = 2|W_\varphi \cap A|. \end{aligned}$$

□

Lemma 4.4 *Let s be an edge of W_φ of length l and let ν be the exterior normal vector to W_φ at s. For $\varepsilon > 0$ let $F_\varepsilon = \{x \in W_\varphi : \langle x, \nu\rangle \le \varphi(\nu) - \varepsilon\}$. Then*

$$P_\varphi(W_\varphi, \mathbb{R}^2) - P_\varphi(F_\varepsilon, \mathbb{R}^2) = \varepsilon l + o(\varepsilon).$$

Proof:
First of all notice that $\varphi(\nu)$ is the distance of the straight line passing through s from the origin. Let x_1, x_2 be the edges of s and let y_1, y_2 be the points of ∂W_φ such that $\langle y_i, \nu\rangle = \varphi(\nu) - \varepsilon$ ($i \in \{1,2\}$). Consider also the points z_1, z_2 determined by the intersection of the line containing s respectively with the lines through y_1 and y_2 parallel to ν. Let now X_1, X_2 be the intersection of W_φ with the triangles with vertices respectively $0, x_1, z_1$ and $0, x_2, z_2$, Y_1, Y_2 be the intersection of W_φ with the triangles with vertices $0, z_1, y_1$ and $0, z_2, y_2$ and Z_1, Z_2 be the quadrilaterals with vertices $0, x_1, z_1, y_1$ and $0, x_2, z_2, y_2$ (see Figure 7).

Notice that $Z_i \setminus (X_i \cup Y_i)$ is contained in a rectangle of sides $|z_i - y_i| = \varepsilon$ and $|z_i - x_i| = o(1)$ (for $\varepsilon \to 0$). Let also δ_i be equal to 0 if the point z_i belongs to s and equal to 1 otherwise.

Since the two triangles Y_1, Y_2 have base with length $|y_i - z_i| = \varepsilon$ and the sum of the heights is $|y_1 - y_2| = l + o(1)$, we get $2(|Y_1| + |Y_2|) = \varepsilon l + o(\varepsilon)$.

Using Lemma Lemma 4.3 we obtain

$$\begin{aligned} P_\varphi(W_\varphi, \mathbb{R}^2) - P_\varphi(F, \mathbb{R}^2) &= 2(|Z_1| + |Z_2| - \delta_1|X_1| - \delta_2|X_2|) \\ &= 2(|Y_1| + |Y_2|) + o(\varepsilon) = \varepsilon l + o(\varepsilon). \end{aligned}$$

□

Theorem 4.5 *Let $\Omega \subset \mathbb{R}^2$ be an open set, $\lambda \neq 0$ constant and $E \in \mathcal{N}_\lambda(\Omega)$. Then, every connected component of $\partial E \cap \Omega$ is contained, up to a translation, in $\frac{1}{\lambda}\partial W_\varphi$.*

Proof:
Notice that if $\lambda > 0$ we have

$$P_\varphi(\lambda E, \lambda\Omega) - |\lambda E \cap \lambda\Omega| = \lambda\left[P_\varphi(E,\Omega) - \lambda|E \cap \Omega|\right]$$

so that $E \in \mathcal{N}_\lambda(\Omega) \Rightarrow \lambda E \in \mathcal{N}_1(\lambda\Omega)$. In a similar way[4] we get (again for $\lambda > 0$)

$$\begin{aligned} & P_\varphi\left(\mathbb{R}^2 \setminus (-\lambda E), -\lambda\Omega\right) - |(\mathbb{R}^2 \setminus (-\lambda E)) \cap (-\lambda\Omega)| \\ & = \lambda\big[P_\varphi(E,\Omega) + \lambda|E \cap \Omega|\big] - \lambda^2|\Omega| \end{aligned}$$

so that

$$E \in \mathcal{N}_{-\lambda}(\Omega) \Rightarrow \mathbb{R}^2 \setminus (-\lambda E) \in \mathcal{N}_1(-\lambda\Omega).$$

So it is enough to prove the theorem for $\lambda = 1$.

We divide the proof into three steps.

Step 1. Fix $x_0 \in (\partial E \cap \Omega) \setminus \Sigma(E)$. We will prove that for some $\rho > 0$ the set $E \cap (x_0 + \rho W_\varphi)$ is equal to $(x_1 + \partial W_\varphi) \cap (z_1 + \rho W_\varphi)$ for some $x_1, z_1 \in \mathbb{R}^2$.

By [4, Lemma 6.1 and Proposition 4.6] we can find a convex open neighborhood U of x_0 such that $\partial E \cap \partial U$ consist of exactly two points and $\overline{E} \cap \partial U$ is connected. We notice that $\overline{E} \cap U$ is convex, otherwise letting E' be the convex hull of $\overline{E} \cap U$ the set $F := E \cup E'$ verifies $F \triangle E \subset U$, and $P_\varphi(F,\Omega) - |F \cap \Omega| < P_\varphi(E,\Omega) - |E \cap \Omega|$. Choose now $\rho \in]0,1[$ such that, letting $B = x_0 + \rho W_\varphi$, we have $B \subset U$. Being $\overline{E} \cap U$ convex, possibly reducing ρ, we can assume that $\partial B \cap \partial E$ consists of exactly two points and $\partial E \cap \overline{B}$ is a graph along some direction.

For $t \in [0,1[$ we can find $x(t) \in \mathbb{R}^2$ and $\eta(t) \in [\rho, +\infty[$ such that

(i) $x(0) = x_0$, $\eta(0) = \rho$;

(ii) $x(t)$, $\eta(t)$ are continuous and $\lim_{t\to 1^-} \eta(t) = +\infty$;

(iii) $\overline{x(t) + \eta(t) W_\varphi} \cap \partial B = \overline{E} \cap \partial B$, for all $t > 0$.

Let also $V(t) = |(x(t) + \eta(t) W_\varphi) \cap B|$ ($t \in [0,1[$) and $V(1) = \lim_{t \to 1^-} V(t)$. We claim that there exists $\bar{t} \in [0,1]$ such that $V(\bar{t}) = |E \cap B|$. Clearly $V(t)$ is continuous and $V(0) > |E \cap B|$. Notice that $x(t) + \eta(t) W_\varphi$ converges, for $t \to 1^-$ to the tangent cone to W_φ in the direction defined by the line through $\partial E \cap \partial B$. Since this tangent cone is a minimal surface by Theorem 3.8, being E a minimum for F_1, we get $V(1) \leq |E \cap B|$. So the claim is proved.

Let $x_1 = x(\bar{t})$ and $\eta = \eta(\bar{t})$. From Theorem 4.1 it follows that $P_\varphi(E, B) \geq P_\varphi(x_1 + \eta W_\varphi, B)$, hence from the minimality of E we get $E \cap B = x_1 + \eta W_\varphi \cap B$.

Since $\partial E \cap \overline{B}$ is a graph along some direction, we can consider a change of coordinates $\Lambda : \mathbb{R}^2 \to \mathbb{R}^2$ which preserves the Lebesgue measure and such that $\Lambda(\partial E \cap \partial B)$ is contained in the x-axis and $\Lambda(\partial E \cap \overline{B})$ is a graph along the y-axis. Then we can apply Lemma 4.2 with φ replaced by $\varphi \cdot \Lambda^{-1}$ and conclude that $\eta = 1$.

[4] notice that $P_\varphi(E,\Omega) = P_\varphi(-(\mathbb{R}^2 \setminus E), -\Omega)$

Step 2. Let us prove that each connected component of $(\partial E \cap \Omega) \setminus \Sigma(E)$ is contained (up to a translation) in ∂W_φ.

Let us consider a covering of a connected component of $(\partial E \cap \Omega) \setminus \Sigma(E)$ given by the open sets defined in Step 1. Observe that given two sets U_1, U_2 of this covering, if $\partial E \cap U_i = (z_i + \partial W_\varphi) \cap U_i$, $i \in \{1, 2\}$ and $\partial E \cap U_1 \cap U_2 \neq \emptyset$, then either $z_1 = z_2$ or $\partial E \cap U_1 \cap U_2$ is a segment (parallel to an edge of W_φ). So it is enough to check that each segment of $\partial E \cap \Omega$, parallel to an edge of W_φ and having the same exterior normal, is not longer than the corresponding edge of W_φ, and is equal when the extremes of the segment are both contained in $\Omega \setminus \Sigma(E)$.

Let $[x_1, x_2]$ be an edge of E parallel to an edge s of W_φ with length l and having the same exterior normal vector ν.

If the length of $[x_1, x_2]$ is $l' > l$, for $\varepsilon, \delta > 0$ sufficiently small it is possible to find $z_1, z_2 \in \mathbb{R}^2$ such that $|z_1 - z_2| = \delta$ and $\{x \in z_i + W_\varphi : \langle x - z_i, \nu\rangle = \varphi(\nu) - \varepsilon\} \subset [x_1, x_2]$. We consider the set E_ε which is equal to E out of a neighbourhood W of $[x_1, x_2]$ and which coincides with $E \cup (z_1 + W_\varphi) \cup (z_2 + W_\varphi)$ in W. If δ is sufficiently small and F_ε is the set defined in Lemma 4.4 we obtain that

$$P_\varphi(E_\varepsilon, \Omega) - P_\varphi(E, \Omega) = P_\varphi(W_\varphi, \mathbb{R}^2) - P_\varphi(F_\varepsilon, \mathbb{R}^2) = \varepsilon l + o(\varepsilon).$$

Moreover, it is easy to check that $|E_\varepsilon \cap \Omega| - |E \cap \Omega| = |E_\varepsilon \setminus E| = (l + \delta)\varepsilon + o(\varepsilon)$. Therefore, for ε sufficiently small the set E_ε contradicts the minimality of E.

Suppose now that $l' < l$. Consider the set $E_\varepsilon := \{x \in E : \mathrm{dist}(x, r) > \varepsilon\}$, where r is the straight line passing through x_1, x_2. Since E is locally equal to W_φ, again by Lemma 4.4 we find that

$$P_\varphi(E, \Omega) - P_\varphi(E_\varepsilon, \Omega) = P_\varphi(W_\varphi, \mathbb{R}^2) - P_\varphi(F_\varepsilon, \mathbb{R}^2) = \varepsilon l + o(\varepsilon).$$

As above, since $|E_\varepsilon \cap \Omega| - |E \cap \Omega| = |E \setminus E_\varepsilon| = l'\varepsilon + o(\varepsilon)$, we get a contradiction.

Step 3. We claim that $\Sigma(E) = \emptyset$. When W_φ is neither a triangle nor a quadrilateral, the statement follows by Theorem 3.4.

Notice that, by Step 2, every connected component of $(\partial E \cap \Omega) \setminus \Sigma(E)$ is a polygonal curve which has a finite number of vertices (no more than the number of vertices of W_φ).

Let $x_0 \in \Sigma(E) \cap \Omega$. By Theorem 3.4, we can find $\rho > 0$ such that $B_\rho(x_0) \subset \Omega$ and $\Sigma(E) \cap B_\rho(x_0) = \{x_0\}$. Moreover, we can assume that the number of connected components of $\partial E \setminus \Sigma(E)$ which intersect $B_\rho(x_0)$ is finite, otherwise this would imply that $P_\varphi(E, B_\rho(x_0)) = \infty$. Therefore, possibly reducing ρ, we can suppose that $\partial E \cap B_\rho(x_0)$ is a cone over x_0. By Step 1 we know that E is "locally equal" to W_φ, so that $E \cap B_\rho(x_0)$ turns out to be a minimal cone (since W_φ is a polygonal set).

Then, recalling Theorem 3.11, we reach a contradiction if we modify the set E in $B_\rho(x_0)$, by adding suitable triangles in such a way that the perimeter does not change and the volume strictly increases, as in Figure 8.

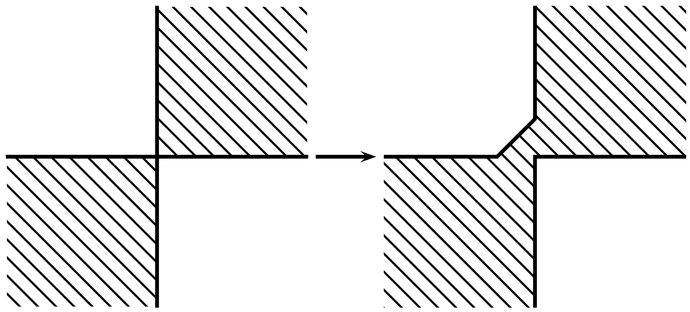

Figure 8: How to add area to a minimal cone.

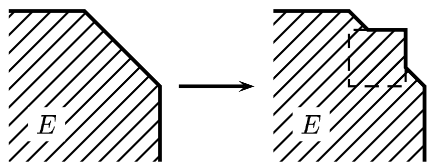

Figure 9: How to increase the volume of E by adding a small square ball.

$\square$

Given a set $E \subset X$ we say that E is locally convex (resp. concave) in $\Omega \subset X$ if for any $x \in \Omega$ there exists $\rho > 0$ such that $\overline{E} \cap B_\rho(x)$ (resp. $\overline{X \setminus E} \cap B_\rho(x)$) is convex.

If E is locally convex or locally concave in Ω then ∂E is the graph of a lipschitz function in a neighborhood of any $x \in \partial E \cap \Omega$.

Theorem 4.6 *Let $E \in \mathcal{N}_H(\Omega)$ and assume $H \geq \lambda > 0$ (resp. $H \leq \mu < 0$). Then, E is locally convex (resp. concave) in Ω. Moreover, $\Sigma(E) = \emptyset$.*

Proof:
Suppose for simplicity $\lambda \geq 1$. Reasoning as in Theorem 4.5, Step 1, we get that E is locally convex in $\Omega \setminus \Sigma(E)$. Suppose now that W_φ is a triangle or a quadrilateral (otherwise $\Sigma(E) = \emptyset$ by Theorem 3.4). We claim that for any $x \in (\partial E \cap \Omega) \setminus \Sigma(E)$ such that ∂E is differentiable in x, $\nu_E(x)$ is a vertex of F_φ. Indeed, if not we can add to E a small ball with center in x in such a way that the perimeter does not change and the volume increases (see Figure 9). This property, together with the local convexity of E, assures that E is locally equal to W_φ in $(\partial E \cap \Omega) \setminus \Sigma(E)$, i.e. for any $x \in (\partial E \cap \Omega) \setminus \Sigma(E)$ there exists $\rho > 0$ and $\overline{x} \in \partial W_\varphi$ such that $(E - x) \cap B_\rho = (W_\varphi - \overline{x}) \cap B_\rho$. This implies that for any $x_0 \in \Sigma(E)$ there exists $\rho > 0$ such that $E \cap B_\rho(x_0)$ coincides in $B_\rho(x_0)$ with a singular cone on x_0 which is minimal for the perimeter. We can now conclude as in Theorem 4.5, Step 3. $\square$

From Theorem 4.6 we easily get the following result.

Corollary 4.7 *Let $E \in \mathcal{N}_H(\Omega)$ and assume $H \in \mathcal{C}^0(\Omega)$. Then, $\Sigma(E) \subseteq \{x \in \Omega\colon H(x) = 0\}$. Moreover, ∂E is the graph of a lipschitz function in a neighborhood of any $x \in \partial E \cap \{x \in \Omega\colon H(x) \neq 0\}$.*

Notice that, by Theorem 4.6 and Corollary 4.7, if E is a minimizer of (2) and $H \in \mathcal{C}^0(\Omega)$ is such that $\mathcal{H}^1(\partial\{x \in \Omega : H(x) = 0\}) = 0$ then ∂E is the graph of a lipschitz function in a neighborhood of almost every $x \in \partial E \cap \Omega$.

References

[1] F. J. Almgren, J. Taylor, and L. Wang. Curvature-driven flows: a variational approach. *SIAM J. Control Optim.*, 31:387–437, 1993.

[2] L. Ambrosio, V. Caselles, S. Masnou, and J. M. Morel. Connected components of sets of finite perimeter and applications to image processing. *Journal of EMS*, 1999. (to appear).

[3] L. Ambrosio, N. Fusco, D. Pallara. *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs, Oxford: Clarendon Press, 2000.

[4] L. Ambrosio, M. Novaga, and E. Paolini. Some regularity results for minimal crystals. *ESAIM Control Optim. Calc. Var.*, 8:69–103, 2002.

[5] G. Bellettini, M. Novaga, and M. Paolini. Facet-breaking for three-dimensional crystals evolving by mean curvature. *Interfaces and Free boundaries*, 1:39–55, 1999.

[6] E. Bombieri. Regularity theory for almost minimal currents. *Arch. Rational Mech. Anal.*, 78:99–130, 1982.

[7] R. Finn. *Equilibrium capillary surfaces.* Springer-Verlag, New York, 1986.

[8] I. Fonseca. The wulff theorem revisited. *Proc. R. Soc. Lond. A*, 432:125–145, 1991.

[9] I. Fonseca and S. Müller. A uniqueness proof for the Wulff theorem. *Proc. R. Soc. Edinb.*, 119A:125–136, 1991.

[10] E. Giusti. *Minimal surfaces and functions of bounded variation*, volume 80 of *Monographs in Mathematics.* Birkhäuser, 1984.

[11] U. Massari. Esistenza e regolarità delle ipersuperfici di curvatura media assegnata in $\mathbb{R}^n$. *Arch. Rat. Mech. Anal.*, 55:357–382, 1974.

[12] F. Morgan, C. French, and S. Greenleaf. Wulff clusters in $\mathbb{R}^2$. *J. Geom. Anal.*, 8(1):97–115, 1998.

[13] C. Pommerenke. *Boundary Behaviour of Conformal Maps*, volume 299 of *Grundlehren der Mathematischen Wissenschaften.* Springer–Verlag, 1992.

[14] J. Taylor. Geometric crystal growth in 3D via facetted interfaces. *In: Computational Crystal Growers Workshop, Selected Lectures in Mathematics, Amer. Math. Soc.*, pages 111–113, 1992.

[15] J. Yunger. *Facet Stepping and Motion By Crystalline Curvature.* PhD thesis, Rutgers University, NJ, 1998.