The $p$-Laplace eigenvalue problem as $p \to 1$ and Cheeger sets in a Finsler metric*

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Dedicated to the memory of Thomas Lachand-Robert

Abstract

We consider the $p$–Laplacian operator on a domain equipped with a Finsler metric. After deriving and recalling relevant properties of its first eigenfunction for $p > 1$, we investigate the limit problem as $p \to 1$.

Keywords: $p$-Laplace, eigenfunction, Finsler metric, Cheeger set, anisotropic isoperimetric inequality

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1 Introduction

Imagine a nonlinear elastic membrane, fixed on a boundary $\partial \Omega$ of a plane domain $\Omega$. If $u(x)$ denotes its vertical displacement, and if its deformation energy is given by $\int_\Omega |\nabla u|^p \, dx$, then a minimizer of the Rayleigh quotient

$$
\frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx}
$$
on $W_0^{1,p}(\Omega)$ satisfies the Euler-Lagrange equation

$$
-\Delta_p u = \lambda_p |u|^{p-2}u \quad \text{in } \Omega,
$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$ is the well-known $p$-Laplace operator. This eigenvalue problem has been extensively studied in the literature. As $p \to 1$, formally the limit equation reads

$$
-\text{div} \left( \frac{\nabla u}{|\nabla u|} \right) = \lambda_1(\Omega) \quad \text{in } \Omega,
$$

$$
u = 0 \quad \text{on } \partial \Omega.
$$

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For a precise interpretation of (1.2) see [22] or [31]. Naturally, here \( \lambda_1(\Omega) := \lim_{p \to 1} \lambda_p(\Omega) \). A somewhat surprising recent result is that the family of eigenfunctions \( \{u_p\} \) converges in \( L^1(\Omega) \) mum grano salis to (a multiple of) the characteristic function \( \chi_C \) of a subset \( C \) of \( \Omega \), a so called Cheeger-set, see [20]. A Cheeger set of \( \Omega \) is characterized as a domain that minimizes

\[
h(\Omega) := \inf_D \frac{|\partial D|}{|D|}
\]

with \( D \) varying over all smooth subdomains of \( \Omega \) whose boundary \( \partial D \) does not touch \( \partial \Omega \), and with \(|\partial D|\) and \(|D|\) denoting \((n - 1)\)- and \(n\)-dimensional Lebesgue measure of \( \partial D \) and \( D \). The existence, uniqueness, regularity and construction of such sets is discussed in [20] and [21] (and partly in [32]) and its continuous dependence on \( \Omega \) in [17]. The paper [24] contains a numerical method for the calculation of \(n\)-dimensional Cheeger sets and some three-dimensional examples. Cheeger sets are of significant importance in the modelling of landslides, see [18], [19], or in fracture mechanics, see [23]. Notice that a set \( D \subseteq \Omega \) is a Cheeger set if and only if it is a minimizer of

\[
|\partial E| - h(\Omega)|E| \quad \text{for } E \subseteq \Omega.
\]

Now suppose that the membrane is not isotropic. It is for instance woven out of elastic strings like a piece of material. Then the deformation energy can be anisotropic, see [5]. Another way to describe this effect is by stating that the Euclidean distance in \( \Omega \) is somehow distorted. It is the purpose of the present paper to generalize the above result on eigenfunctions and their convergence as \( p \to 1 \) to the situation, where \( \Omega \subseteq \mathbb{R}^n \) is no longer equippped with the Euclidean norm, but instead with a general norm \( \phi \). In that case a Lipschitz continuous function \( u : \Omega \to \mathbb{R} \) (in a convex domain \( \Omega \)) has Lipschitz constant \( L = \sup_{z \in \Omega} \phi^*(\nabla u(z)) \), where \( \phi^* \) denotes the dual norm to \( \phi \). Therefore the Rayleigh quotient studied in this paper is given by

\[
R_p(u) := \frac{\int_{\Omega} (\phi^*(\nabla u))^p \, dx}{\int_{\Omega} |u|^p \, dx}
\]

on \( W^{1,p}_0(\Omega) \) and the Cheeger constant by

\[
h(\Omega) := \inf_{D \subseteq \Omega} \frac{P_\phi(D)}{|D|},
\]

with \( P_\phi \) denoting anisotropic perimeter in \( \mathbb{R}^n \) (see (2.10) below). The minimizer \( u_p \) of \( R_p \) satisfies the Euler-Lagrange equation

\[
-Q_p u := -\text{div} \left( (\phi^*(\nabla u))^{p-2} J(\nabla u) \right) \equiv \lambda_p |u|^{p-2} u \quad \text{in } \Omega
\]
in the weak sense [8], i.e.

\[ \int_{\Omega} (\phi^*(\nabla u_p))^{p-2} (\eta, \nabla v) \, dx = \lambda_p \int_{\Omega} |u_p|^{p-2} u_p \cdot v \, dx \quad (1.7) \]

for any \( v \in W^{1,p}_0(\Omega) \) and for a measurable selection \( \eta \in J(\nabla u_p) \), where the function \( J : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n) \) is defined as the subdifferential

\[ J(\xi) := \partial \left( \frac{\phi^*(\xi)^2}{2} \right). \quad (1.8) \]

Note that the function \( J \) is single-valued iff the norm \( \phi \) is strictly convex, i.e. if its unit sphere \( \{ x : \phi(x) = 1 \} \) contains no nontrivial line segments [36, pag. 400]. Note further that \( J(0) = 0 \) and that for the Euclidean norm the duality map reduces to the identity \( J(\nabla u) = \nabla u \).

The paper is organized as follows. In Section 2 we fix some notation. In Section 3 we recall and derive the existence, uniqueness, regularity and log-concavity of solutions for \( p > 1 \). In Section 4 we derive the limit equation for \( p \to 1 \). In Section 5, we discuss in detail the two-dimensional case, proving uniqueness of Cheeger sets in the convex case. In Section 6 we provide some instructive examples.

## 2 Notation

We say that the norm \( \phi \) is regular if \( \phi^2, (\phi^*)^2 \in C^2(\mathbb{R}^n) \). This includes for instance \( \phi(x) = \|x\|_q \) with \( q \in (1, \infty) \) but excludes the crystalline cases \( q = 1 \) or \( q = \infty \), see Section 6.

Given \( E \subset \mathbb{R}^n \) and \( x \in \mathbb{R}^n \), we set

\[ \text{dist}_\phi(x, E) := \inf_{y \in E} \phi(x - y), \quad d^E_\phi(x) := \text{dist}_\phi(x, E) - \text{dist}_\phi(\mathbb{R}^n \setminus E, x). \]

Notice that, at each point where \( d^E_\phi \) is differentiable, there holds

\[ \phi^*(\nabla d^E_\phi) = 1. \quad (2.9) \]

Let us define the (anisotropic) perimeter of \( E \) as

\[ P_\phi(E) := \sup \left\{ \int_E \text{div} \eta \, dx \mid \eta \in C^1_c(\mathbb{R}^n), \phi(\eta) \leq 1 \right\} = \int_{\partial^* E} \phi^*(\nu^E) d\mathcal{H}^{n-1}, \quad (2.10) \]

where \( \partial^* E \) and \( \nu^E \) denote the reduced boundary of \( E \) and the (Euclidean) unit normal to \( \partial^* E \).
Given an open set $\Omega \subseteq \mathbb{R}^n$ we define the $BV$-seminorm of $v \in BV(\Omega)$ as

$$\int_\Omega \phi^*(Du) := \sup \left\{ \int_\Omega v \, \text{div} \eta \, dx \mid \eta \in C^1_c(\mathbb{R}^n), \, \phi(\eta) \leq 1 \right\}.$$ 

Given $\delta > 0$, we define

$$E_+^\delta := \left\{ x \in \mathbb{R}^n \mid d_{\phi}^E < \delta \right\} = E + \delta W_\phi,$$

$$E_-^\delta := \left\{ x \in \mathbb{R}^n \mid d_{\phi}^E > -\delta \right\},$$

$$E_{\pm}^\delta := (E^\delta_{\pm})^\delta \subseteq E,$$

where $W_\phi := \{ x \mid \phi(x) < 1 \}$, also called *Wulff shape*, denotes the unit ball with respect to the norm $\phi$.

Given a compact set $E \subseteq \mathbb{R}^n$ with Lipschitz boundary, we denote by $n_\phi : \partial E \to \mathbb{R}^n$ any Lipschitz vector field satisfying $n_\phi \in J(\nabla d_{\phi}^E)$ a.e. on $\partial E$. Moreover, we set

$$\| \kappa_{\phi} \|_{L^\infty(\partial E)} := \inf_{n_\phi \in J(\nabla d_{\phi}^E)} \| \text{div}_\tau n_\phi \|_{L^\infty(\partial E)},$$

which represents the $L^\infty$-norm of the $\phi$-mean curvature of $\partial E$. Here $\text{div}_\tau$ denotes the tangential divergence operator. We make the convention that $\| \kappa_{\phi} \|_{L^\infty(\partial E)} = +\infty$ if the set $E$ does not admit any Lipschitz vector field $n_\phi \in J(\nabla d_{\phi}^E)$. We say that $E$ is $\phi$-regular if $\| \kappa_{\phi} \|_{L^\infty(\partial E)} < +\infty$.

Notice that in the Euclidean case $E$ is $\phi$-regular iff $\partial E$ is of class $C^{1,1}$. Moreover, the unit ball $W_\phi$ is always $\phi$-regular and $\| \kappa_{\phi} \|_{L^\infty(\partial W_\phi)} = n - 1$. To see this, it is enough to consider the vector field $n_\phi(x) = x/\phi(x)$.

### 3 Existence, uniqueness, regularity and log-concavity of solutions

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. If we minimize the functional

$$I_p(v) := \int_\Omega \phi^*(\nabla v)^p \, dx \quad \text{on} \quad K := \{ v \in W^{1,p}_0(\Omega) ; \, \| v \|_{L^p(\Omega)} = 1 \}, \quad (3.1)$$

then via standard arguments (see [6]) a minimizer $u_p$ exists for every $p > 1$ and it is a weak solution to the equation (1.6), with $\lambda_p = I_p(u_p)$. Note that $\Lambda_p := I_p(u_p)^{1/p}$ is the minimum of the Rayleigh quotient

$$R_p(v) := \left( \int_\Omega (\phi^*(\nabla v))^p \, dx \right)^{1/p} / \| v \|_p$$

(3.2)
on \( W^{1,p}_0(\Omega) \setminus \{0\} \). Without loss of generality we may assume that \( u_p \) is non-negative. Otherwise we can replace it by its modulus.

Moreover, as shown in [6] any nonnegative weak solution of (1.6) is necessarily bounded and positive in \( \Omega \). If \( p > n \), then \( u_p \) is also uniformly Hölder continuous because of the Sobolev-embedding theorem and the equivalence of the usual Sobolev norm with

\[
\|u\|_{1,p} := \left( \int_\Omega |u|^p \, dx \right)^{1/p} + \left( \int_\Omega (\phi^*(\nabla u))^p \, dx \right)^{1/p}.
\]  

(3.3)

If the norm \( \phi \) is regular and \( p > 1 \), one can even show that \( u_p \in C^{1,\alpha}(\Omega) \). Indeed, the function \( u_p \) minimizes

\[
J_p(v) := \int_\Omega (\phi^*(\nabla v))^p - \lambda_p(\Omega) |u|^p \, dx,
\]

and the theory for quasiminima in [16] implies that minimizers are bounded (Thm. 7.5), Hölder continuous (Thm. 7.16) and satisfy a strong maximum principle (Thm. 7.12), because one can easily check that \( u_p \) satisfies (7.71) in [16]. Therefore \( u_p \) is positive. Once positivity is known, the uniqueness follows from a simple convexity argument, see [4] or [6]. Moreover, from the result in [11] one can conclude that \( u_p \in C^{0,\beta}(\Omega) \) for any \( \beta \in (0, 1) \). Finally, if \( \phi \) is regular, then \( u_p \in C^{1,\alpha}(\Omega) \) according to [7], [26], [34], [35] or [12]. Let us summarize these statements.

**Theorem 3.1.** For every \( p \in (1, \infty) \) the nonnegative minimizer \( u_p \) of (3.1) is positive, unique, belongs to \( C^{0,\beta}(\Omega) \) for any \( \beta \in (0, 1) \) and it solves (1.6) in the weak sense. Moreover, if the norm \( \phi \) is regular then \( u_p \) is of class \( C^{1,\alpha}(\Omega) \) for some \( \alpha \in (0, 1) \). Finally, if \( \Omega \) is convex, then \( u_p \) is log-concave and the level sets set \( \{u_p > t\} \subseteq \Omega \) are convex for all \( t > 0 \).

**Proof.** To prove the last statement, we follow Sakaguchi’s approach from [29], first for strictly convex \( \Omega \) and for a smooth norm \( \phi \). The general case follows then from approximation arguments for \( \Omega \) and \( \phi \). Log-concavity of a sequence \( u_{p,n} \) is preserved under pointwise limits as \( n \to \infty \), because the inequality

\[
\log u_{p,n} \left( \frac{x_1 + x_2}{2} \right) \geq \frac{1}{2} \log u_{p,n}(x_1) + \frac{1}{2} \log u_{p,n}(x_2) \quad \text{in } \Omega \times \Omega
\]

is stable under such limits. If \( u_p \) solves (1.6), then \( v_p := \log u_p \) solves

\[
-\text{div} \left( (\phi^*(\nabla v))^p - \lambda_p \right) = (p-1) (\phi^*(\nabla v))^p + \lambda_p \quad \text{in } \Omega
\]  

(3.4)

and this degenerate elliptic equation can be approximated by a nondegenerate one

\[
-\text{div} \left( (\varepsilon + (\phi^*(\nabla v))^2)^{\frac{p-2}{2}} \cdot J(\nabla v) \right)
\]
\[(p - 1 - \varepsilon)(\phi^*(\nabla v))^2(\varepsilon + (\phi^*(\nabla v))^2)^{\frac{p-2}{2}} + \lambda_p. \quad (3.5)\]

Modulo yet another approximation by a right hand side which is strictly monotone in \(v\), equation (3.5) is now amenable to Korevaar's concavity maximum principle which states that the concavity function

\[C(x_1, x_2) := v \left(\frac{x_1 + x_2}{2}\right) - \frac{1}{2} v(x_1) - \frac{1}{2} v(x_2) \in \Omega \times \Omega\]

can attain a negative minimum only on the boundary of \(\Omega \times \Omega\). The latter is ruled out, however, because of the boundary condition. \(\square\)

**Remark 3.2.** We should point out that without uniqueness of \(u_p\) the approximation arguments would only yield log-concavity of a solution and not the solution \(u_p\).

### 4 The limit problem for \(p \to 1\)

The following estimate for \(\lambda_p\) is optimal (as \(p \to 1\)) for any shape of \(\Omega\) (see [6]).

**Theorem 4.1.** (Convergence of eigenvalues) For every \(p \in (1, \infty)\) the eigenvalue \(\lambda_p(\Omega)\) can be estimated from below as follows:

\[\lambda_p(\Omega) \geq \left(\frac{h(\Omega)}{p}\right)^p. \quad (4.1)\]

Here \(h(\Omega)\) is the Cheeger constant of \(\Omega\) as defined in (1.5). Moreover, as \(p \to 1\), the eigenvalue \(\lambda_p(\Omega)\) converges to \(\lambda_1(\Omega) = h(\Omega)\).

**Proof.** In the Euclidean case this is Cheeger’s original estimate [10] when \(p = 2\), and for general \(p\) it can be found in [25], [2], [27] and [33]. For a more general \(\phi\) one can easily modify their proofs by using the generalized coarea formula from [14] or [15]. To prove the limiting behaviour of \(\lambda_p(\Omega)\) as \(p \to 1\) we proceed as in [20] and observe that (4.1) implies \(\liminf_{p \to 1} \lambda_p(\Omega) \geq h(\Omega)\). Therefore it suffices to find a suitable upper bound. Let \(\{D_k\}_{k=1,2,...}\) be a sequence of regular domains for which \(P_\phi(D_k)/|D_k|\) converges to \(h(\Omega)\). We approximate the characteristic function of each \(D_k\) by a function \(w_k\) with the following properties: \(w \equiv 1\) on \(\overline{D_k}\), \(w \equiv 0\) outside an \(\varepsilon\)-neighborhood of \(D_k\) and \(\phi^*(\nabla w_k) = 1/\varepsilon\) in an \(\varepsilon\)-layer outside \(D_k\). For small \(\varepsilon\) the function \(w_k\) is in \(W^{1,\infty}_0(\Omega)\) and provides the upper bound

\[\lambda_p(\Omega) \leq \frac{P_\phi(D_k)}{|D_k|} \varepsilon^{1-p}. \quad (4.2)\]

Now one sends first \(p \to 1\), then \(k \to \infty\) to complete the proof. \(\square\)
Theorem 4.2. (Convergence of eigenfunctions) As $p \to 1$, the eigenfunction $u_p$ converges, up to a subsequence, to a limit function $u_1 \in BV(\Omega)$, with $u_1 \geq 0$ and $\|u_1\|_1 = 1$. Moreover, almost all level sets $\Omega_t := \{u_1 > t\}$ of $u_1$ are Cheeger sets.

Proof. For every $p > 1$ the function $u_p$ minimizes

$$J_p(v) := \int_{\Omega} (\phi^*(\nabla v))^{p} - \lambda_p(\Omega)|v|^p \, dx$$

on $W_0^{1,p}(\Omega)$. If one extends $J_p$ to $L^1(\Omega)$ by setting it $+\infty$ on $L^1(\Omega) \setminus W_0^{1,p}(\Omega)$, the family $J_p$ $\Gamma$-converges (see [13]) with respect to the $L^1(\Omega)$-topology to

$$J_1(v) := \begin{cases} \int_{\Omega} \phi^*(Dv) - h(\Omega) \int_{\Omega} |v| \, dx & v \in BV(\Omega), \\ +\infty & v \in L^1(\Omega) \setminus BV(\Omega). \end{cases}$$

Indeed, since $J_1$ is lower semicontinuous on $L^1(\Omega)$, it is enough to prove the $\Gamma$-limsup inequality on the subset $C^1(\overline{\Omega}) \subset L^1(\Omega)$ (which is dense both in topology and in energy), where it becomes trivial.

Let us now prove the $\Gamma$-liminf inequality. Notice that, if $u_{p_n} \to u$ in $L^1(\Omega)$, then either there exists a subsequence $u_{p_k}$ which is equibounded in $BV(\Omega)$, or $J_{p_n}(u_{p_n})$ goes to $+\infty$. If $u_k := u_{p_{n_k}}$ is bounded in $BV(\Omega)$, letting $p_k := p_{n_k}$ and $\lambda_k := J_{u_{p_k}}(u_{p_k})$, we have

$$J_1(u_k) = \int_{\Omega} \phi^*(\nabla u_k) - h(\Omega)|u_k| \, dx$$

$$\leq \left[ \int_{\Omega} (\phi^*(\nabla u_k))^{p_k} \, dx \right]^{\frac{1}{p_k}} |\Omega|^{\frac{p_k-1}{p_k}} - h(\Omega) \int_{\Omega} |u_k| \, dx$$

$$\leq \frac{1}{p_k} \int_{\Omega} (\phi^*(\nabla u_k))^{p_k} \, dx + \left( \frac{p_k-1}{p_k} - 1 \right) h(\Omega) \int_{\Omega} |u_k| \, dx$$

$$+ \lambda_k(\Omega) \int_{\Omega} |u_k|^{p_k} \, dx - \lambda_k(\Omega) \int_{\Omega} |u_k|^{p_k} \, dx$$

$$\leq J_k(u_k) + \left( \frac{p_k-1}{p_k} - 1 \right) h(\Omega) \int_{\Omega} |u_k|^{p_k} \, dx$$

$$= J_k(u_k) + \omega_k,$$  \hspace{1cm} (4.3)

where $\lim_k \omega_k = 0$. It follows

$$J_1(u) \leq \liminf_{k \to \infty} J_1(u_k) \leq \liminf_{k \to \infty} J_k(u_k).$$

Since $J_p \geq 0$ on $W_0^{1,p}(\Omega)$, we get $J_1 \geq 0$ on $BV(\Omega)$. Moreover $u_p$ forms a minimizing sequence for $J_1$ since, from the last inequality in (4.3), we have

$$\int_{\Omega} \phi^*(\nabla u_p) \, dx \leq \frac{p-1}{p} |\Omega| + \lambda_p(\Omega),$$

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where we have used the fact that \( J_p(u_p) = 0 \) and \( \|u_p\|_p = 1 \). As a consequence, the family \( \{u_p\}_{p>1} \) is bounded in \( BV(\Omega) \) and, after possibly passing to a subsequence, it converges strongly in \( L^1(\Omega) \) to a limit function \( u_1 \in BV(\Omega) \) such that \( J_1(u_1) = 0, u_1 \geq 0 \) and \( \|u_1\|_1 = 1 \). Using the coarea formula, one can see that for all \( t \in [0, \max_{\Omega} u_1] \) the level set \( \Omega_t := \{u_1 > t\} \) is a Cheeger set. 

\[ \square \]

**Remark 4.3.** As a consequence of Theorem 4.2 and the logconcavity of \( u_p \), for convex \( \Omega \) (Theorem 3.1) there exists a convex Cheeger set. Moreover, it follows from the results of [9] that there exists a convex Cheeger set \( D \subseteq \Omega \) which is maximal, in the sense that any other Cheeger set of \( \Omega \) must be contained in \( D \). The uniqueness of Cheeger sets is in general not true for nonconvex domains (see [21]).

## 5 The planar case

In this section we derive further properties of the function \( u_1 \), under the additional assumption \( n = 2 \). Let us begin with the following theorem, which extends the analogous result in the Euclidean case [21, Th. 1].

**Theorem 5.1.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded open convex set. Then, there exists a unique Cheeger set \( D \subseteq \Omega \). Moreover, \( D \) is convex and we have

\[
h(\Omega) = \frac{1}{t^*}, \quad D = \Omega^*_t,
\]

where \( t^* > 0 \) is the (unique) value \( t \) such that \( \|\Omega_t^\|= t^2|W_\phi| \).

**Proof.** Let \( D \) be a Cheeger set of \( \Omega \). Notice first that \( D \) is a convex set, since otherwise we could replace it by its convex hull and reduce (1.3) (see [3, Th. 7.1]). Moreover, from the first variation of (1.3) it follows that the anisotropic curvature of \( \partial D \) is bounded by \( h(\Omega) \), and each connected component of \( \partial D \cap \Omega \) is contained up to translation in \( \frac{1}{\pi(\Omega)} \partial W_\phi \) (see [28, Theorem 4.3]). Let \( \bar{D} \) be the open maximal Cheeger set of \( \Omega \) (recall Remark 4.3), and let \( \Gamma \subset \frac{1}{\pi(\Omega)} \partial W_\phi \) be a connected component of \( \partial D \cap \bar{D} \). We denote by \( x, y \in \Gamma \cap \partial \bar{D} \) the extremal points of \( \Gamma \), and we let \( \Gamma' \) be the arc of \( \partial \bar{D} \) with extremal \( x, y \) and lying in the same halfplane of \( \Gamma \) with respect to the straight line \( r \) passing through \( x, y \) (see Figure 1). Reasoning as in [3, Lemma 7.3], it is easy to show that both \( \Gamma \) and \( \Gamma' \) can be written as graphs on \( r \) along some directions. More precisely, there exists a vector \( v \in \mathbb{R}^2 \), with \( |v| = 1 \), and two functions \( f_1, f_2 : r \to \mathbb{R} \) such that \( 0 \leq f_1 \leq f_2 \) on \( [x, y] \), that \( \min\{f_2(x), f_2(y)\} = 0 \), and that \( \Gamma = F_1([x, y]) \) and \( \Gamma' = F_2([x, y]) \), with \( F_i(x) := f_i(x)v \), for \( i = 1, 2 \).
Without loss of generality, we shall assume that $v \perp r$. Since $D$ and $\tilde{D}$ are both minimizers of (1.3), it follows that both $f_1$ and $f_2$ are minimizers of

$$G(f) := \int_{[x,y]} \phi^*(-f'(s), 1) - h(\Omega) f(s) \, ds. \quad (5.2)$$

If $\phi$ is a regular norm, then the functional $G$ is strictly convex, which implies $f_1 = f_2$, i.e. $D = \tilde{D}$. For a general norm, one has to be more careful, since the functional $G$ is not strictly convex, but only convex. However, reasoning as in [3, Lemma 8.2], the inclusion $\Gamma \subset \frac{1}{h(\Omega)} \partial W_\phi$ and the inequality $f_1 \leq f_2$ imply $\|\kappa_\phi\|_{L^\infty(\Gamma')} \geq h(\Omega)$, with equality iff $\Gamma = \Gamma'$, which proves the uniqueness of the Cheeger set $D$.

Let us now prove (5.1), reasoning as in [21, Th. 1]. It has been proved in [3] that the convex set $D = \Omega^{1/h(\Omega)}$ is a Cheeger set of $\Omega$, hence it is the unique Cheeger set of $\Omega$. Therefore, it remains to prove that $t^* = 1/h(\Omega)$, i.e.

$$|\Omega_{-\frac{1}{h(\Omega)} \kappa_\phi}| = \frac{|W_\phi|}{h(\Omega)^2}.$$ 

Let us recall from [1, Section 2.7],[30] the following Steiner-type formulae

$$|C^\delta| = |C| + \delta P_\phi(C) + \delta^2 |W_\phi|,$$

$$P_\phi(C^\delta) = P_\phi(C) + \delta P_\phi(W_\phi). \quad (5.3)$$

Incidentally, the second equation follows from the first one and, as in the Euclidean case, $P_\phi(W_\phi) = 2|W_\phi|$. This follows from integrating $\text{div} \, x$ on $W_\phi$. 

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Applying (5.3) to \( C = D_1^1(\Omega) \) and recalling that \( h(\Omega) = P_0(D)/|D| \), we get
\[
|D_1^1(\Omega)| = \frac{|W_0|}{h(\Omega)^2}.
\]
The claim now follows if we observe that
\[
\Omega_1^1(\Omega) = D_1^1(\Omega).
\]
\[\square\]

**Corollary 5.2.** If \( n = 2 \) and \( \Omega \) is a bounded convex set, then the sequence of functions \( u_p \) converges to a multiple of the characteristic function of \( D \). Moreover, \( D = \Omega \) if and only if
\[
\|\kappa_\phi\|_{L^\infty(\partial \Omega)} \leq h(\Omega) .
\] (5.4)

In particular, (5.4) always holds in the case \( \Omega = W_0 \).

### 6 Example and concluding remarks

If the norm under consideration for \( x \in \Omega \) is the usual \( \ell_q \) norm, i.e. for 
\[
\phi_q(x) = (\sum_{i=1}^n |x_i|^q)^{1/q}, \quad q \geq 1.
\]
When \( q > 1 \), the dual norm of \( \phi_q \) is given by
\[
\phi_q^* = \phi_{q'}, \quad \text{with } q' = q/(q-1),
\]
and the duality map according to (1.8) is
\[
J_i(y) = (|y|_{q'})^{2-q'}|y_i|^{q'-2}y_i.
\]

Then the \( p \)-Laplace operator in this metric is given by (see [6])
\[
Q_{p,q}u = \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \phi_{q'}(\nabla u)^{p-q'} \left| \frac{\partial u}{\partial x_i} \right|^{q'-2} \frac{\partial u}{\partial x_i} \right),
\]
and for \( q = 2 = q' \) the norm \( \phi_{q'} \) is just the Euclidean norm and \( Q_{p,q} \) reduces to the well-known \( p \)-Laplace Operator
\[
Q_{p,q}u = \Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u).
\]

For general \( q \) and \( p \to 1 \) the operator \( Q_{1,q} \) is formally given by
\[
Q_{1,q}u = \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \left[ \frac{|u_{x_i}|}{\phi_{q'}(\nabla u)} \right]^{q'-2} \frac{u_{x_i}}{\phi_{q'}(\nabla u)} \right).
\]

Again for \( q = 2 = q' \) this expression shrinks down to the customary
\[
Q_{1,2}u = \Delta_1 u = \text{div} \left( \frac{\nabla u}{|\nabla u|} \right).
\]
We complete this section with the construction of a particular Cheeger set for a nonregular anisotropy. Let us fix $n = 2$ and consider the norm $\phi = \phi_1$. Notice that in this case the Wulff Shape $W_\phi$ has the shape of a rhombus. To be precise, it is square of sidelength $\sqrt{2}$, centered in the origin and rotated by $\pi/2$ with respect to the coordinate axes. Moreover, the dual norm $\phi^*$ is given by $\phi^*(y) = \max\{|y_1|, |y_2|\}$. To better illustrate the results of Section 5, let us compute the Cheeger set (and Cheeger constant) of a square $Q$ of sidelength 1 (see Figure 2).

Since in this case $|W_\phi| = 2$ and $Q_L^-$ is a square of sidelength $1 - 2t$, from Theorem 5.1 we get $t^* = 1 - \sqrt{2}/2$ and $h(Q) = 2 + \sqrt{2}$. It is interesting to note that the Cheeger set of $Q$ is a regular octahedron.
Figure 2: The Cheeger set of a square with respect to the norm $\phi_1$.

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