On the existence of connecting orbits for critical values of the energy

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Abstract
We consider an open connected set Ω and a smooth potential $U$ which is positive in Ω and vanishes on $\partial \Omega$. We study the existence of orbits of the mechanical system

$$\ddot{u} = U_x(u),$$

that connect different components of $\partial \Omega$ and lie on the zero level of the energy. We allow that $\partial \Omega$ contains a finite number of critical points of $U$. The case of symmetric potential is also considered.

1 Introduction

Let $U : \mathbb{R}^n \to \mathbb{R}$ be a function of class $C^2$. We assume that $\Omega \subset \mathbb{R}^n$ is a connected component of the set $\{x \in \mathbb{R}^n : U(x) > 0\}$ and that $\partial \Omega$ is compact and is the union of $N \geq 1$ distinct nonempty connected components $\Gamma_1, \ldots, \Gamma_N$. We consider the following situations

$\mathbf{H}$ $N \geq 2$ and, if $\Omega$ is unbounded, there is $r_0 > 0$ and a non-negative function $\sigma : [r_0, +\infty) \to \mathbb{R}$ such that

$$\sqrt{U(x)} \geq \sigma(|x|), \quad x \in \Omega, \ |x| \geq r_0. \quad (1.1)$$

$\mathbf{H}_s$ $\Omega$ is bounded, the origin $0 \in \mathbb{R}^n$ belongs to $\Omega$ and $U$ is invariant under the antipodal map

$$U(-x) = U(x), \quad x \in \Omega.$$  

Condition (1.1) was first introduced in [7]. A sufficient condition for (1.1) is that $\lim \inf_{|x| \to \infty} U(x) > 0$.

We study non constant solutions $u : (T_-, T_+) \to \Omega$, of the equation

$$\ddot{u} = U_x(u), \quad U_x = \left(\frac{\partial U}{\partial x}\right)^T, \quad (1.2)$$

that satisfy

$$\lim_{t \to T_\pm} d(u(t), \partial \Omega) = 0, \quad (1.3)$$

with $d$ the Euclidean distance, and lie on the energy surface

$$\frac{1}{2} |\dot{u}|^2 - U(u) = 0. \quad (1.4)$$

We allow that the boundary $\partial \Omega$ of $\Omega$ contains a finite set $P$ of critical points of $U$ and assume

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\textbf{H}_1 \text{ If } \Gamma \in \{\Gamma_1, \ldots, \Gamma_N\} \text{ has positive diameter and } p \in P \cap \Gamma \text{ then } p \text{ is a hyperbolic critical point of } U.

If \Gamma has positive diameter, then hyperbolic critical points \( p \in \Gamma \) correspond to saddle-center equilibrium points in the zero energy level of the Hamiltonian system associated to (1.2). These points are organizing centers of complex dynamics, see [6].

Note that \textbf{H}_1 does not exclude that some of the \(\Gamma_j\) reduce to a singleton, say \(\{p\}\), for some \(p \in P\). In this case nothing is required on the behavior of \( U \) in a neighborhood of \( p \) aside from being \(C^2\).

A comment on \textbf{H} and \textbf{H}_1 is in order. If \( P \) is nonempty \( u \equiv p \) for \( p \in P \) is a constant solution of (1.2) that satisfies (1.3) and (1.4). To avoid trivial solutions of this kind we require \( N \geq 2 \) in \textbf{H}, and look for solutions that connect different components of \( \partial \Omega \). In \textbf{H}, we do not exclude that \( \partial \Omega \) is connected (\( N = 1 \)) and avoid trivial solutions by restricting to a symmetric context and to solutions that pass through \( 0 \).

We prove the following results.

\textbf{Theorem 1.1}. Assume that \textbf{H} and \textbf{H}_1 hold. Then for each \( \Gamma_- \in \{\Gamma_1, \ldots, \Gamma_N\} \) there exist \( \Gamma_+ \in \{\Gamma_1, \ldots, \Gamma_N\} \setminus \{\Gamma_-\} \) and a map \( u^* : (T_-, T_+) \to \Omega \), with \( -\infty \leq T_- < T_+ \leq +\infty \), that satisfies (1.2), (1.4) and

\[
\lim_{t \to T_{\pm}} d(u^*(t), \Gamma_{\pm}) = 0. \tag{1.5}
\]

Moreover, \( T_- > -\infty \) (resp. \( T_+ < +\infty \)) if and only if \( \Gamma_- \) (resp. \( \Gamma_+ \)) has positive diameter. If \( T_- > -\infty \) it results

\[
\lim_{t \to T_-} u^*(t) = x_-,
\]
\[
\lim_{t \to T_-} u^*(t) = 0,
\]

for some \( x_- \in \Gamma_- \setminus P \). An analogous statement holds if \( T_+ < +\infty \).

\textbf{Theorem 1.2}. Assume that \textbf{H}_a and \textbf{H}_1 hold. Then there exist \( \Gamma_+ \in \{\Gamma_1, \ldots, \Gamma_N\} \) and a map \( u^* : (0, T_+) \to \Omega \), with \( 0 < T_+ \leq +\infty \), that satisfies (1.2), (1.4) and

\[
\lim_{t \to T_+} d(u^*(t), \Gamma_+) = 0. \tag{1.6}
\]

Moreover, \( T_+ < +\infty \) if and only if \( \Gamma_+ \) has positive diameter. If \( T_+ < +\infty \) it results

\[
\lim_{t \to T_+} u^*(t) = x_+,
\]
\[
\lim_{t \to T_+} u^*(t) = 0,
\]

for some \( x_+ \in \Gamma_+ \setminus P \).

We list a few straightforward consequences of Theorems 1.1 and 1.2.

\textbf{Corollary 1.3}. Theorem 1.1 implies that, if \( \partial \Omega = P \), given \( p_- \in P \) there is \( p_+ \in P \setminus \{p_-\} \) and a heteroclinic connection between \( p_- \) and \( p_+ \), that is a solution \( u^* : \mathbb{R} \to \mathbb{R}^n \) of (1.2) and (1.4) that satisfies

\[
\lim_{t \to \pm \infty} u^*(t) = p_{\pm}.
\]

The problem of the existence of heteroclinic connections between two isolated zeros \( p_+ \) of a non-negative potential has been recently reconsidered by several authors. In [1] existence was established under a mild monotonicity condition on \( U \) near \( p_{\pm} \). This condition was removed in [8], see also [2].

The most general results, equivalent to the consequence of Theorem 1.1 discussed in Section 2.1, were recently obtained in [7] and in [11], see also [3]. All these papers establish existence by a variational approach. In [1], [8] and [2] by minimizing the action functional, and in [7] and [11] by minimizing the Jacobi functional.
**Corollary 1.4.** Theorem 1.1 implies that, if \( \Gamma_\pm = \{ p \} \) for some \( p \in P \) and the elements of \( \{ \Gamma_1, \ldots, \Gamma_N \} \setminus \{ \Gamma_- \} \) have all positive diameter, there exists a nontrivial orbit homoclinic to \( p \) that satisfies (1.2), (1.4).

**Proof.** Let \( v^* : \mathbb{R} \to \Omega \cup \{ x_+ \} \) be the extension defined by

\[
v^*(T_+ + t) = u^*(T_+ - t), \quad t \in (0, +\infty), \quad v^*(T_+) = x_+,\]

of the solution \( u^* : (-\infty, T_+) \to \Omega \) given by Theorem 1.1. The map \( v^* \) so defined is a smooth non-constant solution of (1.2) that satisfies

\[
\lim_{t \to \pm \infty} v^*(t) = p.
\]

\[\Box\]

**Corollary 1.5.** Theorem 1.1 implies that, if all the sets \( \Gamma_1, \ldots, \Gamma_N \) have positive diameter, given \( \Gamma_- \in \{ \Gamma_1, \ldots, \Gamma_N \} \), there exist \( \Gamma_+ \in \{ \Gamma_1, \ldots, \Gamma_N \} \setminus \{ \Gamma_- \} \) and a periodic solution \( v^* : \mathbb{R} \to \Omega \) of (1.2) and (1.4) that oscillates between \( \Gamma_- \) and \( \Gamma_+ \). This solution has period \( T = 2(T_+ - T_-) \).

**Proof.** The solution \( v^* \) is the \( T \)-periodic extension of the map \( w^* : [T_-, 2T_+ - T_-] \to \Omega \) defined by \( w^*(t) = u^*(t) \) for \( t \in (T_-, T_+) \), where \( u^* \) is given by Theorem 1.1, and

\[
\begin{align*}
w^*(T_+) &= x_+, \\
w^*(T_+ + t) &= u^*(T_- - t), \quad t \in (0, T_+ - T_-).
\end{align*}
\]

\[\Box\]

The problem of existence of heteroclinic, homoclinic and periodic solutions of (1.2), in a context similar to the one considered here, was already discussed in [2] where \( \partial \Omega \) is allowed to include continua of critical points. Our result concerning periodic solutions extends a corresponding result in [2] where existence was established under the assumption that \( P = \emptyset \).

The following result is a direct consequence of Theorem 1.2.

**Corollary 1.6.** Theorem 1.2 implies that, if all the sets \( \Gamma_1, \ldots, \Gamma_N \) have positive diameter, there exists \( \Gamma_+ \in \{ \Gamma_1, \ldots, \Gamma_N \} \) and a periodic solution \( v^* : \mathbb{R} \to \Omega \) of (1.2) and (1.4) that satisfies

\[
v^*(-t) = -v^*(t), \quad t \in \mathbb{R}.
\]

This solution has period \( T = 4T_+ \), with \( T_+ \).

**Proof.** The solution \( v^* \) is the \( T \)-periodic extension of the map \( w^* : [-2T_+, 2T_+] \to \Omega \) defined by \( w^*(t) = u^*(t) \) for \( t \in (0, T_+) \), where \( u^* \) is given by Theorem 1.2, and by

\[
\begin{align*}
w^*(t) &= -w^*(-t), \quad t \in (-T_+, 0), \\
w^*(0) &= 0, \quad w^*(\pm T_+) = \pm x_+, \\
w^*(T_+ + t) &= u^*(T_+ - t), \quad t \in (0, T_+], \\
w^*(-T_+ + t) &= u^*(-T_- - t), \quad t \in [-T_+, 0).
\end{align*}
\]

In particular the solution oscillates between \( x_+ \) and \(-x_+ \) and this is true also when \( \partial \Omega \) is connected \((N = 1)\). \[\Box\]
2 Proof of Theorems 1.1 and 1.2

We recall a classical result.

**Lemma 2.1.** Let \( G : \mathbb{R}^n \rightarrow \mathbb{R} \) be a smooth bounded and non-negative potential, \( I = (a, b) \) a bounded interval. Define the Jacobi functional

\[
J_G(q, I) = \sqrt{2} \int_I \sqrt{G(q(t))} |\dot{q}(t)| \, dt
\]

and the action functional

\[
A_G(q, I) = \int_I \left( \frac{1}{2} |\dot{q}(t)|^2 + G(q(t)) \right) \, dt.
\]

Then

(i) \( J_G(q, I) \leq A_G(q, I), \quad q \in W^{1,2}(I; \mathbb{R}^n) \)

with equality sign if and only if

\[
\frac{1}{2} |\dot{q}(t)|^2 - G(q(t)) = 0, \quad t \in I.
\]

(ii) \( \min_{q \in Q} J_G(q, I) = \min_{q \in Q} A_G(q, I) \),

where

\[
Q = \{ q \in W^{1,2}(I; \mathbb{R}^n) : q(a) = q_a, q(b) = q_b \}.
\]

When \( G = U \) we shall simply write \( J, A \) for \( J_U, A_U \).

We now start the proof of Theorem 1.1. Choose \( \Gamma_- \in \{ \Gamma_1, \ldots, \Gamma_N \} \) and set

\[
d = \min\{|x - y| : x \in \Gamma_-, y \in \partial \Omega \setminus \Gamma_-\}.
\]

For small \( \delta \in (0, d) \) let \( O_\delta = \{ x \in \Omega : d(x, \Gamma_-) < \delta \} \) and let \( U_0 = \frac{1}{2} \min_{x \in \partial O_\delta \cap \Omega} U(x) \). We note that \( U_0 > 0 \) and define the admissible set

\[
\mathcal{U} = \{ u \in W^{1,2}((T_\alpha^u, T_\beta^u); \mathbb{R}^n) : -\infty < T_\alpha^u < T_\beta^u < +\infty, \quad u((T_\alpha^u, T_\beta^u)) \subset \Omega, \quad U(u(0)) = U_0, \quad u(T_-^u) \in \Gamma_-, \quad u(T_+^u) \in \partial \Omega \setminus \Gamma_- \}.
\]

We determine the map \( u^* \) in Theorem 1.1 as the limit of a minimizing sequence \( \{ u_j \} \subset \mathcal{U} \) of the action functional

\[
A(u, (T_\alpha^u, T_\beta^u)) = \int_{T_\alpha^u}^{T_\beta^u} \left( \frac{1}{2} |\dot{u}(t)|^2 + U(u(t)) \right) \, dt.
\]

Note that in the definition of \( \mathcal{U} \) the times \( T_\alpha^u \) and \( T_\beta^u \) are not fixed but, in general, change with \( u \). Note also that the condition \( U(u(0)) = U_0 \) in (2.1) is a normalization which can always be imposed by a translation of time and has the scope of eliminating the loss of compactness due to translation invariance. Let \( \bar{x}_- \in \Gamma_- \) and \( \bar{x}_+ \in \partial \Omega \setminus \Gamma_- \) be such that \( |\bar{x}_+ - \bar{x}_-| = d \) and set

\[
\dot{u}(t) = (1 - (t + \tau))\bar{x}_- + (t + \tau)\bar{x}_+, \quad t \in [-\tau, 1 - \tau],
\]

where \( \tau \in (0, 1) \) is chosen so that \( U(\dot{u}(0)) = U_0 \). Then \( \bar{u} \in \mathcal{U}, \ T_-^\bar{u} = -\tau, \ T_+^\bar{u} = 1 - \tau \) and

\[
A(\bar{u}, (-\tau, 1 - \tau)) = a < +\infty.
\]
Next we show that there are constants $M > 0$ and $T_0 > 0$ such that each $u \in \mathcal{U}$ with
\[ \mathcal{A}(u, (T_u^-, T_u^+)) \leq a, \] (2.2)
satisfies
\[ \|u\|_{L^\infty((T_u^-, T_u^+); \mathbb{R}^n)} \leq M, \]
\[ T_u^- \leq -T_0 < T_0 \leq T_u^+. \] (2.3)

The $L^\infty$ bound on $u$ follows from $H$ and from Lemma 2.1, in fact, if $\Omega$ is unbounded, $|u(i)| = M$ for some $i \in (T_u^-, T_u^+)$ implies
\[ a \geq \mathcal{A}(u, (T_u^-, i)) \geq \int_{T_u^-}^{i} \sqrt{2U(u(t))|\dot{u}(t)|} dt \geq \sqrt{2} \int_{r_0}^{M} \sigma(s) ds. \]

The existence of $T_0$ follows from
\[ \frac{d^2}{|T_u^-|} \leq \int_{T_u^-}^{0} |\dot{u}(t)|^2 dt \leq 2a, \quad \frac{d^2}{T_u^+} \leq \int_{0}^{T_u^+} |\dot{u}(t)|^2 dt \leq 2a, \]
where $d_1 = d(\partial \Omega, \{x : U(x) > U_0\})$.

Let $\{u_j\} \subset \mathcal{U}$ be a minimizing sequence
\[ \lim_{j \to +\infty} \mathcal{A}(u_j, (T_u^{j-}, T_u^{j+})) = \inf_{u \in \mathcal{U}} \mathcal{A}(u, (T_u^-, T_u^+)) := a_0 \leq a. \] (2.4)

We can assume that each $u_j$ satisfies (2.2) and (2.3). By considering a subsequence, that we still denote by $\{u_j\}$, we can also assume that there exist $T_u^\infty, T_u^\infty$ with $-\infty \leq T_u^\infty \leq -T_0 < T_0 \leq T_u^\infty \leq +\infty$ and a continuous map $u^* : (T_u^\infty, T_u^\infty) \to \mathbb{R}^n$ such that
\[ \lim_{j \to +\infty} T_u^{j-} = T_u^{-}, \]
\[ \lim_{j \to +\infty} u_j(t) = u^*(t), \quad t \in (T_u^-, T_u^\infty), \] (2.5)
and in the last limit the convergence is uniform on bounded intervals. This follows from (2.3) which implies that the sequence $\{u_j\}$ is equi-bounded and from (2.2) which implies
\[ |u_j(t_1) - u_j(t_2)| \leq \left| \int_{t_1}^{t_2} |\dot{u}_j(t)| dt \right| \leq \sqrt{a}|t_1 - t_2|^{\frac{1}{2}}, \] (2.6)
so that the sequence is also equi-continuous.

By passing to a further subsequence we can also assume that $u_j \to u^*$ in $W^{1,2}((T_1, T_2); \mathbb{R}^n)$ for each $T_1, T_2$ with $T_u^\infty < T_1 < T_2 < T_u^\infty$. This follows from (2.2), which implies
\[ \frac{1}{2} \int_{T_u^{j-}}^{T_u^{j+}} |\dot{u}_j|^2 dt \leq \mathcal{A}(u_j, (T_u^{j-}, T_u^{j+})) \leq a, \]
and from the fact that each map $u_j$ satisfies (2.3) and therefore is bounded in $L^2((T_u^{j-}, T_u^{j+}); \mathbb{R}^n)$.

We also have
\[ \mathcal{A}(u^*, (T_u^\infty, T_u^\infty)) \leq a_0. \] (2.7)

Indeed, from the lower semicontinuity of the norm, for each $T_1, T_2$ with $T_u^\infty < T_1 < T_2 < T_u^\infty$ we have
\[ \int_{T_1}^{T_2} |\dot{u}_j|^2 dt \leq \liminf_{j \to +\infty} \int_{T_1}^{T_2} |\dot{u}_j|^2 dt. \]
This and the fact that \( u_j \) converges to \( u^* \) uniformly in \([T_1, T_2]\) imply
\[
A(u^*, (T_1, T_2)) \leq \liminf_{j \to +\infty} A(u_j, (T_1, T_2)) \leq \liminf_{j \to +\infty} A(u_j, (T_-^{u_j}, T_+^{u_j})) = a_0.
\]
Since this is valid for each \( T_-^\infty < T_1 < T_2 < T_+^\infty \) the claim (2.7) follows.

**Lemma 2.2.** Define \( T_-^\infty \leq T_- \leq -T_0 < T_0 \leq T_+ \leq T_+^\infty \) by setting
\[
T_- = \inf \{ t \in (T_-^\infty, 0] : u^*((t, 0]) \subset \Omega \} \\
T_+ = \sup \{ t \in (0, T_+^\infty) : u^*([0, t)) \subset \Omega \}.
\]
Then
\[
(i) \quad A(u^*, (T_-, T_+)) = a_0. \tag{2.8}
\]
\[
(ii) \ T_+ < +\infty \text{ implies } \lim_{t \to T_+} u^*(t) = x_+ \text{ for some } x_+ \in \Gamma_+ \text{ and } \Gamma_+ = \{ \Gamma_1, \ldots, \Gamma_N \} \setminus \{ \Gamma_- \}.
\]
\[
(iii) \ T_+ = +\infty \text{ implies } \lim_{t \to +\infty} d(u^*(t), \Gamma_+) = 0, \tag{2.9}
\]
for some \( \Gamma_+ \in \{ \Gamma_1, \ldots, \Gamma_N \} \setminus \{ \Gamma_- \} \).

Corresponding statements apply to \( T_- \).

**Proof.** We first prove (ii), (iii). If \( T_+ < +\infty \) the existence of \( \lim_{t \to T_+} u^*(t) \) follows from (2.6) which implies that \( u^* \) is a \( C^{0, \frac{1}{2}} \) map. The limit \( x_+ \) belongs to \( \partial \Omega \) and therefore to \( \Gamma_+ \) for some \( \Gamma_+ \in \{ \Gamma_1, \ldots, \Gamma_N \} \). Indeed, \( x_+ \not\in \partial \Omega \) would imply the existence of \( \tau > 0 \) such that, for \( j \) large enough,
\[
d(u_j([T_+, T_+ + \tau]), \partial \Omega) \geq \frac{1}{2} d(x_+, \partial \Omega),
\]
in contradiction with the definition of \( T_+ \). If \( T_+ = +\infty \) and (iii) does not hold there is \( \delta > 0 \) and a diverging sequence \( \{ t_j \} \) such that
\[
d(u^*(t_j), \partial \Omega) \geq \delta.
\]
Set \( U_m = \min_{(x, \partial \Omega) = \delta} U(x) > 0 \). From the uniform continuity of \( U \) in \( \{|x| \leq M\} \) (\( M \) as in (2.3)) it follows that there is \( l > 0 \) such that
\[
|U(x_1) - U(x_2)| \leq \frac{1}{2} U_m, \quad \text{for } |x_1 - x_2| \leq l, \ x_1, x_2 \in \{|x| \leq M\}.
\]
This and \( u^* \in C^{0, \frac{1}{2}} \) imply
\[
U(u^*(t)) \geq \frac{1}{2} U_m, \quad t \in I_j = \left( t_j - \frac{l^2}{a}, t_j + \frac{l^2}{a} \right),
\]
and, by passing to a subsequence, we can assume that the intervals \( I_j \) are disjoint. Therefore for each \( T > 0 \) we have
\[
\sum_{t_j \leq T} \frac{l^2 U_m}{a} \leq \int_0^T U(u^*(t)) dt \leq a_0,
\]
which is impossible for \( T \) large. This establishes (2.9) for some \( \Gamma_+ \in \{ \Gamma_1, \ldots, \Gamma_N \} \). It remains to show that \( \Gamma_+ \neq \Gamma_- \). This is a consequence of the minimizing character of \( \{ u_j \} \). Indeed, \( \Gamma_+ = \Gamma_- \) would imply the existence of a constant \( c > 0 \) such that \( \lim_{j \to +\infty} A(u_j, (T_-^{u_j}, T_+^{u_j})) \geq a_0 + c \).
Now we prove (i). $T_+ - T_- < +\infty$, implies that $u^*$ is an element of $U$ with $T_{\pm}^u = T_\pm$. It follows that $A(u^*, (T_-, T_+)) \geq a_0$, which together with (2.7) imply (2.8). Assume now $T_+ - T_- = +\infty$. If $T_+ = +\infty$, (2.9) implies that, given a small number $\epsilon > 0$, there are $t_\epsilon$ and $\bar{x}_\epsilon \in \partial\Omega$ such that $|u^*(t_\epsilon) - \bar{x}_\epsilon| = \epsilon$ and the segment joining $u^*(t_\epsilon)$ to $\bar{x}_\epsilon$ belongs to $\Omega$. Set

$$v_\epsilon(t) = (1 - (t - t_\epsilon))u^*(t_\epsilon) + (t - t_\epsilon)\bar{x}_\epsilon, \quad t \in (t_\epsilon, t_\epsilon + 1].$$

From the uniform continuity of $U$ there is $\eta_\epsilon > 0$, $\lim_{\epsilon \to 0} \eta_\epsilon = 0$, such that $U(v_\epsilon(t)) \leq \eta_\epsilon$, for $t \in [t_\epsilon, t_\epsilon + 1]$. Therefore we have

$$A(v_\epsilon, (t_\epsilon, t_\epsilon + 1)) \leq \frac{1}{2} \epsilon^2 + \eta_\epsilon.$$

If $T_+ > -\infty$ the map $u_\epsilon = \mathbb{1}_{[T_-, t_\epsilon]}u^* + \mathbb{1}_{(t_\epsilon, t_\epsilon + 1]}v_\epsilon$ belongs to $U$ and it results

$$a_0 \leq A(u_\epsilon, (T_-, t_\epsilon + 1)) = A(u^*, (T_-, t_\epsilon)) + A(v_\epsilon, (t_\epsilon, t_\epsilon + 1)) \leq A(u^*, (T_-, T_+)) + \frac{1}{2} \epsilon^2 + \eta_\epsilon.$$

Since this is valid for all small $\epsilon > 0$ we get

$$a_0 \leq A(u^*, (T_-, T_+)),$$

that together with (2.7) establishes (2.8) if $T_- > -\infty$ and $T_+ = +\infty$. The discussion of the other cases where $T_+ - T_- = +\infty$ is similar. \hfill \Box

We observe that there are cases with $T_+ < T_+^\infty$ and/or $T_- > T_-^\infty$, see Remark 2.

**Lemma 2.3.** The map $u^*$ satisfies (1.2) and (1.4) in $(T_-, T_+)$. 

**Proof.** 1. We first show that for each $T_1$, $T_2$ with $T_- < T_1 < T_2 < T_+$ we have

$$A(u^*, (T_1, T_2)) = \inf_{v \in \mathcal{V}} A(v, (T_1, T_2)), \tag{2.10}$$

where

$$\mathcal{V} = \{v \in W^{1,2}((T_1, T_2); \mathbb{R}^n) : v(T_i) = u^*(T_i), i = 1, 2; v([T_1, T_2]) \subset \Omega\}.$$

Suppose instead that there are $\eta > 0$ and $v \in \mathcal{V}$ such that

$$A(v, (T_1, T_2)) = A(u^*, (T_1, T_2)) - \eta.$$

Set $w_j : (T_-^{u_j}, T_+^{u_j}) \to \Omega$ defined by

$$w_j(t) = \begin{cases} 
  u_j(t), & t \in (T_-^{u_j}, T_1) \cup [T_2, T_+^{u_j}), \\
  v(t) + \frac{T_2 - t}{T_2 - T_1} \delta_{ij} + \frac{t - T_1}{T_2 - T_1} \delta_{2j}, & t \in (T_1, T_2),
\end{cases}$$

where $\delta_{ij} = u_j(T_i) - u^*(T_i)$, $i = 1, 2$, with $u_j$ as in (2.4). Define $v_j : [T_-^{u_j}, T_+^{u_j}] \to \mathbb{R}^n$ by

$$v_j(t) = w_j(t - \tau_j),$$

where $\tau_j$ is such that $U(v_j(0)) = U_0$, as in (2.1). Note that

$$A(v_j, (T_-^{u_j}, T_+^{u_j})) = A(w_j, (T_-^{u_j}, T_+^{u_j})). \tag{2.11}$$

From (2.5) we have $\lim_{j \to +\infty} \delta_{ij} = 0$, $i = 1, 2$, so that

$$\lim_{j \to +\infty} A(w_j, (T_1, T_2)) = A(v, (T_1, T_2)) = A(u^*, (T_1, T_2)) - \eta \leq \lim_{j \to +\infty} \inf A(u_j, (T_1, T_2)) - \eta.$$
Therefore we have

\[ \lim_{j \to +\infty} A(w_j, (T_+^u, T_+^u)) = \lim_{j \to +\infty} A(w_j, (T_1, T_2)) + \lim_{j \to +\infty} A(u_j, (T_+^u, T_1) \cup (T_2, T_+^u)) \]

\[ \leq \lim_{j \to +\infty} A(u_j, (T_1, T_2)) - \eta + \lim_{j \to +\infty} A(u_j, (T_+^u, T_1) \cup (T_2, T_+^u)) \leq a_0 - \eta, \]

that, given (2.11), is in contradiction with the minimizing character of the sequence \( \{u_j\} \).

The fact that \( u^* \) satisfies (1.2) follows from (2.10) and regularity theory; see [5]. To show that \( u^* \) satisfies (1.4) we distinguish the case \( T_+ - T_- < +\infty \) from the case \( T_+ - T_- = +\infty \).

2. \( T_+ - T_- < +\infty \). Given \( t_0, t_1 \) with \( T_- < t_0 < t_1 < T_+ \), let \( \phi : [t_0, t_1 + \tau] \to [t_0, t_1] \) be linear, with \( |\tau| \) small, and let \( \psi : [t_0, t_1] \to [t_0, t_1 + \tau] \) be the inverse of \( \phi \). Define \( u_\tau : [T_-, T_+ + \tau] \to \mathbb{R}^n \) by setting

\[
u_\tau(t) = \begin{cases} u^*(t), & t \in [T_-, t_0], \\ u^*(\phi(t)), & t \in [t_0, t_1 + \tau], \\ u^*(t - \tau), & t \in (t_1 + \tau, T_+ + \tau) \end{cases}
\]

(2.12)

Note that \( u_\tau \in \mathcal{U} \) with \( T_+^{u_\tau} = T_- \) and \( T_-^{u_\tau} = T_+ + \tau \). Since \( u^* \) is a minimizer we have

\[
\frac{d}{d\tau} A(u_\tau, (T_-^{u_\tau}, T_+^{u_\tau}))|_{\tau = 0} = 0.
\]

(2.13)

From (2.12), using also the change of variables \( t = \psi(s) \), it follows

\[
\begin{align*}
A(u_\tau, (T_-^{u_\tau}, T_+^{u_\tau})) - A(u^*, (T_-^u, T_+^u)) &= \int_{t_0}^{t_1 + \tau} \left( \frac{\phi^2(s)}{2} |\dot{u}^*(\phi(t))|^2 + U(u^*(\phi(t))) \right) dt - \int_{t_0}^{t_1} \left( \frac{1}{2} |\dot{u}^*(t)|^2 + U(u^*(t)) \right) dt \\
&= \int_{t_0}^{t_1} \left( \frac{1}{2} |\dot{u}^*(t)|^2 + (\dot{\phi}(t) - 1)U(u^*(t)) \right) dt \\
&= \int_{t_0}^{t_1} \left( \frac{\frac{1}{2} \tau}{t_1 - t_0} |\dot{u}^*(t)|^2 + \frac{\tau}{t_1 - t_0} U(u^*(t)) \right) dt \\
&= - \frac{\tau}{t_1 - t_0} \int_{t_0}^{t_1} \left( \frac{1}{2} |\dot{u}^*(t)|^2 - U(u^*(t)) \right) dt.
\end{align*}
\]

This and (2.13) imply

\[
\int_{t_0}^{t_1} \left( \frac{1}{2} |\dot{u}^*(t)|^2 - U(u^*(t)) \right) dt = 0.
\]

(2.14)

Since this holds for all \( t_0, t_1 \), with \( T_- < t_0 < t_1 < T_+ \), then (1.4) follows.

3. \( T_+ - T_- = +\infty \). We only consider the case \( T_+ = +\infty \). The discussion of the other cases is similar.

Let \( T \in (T_-, +\infty) \), let \( T_- < t_0 < t_1 < T \) and let \( \phi : [t_0, T] \to [t_0, T] \) be linear in the intervals \([t_0, t_1 + \tau], [t_1 + \tau, T] \), with \( |\tau| \) small, and such that \( \phi([t_0, t_1 + \tau]) = [t_0, t_1] \). Define \( u_\tau : (T_-, +\infty) \to \mathbb{R}^n \) by setting

\[
u_\tau(t) = \begin{cases} u^*(t), & t \in (T_-, t_0) \cup [T, +\infty) \\ u^*(\phi(t)), & t \in [t_0, T] \end{cases}
\]

We have

\[
\begin{align*}
A(u_\tau, (T_-^u, T_+^u)) - A(u^*, (T_-^u, T_+^u)) &= \int_{t_0}^{t_1} \left( \frac{\tau}{t_1 - t_0} |\dot{u}^*(t)|^2 + \frac{\tau}{t_1 - t_0} U(u^*(t)) \right) dt + \int_{t_1}^{T} \left( \frac{\tau}{T - t_1} |\dot{u}^*(t)|^2 - \frac{\tau}{T - t_1} U(u^*(t)) \right) dt.
\end{align*}
\]
Since \( u^* \) restricted to the interval \([t_0, T]\) is a minimizer of (2.10), by differentiating with respect to \( \tau \) and setting \( \tau = 0 \) we obtain
\[
-\frac{1}{t_1-t_0} \int_{t_0}^{t_1} \left( \frac{1}{2} |\dot{u}^*(t)|^2 - U(u^*(t)) \right) dt + \frac{1}{T-t_1} \int_{t_1}^{T} \left( \frac{1}{2} |\dot{u}^*(t)|^2 - U(u^*(t)) \right) dt = 0.
\]
From (2.7) it follows that the second term in this expression converges to zero when \( T \to +\infty \).
Therefore, after taking the limit for \( T \to +\infty \), we get back to (2.14) and, as before, we conclude that (1.4) holds. \( \square \)

**Lemma 2.4.** Assume that \( \lim_{t \to T_+} u^*(t) = p \in P \). Then
\[ T_+ = +\infty. \]

**Proof.** Since \( U \) is of class \( C^2 \) and \( p \) is a critical point of \( U \) there are constants \( c > 0 \) and \( \rho > 0 \) such that
\[ U(x) \leq c|x-p|^2, \quad x \in B_\rho(p) \cap \Omega. \]
Fix \( t_\rho \) so that \( u^*(t) \in B_\rho(p) \cap \Omega \) for \( t \geq t_\rho \). Then \( T_+ = +\infty \) follows from (1.4) and
\[
\frac{d}{dt} |u^* - p| \geq -|\dot{u}^*| = -\sqrt{2U(u^*)} \geq -\sqrt{2c} |u^* - p|, \quad t \geq t_\rho.
\]
\( \square \)

We now show that if \( \Gamma_+ \) has positive diameter then \( T_+ < +\infty \). To prove this we first show that \( T_+ = +\infty \) implies \( u^*(t) \to p \in P \) as \( t \to +\infty \), then we conclude that this is in contrast with (2.8).

**Lemma 2.5.** If \( T_+ = +\infty \), then there is \( p \in P \) such that
\[ \lim_{t \to +\infty} u^*(t) = p. \] (2.15)

An analogous statement applies to \( T_- \).

**Proof.** If \( \Gamma_+ = \{p\} \) for some \( p \in P \), then (2.15) follows by (2.9). Therefore we assume that \( \Gamma_+ \) has positive diameter. The idea of the proof is to show that if \( u^*(t) \) gets too close to \( \partial \Gamma_+ \setminus P \) it is forced to end up on \( \Gamma_+ \setminus P \) in a finite time in contradiction with \( T_+ = +\infty \).
If (2.15) does not hold there is \( q > 0 \) and a sequence \( \{\tau_j\} \), with \( \lim_{j \to \infty} \tau_j = +\infty \), such that \( d(u^*(\tau_j), P) \geq q \), for all \( j \in \mathbb{N} \). Since, by (2.3) \( u^* \) is bounded, using also (2.9), we can assume that
\[ \lim_{j \to +\infty} u^*(\tau_j) = \bar{x}, \quad \text{for some } \bar{x} \in \Gamma_+ \setminus \bigcup_{p \in P} B_{\bar{r}}(p). \] (2.16)

The smoothness of \( U \) implies that there are positive constants \( \bar{r}, r, c \) and \( C \) such that
(i) the orthogonal projection on \( \pi : B_r(\bar{x}) \to \partial \Omega \) is well defined and \( \pi(B_r(\bar{x})) \subset \partial \Omega \setminus P \);
(ii) we have
\[ B_{\bar{r}}(x_0) \subset B_r(\bar{x}), \quad \text{for all } x_0 \in \partial \Omega \cap B_{\bar{r}}(\bar{x}); \]
(iii) if \( (\xi, s) \in \mathbb{R}^{n-1} \times \mathbb{R} \) are local coordinates with respect to a basis \( \{e_1, \ldots, e_n\} \), \( e_j = e_j(x_0) \), with \( e_n(x_0) \) the unit interior normal to \( \partial \Omega \) at \( x_0 \in \partial \Omega \cap B_{\bar{r}}(\bar{x}) \) it results
\[
\frac{1}{2}cs \leq U(x(x_0, (\xi, s))) \leq 2cs, \quad |\xi|^2 + s^2 \leq r^2, \quad s \geq h(x_0, \xi),
\] (2.17)
where
\[
x = x(x_0, (\xi, s)) = x_0 + \sum_{j=1}^{n} \xi_j e_j(x_0) + se_n(x_0),
\]
and \( h : \partial \Omega \cap B_{\bar{r}}(\bar{x}) \times \{|\xi| \leq r \} \to \mathbb{R}, \quad |h(x_0, \xi)| \leq C|\xi|^2, \) for \( |\xi| \leq r \), is a local representation of \( \partial \Omega \) in a neighborhood of \( x_0 \), that is \( U(x(x_0, (\xi, h(x_0, \xi)))) = 0 \) for \( |\xi| \leq r \).
Fix a value $j_0$ of $j$ and set $t_0 = \tau_{j_0}$. If $j_0$ is sufficiently large, setting $t_0 = \tau_{j_0}$ we have that $x_0 = \pi(u^*(t_0))$ is well defined. Moreover $x_0 \in \partial \Omega \cap B_r(\bar{x})$ and

$$u^*(t_0) = x_0 + \delta e_n(x_0), \quad \delta = |u^*(t_0) - x_0|.$$ 

For $k = \frac{8}{3} \sqrt{2}$ let $Q_0$ be the set

$$Q_0 = \{x(x_0,(\xi,s)) : |\xi|^2 + (s-\delta)^2 < k^2 \delta^2, \ s > \delta/2\}.$$ 

Since $\delta \to 0$ as $j_0 \to +\infty$ we can assume that $\delta > 0$ is so small (\(\delta < \min\{\frac{1}{2\sqrt{\gamma}}, \frac{r}{1+k}\}\) suffices) that $\overline{Q_0} \subset \Omega \cap B_r(x_0)$.

**Claim 1.** $u^*(t)$ leaves $\overline{Q_0}$ through the disc $D_0 = \partial Q_0 \setminus \partial B_{k\delta}(u^*(t_0))$.

From (2.4) we have $a_0 \leq \mathcal{A}(v,(T_-,T^+_v))$ for each $W^{1,2}$ map $v: (T_-,T^+_v) \to \mathbb{R}^n$ that coincides with $u^*$ for $t \leq t_0$, and satisfies $v((t_0,T^+_v)) \subset \Omega$, $v(T^+_v) \in \partial \Omega$ and (1.4). Therefore if we set

$$w(s) = x_0 + se_n(x_0),$$

$s \in [0,\delta]$, we have

$$a_0 \leq \mathcal{A}(u^*,(T_-,t_0)) + \mathcal{J}(w,(0,\delta)).$$

(2.18)

On the other hand, if $u^*(t'_0) \in \partial Q_0(x_0) \cap \partial B_{k\delta}(u^*(t_0))$, where

$$t'_0 = \sup\{t > t_0 : u^*(t,t) \subset \overline{Q_0} \setminus \partial B_{k\delta}(u^*(t_0))\},$$

from (2.7) it follows

$$\mathcal{A}(u^*,(T_-,t_0)) + \mathcal{J}(u^*,(t_0,t'_0)) \leq a_0.$$ 

(2.19)

Using (2.17) we obtain

$$\mathcal{J}(w,(0,\delta)) \leq \frac{4}{3} c^2 \delta^2,$$

(2.20)

and, since

$$c^2 \delta^2 \leq U(x(x_0,(\xi,s))), \quad (\xi,s) \in \overline{Q_0}(x_0),$$
we also have, with $k$ defined above,

$$
\frac{8}{3} c^2 \delta^2 = \frac{k}{\sqrt{2}} c^2 \delta^2 \leq \frac{c^2 \delta^2}{\sqrt{2}} \int_{t_0}^{t_1} |\dot{u}^*(t)| dt \leq \sqrt{2} \int_{t_0}^{t_1} \sqrt{U(u^*(t))} |\dot{u}^*(t)| dt.
$$

(2.21)

From (2.20) and (2.21) it follows

$$
\mathcal{J}(u, (0, \delta)) \leq \frac{1}{2} \mathcal{J}(u^*, (t_0, t'_0)),
$$

and therefore (2.18) and (2.19) imply the absurd inequality $a_0 < a_0$. This contradiction proves the claim.

From Claim 1 it follows that there is $t_1 \in (t_0, +\infty)$ with the following properties:

$$
u^*([t_0, t_1)) \subset Q_0(x_0),
\nu(t_1) \in D_0.
$$

Set $x_{0,1} = \pi(u^*(t_1))$ and $\delta_1 = |u^*(t_1) - x_{0,1}|$. Since $h(x_0, 0) = h_\xi(x_0, 0) = 0$ and the radius $\rho_0 = (k^2 - \frac{1}{4})^{1/2} \delta$ of $D_0$ is proportional to $\delta$, we can assume that $\delta$ is so small that the ratio $\frac{2\delta_1}{\delta}$ and $\frac{|x_{0,1} - x_0|}{|u^*(t_1) - x(x_0, \pi/2)|}$ are near 1 so that we have

$$
\delta_1 \leq \rho \delta, \text{ for some } \rho < 1,
|x_{0,1} - x_0| \leq \kappa \delta.
$$

We also have

$$
t_1 - t_0 \leq k' \delta^{1/2}, \quad k' = \frac{8k}{c^2}.
$$

This follows from

$$(t_1 - t_0)\frac{c}{4} \delta \leq A(u^*(t_0, t_1)) = \mathcal{J}(u^*, (t_0, t_1))
= \sqrt{2} \int_{t_0}^{t_1} \sqrt{U(u^*(t))} |\dot{u}^*(t)| dt \leq 2\sqrt{c} |u^*(t_1) - u^*(t_0)| \leq 2\sqrt{\kappa} k \delta^{1/2}.
$$

where we used (2.17) to estimate $\mathcal{J}$ on the segment joining $u^*(t_0)$ with $u^*(t_1)$.

We have $u^*(t_1) = x_{0,1} + \delta_1 e_n(x_{0,1})$ and we can apply Claim 1 to deduce that there exists $t_2 > t_1$ such that

$$
u^*([t_1, t_2)) \subset Q_1(x_{0,1}),
\nu^*(t_2) \in D_1,
$$

where $Q_1$ and $D_1$ are defined as $Q_0$ and $D_0$ with $\delta_1$ and $\pi(x_{0,1}, (\xi, s))$ instead of $\delta$ and $\pi(x_0, (\xi, s))$. Therefore an induction argument yields sequences $\{t_j\}$, $\{x_{0,j}\}$, $\{\delta_j\}$ and $\{Q_j(x_{0,j})\}$ such that

$$
u^*(t_j) = x_{0,j} + \delta_j e_n(x_{0,j}) \subset Q_j(x_{0,j}), \quad x_{0,j} = \pi(u^*(t_j)),
\delta_{j+1} \leq \rho \delta_j \leq \rho \delta^{1/2} \delta,
|x_{0,j+1} - x_{0,j}| \leq k \delta_j \leq k \delta^{j+1} \delta,
(t_{j+1} - t_j) \leq k \delta^{j+1/2} \leq \kappa \delta_j^{1/2} \delta^{1/2},
u^*(t_j) = x_{0,j} + \delta_j e_n(x_{0,j}) \in D_j.
$$

(2.22)

We can also assume that $Q_j(x_{0,j}) \subset \Omega \cap B_r(x_0)$, for all $j \in \mathbb{N}$. This follows from $|u^*(t_{j+1}) - u^*(t_j)| \leq k \delta_j \leq k \rho \delta$.
From (2.22) we obtain that there exists $T$ with $t_0 < T \leq \frac{k\delta^2}{1-\rho^2}$ such that
\[
 u^*(T) = \lim_{t \to T} u^*(t) = \lim_{j \to +\infty} x_{0,j} \in \partial \Omega \setminus P,
\]
\[
 |u^*(T) - x_0| \leq \frac{k\delta}{1-\rho}.
\]
This contradicts the existence of the sequence $\{\tau_j\}$, with $\lim_{j \to \infty} \tau_j = +\infty$, appearing in (2.16) and establishes (2.15). The proof of the lemma is complete.

We continue by showing (2.15) contradicts (2.8).

**Lemma 2.6.** Assume that $\Gamma_+$ has positive diameter. Then
\[
 T_+ < +\infty.
\]
An analogous statement applies to $\Gamma_-$ and $T_-$. 

**Proof.** From Lemma 2.5, if $T_+ = +\infty$ there exists $p \in P$ such that $\lim_{t \to +\infty} u^*(t) = p$. We use a local argument to show that this is impossible if $\Gamma_+$ has positive diameter. By a suitable change of variable we can assume that $p = 0$ and that, in a neighborhood of $0 \in \mathbb{R}^n$, $U$ reads
\[
 U(u) = V(u) + W(u),
\]
where $V$ is the quadratic part of $U$:
\[
 V(u) = \frac{1}{2} \left( -\sum_{i=1}^{m} \lambda_i^2 u_i^2 + \sum_{i=m+1}^{n} \lambda_i^2 u_i^2 \right), \quad \lambda_i > 0 \quad (2.23)
\]
and $W$ satisfies,
\[
 |W(u)| \leq C|u|^3, \quad |W_x(u)| \leq C|u|^2, \quad |W_{xx}(u)| \leq C|u|. \quad (2.24)
\]
Consider the Hamiltonian system with
\[
 H(p, q) = \frac{1}{2} |p|^2 - U(q), \quad p \in \mathbb{R}^n, \quad q \in \Omega \subset \mathbb{R}^n.
\]
For this system the origin of $\mathbb{R}^{2n}$ is an equilibrium point that corresponds to the critical point $p = 0$ of $U$. Set $D = \text{diag}(-\lambda_1^2, \ldots, -\lambda_m^2, \lambda_{m+1}^2, \ldots, \lambda_n^2)$. The eigenvalues of the symplectic matrix
\[
 \begin{pmatrix}
 0 & D \\
 I & 0
 \end{pmatrix}
\]
are
\[
 -\lambda_i, \quad i = m + 1, \ldots, n \\
 \lambda_i, \quad i = m + 1, \ldots, n \\
 \pm i\lambda_i, \quad i = 1, \ldots, m.
\]
Let $(e_1, 0), \ldots, (e_n, 0), (0, e_1), \ldots, (0, e_n)$ be the basis of $\mathbb{R}^{2n}$ defined by $e_j = (\delta_{j1}, \ldots, \delta_{jn})$, where $\delta_{ji}$ is Kronecker’s delta. The stable $S^s$, unstable $S^u$ and center $S^c$ subspaces invariant under the flow of the linearized Hamiltonian system at $0 \in \mathbb{R}^{2n}$ are
\[
 S^s = \text{span}\{(-\lambda_j e_j, e_j)\}_{j=m+1}^{n}, \\
 S^u = \text{span}\{(\lambda_j e_j, e_j)\}_{j=m+1}^{n}, \\
 S^c = \text{span}\{(e_j, 0), (0, e_j)\}_{j=1}^{m}.
\]
From (2.15) and (1.4) we have
\[
\lim_{t \to +\infty} (\dot{u}^*(t), u^*(t)) = 0 \in \mathbb{R}^{2n}.
\]
Let \( W^s \) and \( W^u \) be the local stable and unstable manifold and let \( W^c \) be a local center manifold at \( 0 \in \mathbb{R}^{2n} \). From the center manifold theorem [4], [10], there is a constant \( \lambda_0 > 0 \) such that, for each solution \((p(t), q(t))\) that remains in a neighborhood of \( 0 \in \mathbb{R}^{2n} \) for positive time, there is a solution \((p^*(t), q^*(t)) \in W^c \) that satisfies
\[
|\((p(t), q(t)) - (p^*(t), q^*(t))\)| = O(e^{-\lambda_0 t}).
\]
(2.25)
Since \( W^c \) is tangent to \( S^c \) at \( 0 \in \mathbb{R}^{2n} \), the projection \( W^c_0 \) on the configuration space is tangent to \( S^c_0 = \text{span}\{e_j\}_{j=1}^m \), which is the projection of \( S^c \) on the configuration space. Therefore, if \((p^*, q^*) \neq 0\), given \( \gamma > 0 \), by (2.25) there is \( t_0 \) such that \( d(q(t), S^c_0) \leq \gamma |q(t)| \), for \( t \geq t_0 \). For \( \gamma \) small, this implies that \( q(t) \notin \Omega \) for \( t \geq t_0 \). It follows that \((p^*, q^*) \equiv 0\) and from (2.25) \((p(t), q(t))\) converges to zero exponentially. This is possible only if \((p(t), q(t)) \in W^s\) and, in turn, only if \( q(t) \in W^s_0 \), the projection of \( W^s \) on the configuration space. This argument leads to the conclusion that the trajectory of \( u^* \) in a neighborhood of \( 0 \) is of the form
\[
u^*(t(s)) = u^*(s) = s \eta + z(s),
\]
(2.26)
where
\[
\eta = \sum_{i=m+1}^{n} \eta e_i
\]
is a unit vector\(^1\), \( s \in [0, s_0) \) for some \( s_0 > 0 \), and \( z(s) \) satisfies
\[
z(s) \cdot \eta = 0, \quad |z(s)| \leq c |s|^2, \quad |z'(s)| \leq c |s|
\]
(2.27)
for a positive constant \( c \).

We are now in the position of constructing our local perturbation of \( u \). We first discuss the case \( U = V, z(s) = 0 \). We set
\[
u(s) = s \eta
\]
and, in some interval \([1, s_1]\), construct a competing map \( \bar{v} : [1, s_1] \to \mathbb{R}^n \),
\[
\bar{v} = \bar{u} + g e_1, \quad g : [1, s_1] \to \mathbb{R},
\]
with the following properties:
\[
V(\bar{v}(1)) = 0,
\]
\[
\bar{v}(s_1) = \bar{u}(s_1),
\]
\[
\mathcal{J}_V(\bar{v}, [1, s_1]) < \mathcal{J}_V(\bar{u}, [0, s_1]).
\]
(2.28)
The basic observation is that, if we move from \( \bar{u} \) in the direction of one of the eigenvectors \( e_1, \ldots, e_m \) corresponding to negative eigenvalues of the Hessian of \( V \), the potential \( V \) decreases and therefore, for each \( s_0 \in (1, s_1) \) we can define the function \( g \) in the interval \([1, s_0]\) so that
\[
\mathcal{J}_V(\bar{u} + g e_1, (1, s_0)) = \mathcal{J}_V(\bar{u}, (1, s_0)).
\]
(2.29)
Indeed it suffices to impose that \( g : (1, s_0] \to \mathbb{R} \) satisfies the condition
\[
\sqrt{V(\bar{u}(s))} = \sqrt{1 + g^2(s) \sqrt{V(\bar{u}(s) + g(s)e_1)}}, \quad s \in (1, s_0].
\]
\(^1\)Actually \( \eta \) coincides with one of the eigenvectors of \( U''(0) \).

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According with this condition we take \( g \) as the solution of the problem

\[
\begin{align*}
  g' &= -\frac{\lambda_1 g}{\sqrt{s^2 \lambda^2_\eta - \lambda_1^2 g^2}} = -\frac{\lambda_1 g}{\sqrt{1 - \frac{\lambda_1^2 g^2}{s^2 \lambda^2_\eta}}}, \\
  g(1) &= \frac{\lambda_\eta}{\chi_1}
\end{align*}
\]  

(2.30)

where we have used (2.23) and set

\[ \lambda_\eta = \sqrt{\sum_{i=m+1}^{n} \frac{\eta_i^2}{\chi_i^2}}. \]

Note that the initial condition in (2.30) implies \( V(\bar{v}(1)) = 0 \). The solution \( g \) of (2.30) is well defined in spite of the fact that the right hand side tends to \(-\infty\) as \( s \to 1 \). Since \( g \) defined by (2.30) is positive for \( s \in [1, +\infty) \), to satisfy the condition \( \bar{v}(s_1) = \bar{u}(s_1) \), we give a suitable definition of \( g \) in the interval \([s_0, s_1]\) in order that \( g(s_1) = 0 \). Choose a number \( \alpha \in (0, 1) \) and extend \( g \) with continuity to the interval \([s_0, s_1]\) by imposing that

\[
\sqrt{V(\bar{u}(s))} = \alpha \sqrt{1 + g'^2(s)} \sqrt{V(\bar{u}(s) + g(s)e_1)}, \quad s \in (s_0, s_1].
\]

(2.31)

Therefore, in the interval \((s_0, s_1]\), we define \( g \) by

\[
g' = -\frac{1}{\alpha} \sqrt{\frac{1 - \alpha^2 + \alpha^2 \frac{\lambda_1^2 g^2}{s^2 \lambda^2_\eta}}{1 - \frac{\lambda_1^2 g^2}{s^2 \lambda^2_\eta}}} \leq -\frac{\sqrt{1 - \alpha^2}}{\alpha}.
\]

(2.32)

Since (2.31) implies

\[ \mathcal{J}_V(\bar{v}, [s_0, s_1]) = \frac{1}{\alpha} \mathcal{J}_V(\bar{u}, [s_0, s_1]), \]

from (2.29) we see that \( \bar{v} \) satisfies also the requirement (2.28) above if we can choose \( \alpha \in (0, 1) \) and \( 1 < s_0 < s_1 \) in such a way that

\[ \mathcal{J}_V(\bar{u}, (0, 1)) > \frac{1 - \alpha}{\alpha} \mathcal{J}_V(\bar{u}, (s_0, s_1)). \]

Since (2.32) implies \( s_1 < s_0 + \frac{\alpha g(s_0)}{\sqrt{1 - \alpha^2}} \) a sufficient condition for this is

\[
\mathcal{J}_V(\bar{u}, (0, 1)) > \frac{1 - \alpha}{\alpha} \mathcal{J}_V(\bar{u}, (s_0, s_0 + \frac{\alpha g(s_0)}{\sqrt{1 - \alpha^2}})).
\]
or equivalently
\[ 1 > \frac{1 - \alpha}{\alpha} \left( \left( s_0 + \frac{\alpha g(s_0)}{\sqrt{1 - \alpha^2}} \right)^2 - s_0^2 \right) = 2s_0 g(s_0) \sqrt{\frac{1 - \alpha}{1 + \alpha} + \frac{\alpha g^2(s_0)}{1 + \alpha}}. \] (2.33)

By a proper choice of \( s_0 \) and \( \alpha \) the right hand side of (2.33) can be made as small as we like. For instance we can fix \( s_0 \) so that \( g(s_0) \leq \frac{1}{4} \) and then choose \( \alpha \) in such a way that \( \frac{1}{2}s_0 \sqrt{\frac{1 - \alpha}{1 + \alpha}} \leq \frac{1}{4} \) and conclude that (2.28) holds.

Next we use the function \( g \) to define a comparison map \( v \) that coincides with \( u^* \) outside an \( \epsilon \)-neighborhood of 0 and show that the assumption that the trajectory of \( u^* \) ends up in some \( P \) must be rejected. For small \( \epsilon > 0 \) we define
\[ v(\epsilon s) = \epsilon s \eta + z(\epsilon s) + \epsilon g(s - \sigma) e_1, \quad s \in [1 + \sigma, s_1 + \sigma], \] (2.34)
where \( \sigma = \sigma(\epsilon) \) is determined by the condition
\[ U(v(\epsilon(1 + \sigma))) = 0, \]
which, using (2.23), (2.24), (2.27) and \( g(1) = \frac{\lambda_n}{\lambda_1} \), after dividing by \( \epsilon^2 \), becomes
\[ \frac{1}{2} \epsilon^2 ((1 + \sigma)^2 - 1) = \epsilon f(\sigma, \epsilon), \] (2.35)
where \( f(\sigma, \epsilon) \) is a smooth bounded function defined in a neighborhood of \((0, 0)\). For small \( \epsilon > 0 \), there is a unique solution \( \sigma(\epsilon) = O(\epsilon) \) of (2.35). Note also that (2.34) implies that
\[ v(\epsilon(s_1 + \sigma)) = u^*(\epsilon(s_1 + \sigma)). \]

We now conclude by showing that, for \( \epsilon > 0 \) small, it results
\[ \mathcal{J}_U(u^*(\epsilon), (0, s_1 + \sigma)) > \mathcal{J}_U(v(\epsilon), (1 + \sigma, s_1 + \sigma)). \] (2.36)
From (2.26) and (2.34) we have
\[ \lim_{\epsilon \to 0^+} \epsilon^{-1} \left| \frac{d}{ds} u^*(\epsilon s) \right| = 1, \quad \lim_{\epsilon \to 0^+} \epsilon^{-1} \left| \frac{d}{ds} v(\epsilon s) \right| = \sqrt{1 + g'^{2}(s)}, \] (2.37)
and, using also (2.24) and \( \sigma = O(\epsilon) \),
\[ \lim_{\epsilon \to 0^+} \epsilon^{-2} U(u^*(\epsilon s)) = V(\bar{u}(s)), \quad s \in (0, s_1), \]
\[ \lim_{\epsilon \to 0^+} \epsilon^{-2} U(v(\epsilon s)) = V(\bar{v}(s)), \quad s \in (1, s_1) \] (2.38)
uniformly in compact intervals.

The limits (2.37) and (2.38) imply
\[ \lim_{\epsilon \to 0^+} \epsilon^{-2} \mathcal{J}_U(u^*(\epsilon), (0, s_1 + \sigma)) = \lim_{\epsilon \to 0^+} \sqrt{2} \int_{0}^{s_1 + \sigma} \sqrt{\epsilon^{-2} U(u^*(\epsilon s))} \epsilon^{-1} \left| \frac{d}{ds} u^*(\epsilon s) \right| ds, \]
\[ = \sqrt{2} \int_{0}^{s_1} \sqrt{V(\bar{u}(s))} ds = \mathcal{J}_U(\bar{u}, (0, s_1)) \]
\[ \lim_{\epsilon \to 0^+} \epsilon^{-2} \mathcal{J}_U(v(\epsilon), (1 + \sigma, s_1 + \sigma)) = \lim_{\epsilon \to 0^+} \sqrt{2} \int_{1 + \sigma}^{s_1 + \sigma} \sqrt{\epsilon^{-2} U(v(\epsilon s))} \epsilon^{-1} \left| \frac{d}{ds} v(\epsilon s) \right| ds, \]
\[ = \sqrt{2} \int_{1}^{s_1} \sqrt{V(\bar{v}(s))} \sqrt{1 + g'^{2}(s)} ds = \mathcal{J}_U(\bar{v}, (1, s_1)). \]

This and (iii) above imply that, indeed, the inequality (2.36) holds for small \( \epsilon > 0 \). The proof is complete. \( \square \)
We can now complete the proof of Theorem 1.1. We show that the map $u^* : (T_-, T_+) \to \mathbb{R}^n$ possesses all the required properties. The fact that $u^*$ satisfies (1.2) and (1.4) follows from Lemma 2.3. Lemma 2.2 implies (1.5) and, if $T_- \to -\infty$, also (1.6). The fact that $x_- \in \Gamma_+ \setminus P$ is a consequence of Lemma 2.4 and implies that $\Gamma_-$ has positive diameter. Viceversa, if $\Gamma_-$ has positive diameter, Lemmas 2.5 and 2.6 imply that $T_- \to -\infty$ and that (1.6) holds for some $x_- \in \Gamma_\pm \setminus P$. The proof of Theorem 1.1 is complete.

Remark. From Theorem 1.1 it follows that if $N$ is even then there are at least $N/2$ distinct orbits connecting different elements of $\{\Gamma_1, \ldots, \Gamma_N\}$. If $N$ is odd there are at least $(N + 1)/2$. Simple examples show that, given distinct $\Gamma_i, \Gamma_j \in \{\Gamma_1, \ldots, \Gamma_N\}$, an orbit connecting them does not always exist. Let

$$U_{ij} = \{u \in W^{1,2}((T_-^u, T_+^u); \mathbb{R}^n) : u((T_-^u, T_+^u)) \subset \Omega, u(T^-_u) \in \Gamma_i, u(T^+_u) \in \Gamma_j\}$$

with $i \neq j$ and

$$d_{ij} = \inf_{u \in U_{ij}} \mathcal{A}(u, (T_-^u, T_+^u)).$$

An orbit connecting $\Gamma_i$ and $\Gamma_j$ exists if

$$d_{ij} < d_{ik} + d_{kj}, \quad \forall k \neq i, j.$$

The proof of Theorem 1.2 uses, with obvious modifications, the same arguments as in the proof of Theorem 1.1 to characterize $u^*$ as the limit of a minimizing sequence $\{u_j\}$ of the action functional

$$\mathcal{A}(u, (0, T_u)) = \int_0^{T_u} \left(\frac{1}{2} |u(t)|^2 + U(u(t))\right) dt.$$  \hspace{1cm} (2.39)

in the set

$$U = \{u \in W^{1,2}((0, T_u); \mathbb{R}^n) : 0 < T_u^< + \infty, u(0) = 0, u([0, T_u^>]) \subset \Omega, u(T_u^<) \in \partial \Omega\}.  \hspace{1cm} (2.40)$$

Remark. In the symmetric case of Theorem 1.2 it is easy to construct an example with $T_+ < T_+^\infty$. For $U(x) = 1 - |x|^2$, $x \in \mathbb{R}^2$, the solution $u : [0, \pi/2] \to \mathbb{R}^2$ of (1.2) determined by (1.4) and $u([0, \pi/2]) = \{(s, 0) : s \in [0, 1]\}$ is a minimizer of $\mathcal{A}$ in $U$. For $\epsilon$ small, let $t_\epsilon = \arcsin(1 - \epsilon)$ and define $u_\epsilon : [0, T_u^\infty] \to \mathbb{R}^2$ as the map determined by (1.4), $u_\epsilon([0, t_\epsilon]) = \{(s, 0) : s \in [0, 1 - \epsilon]\}$ and $u_\epsilon((t_\epsilon, T_u^\infty)) = \{(1 - \epsilon, s) : s \in (0, \sqrt{2\epsilon - \epsilon^2}]\}$. In this case $T_+ = \pi/2$ and $T_\infty^\infty = 3\pi/4$.

2.1 On the existence of heteroclinic connections

Corollary 1.3 states the existence of heteroclinic connections under the assumptions of Theorem 1.1 and, in particular, that $U \in C^2$. Actually, by examining the proof of Theorem 1.1 we can establish an existence result under weaker hypotheses. In the special case $\partial \Omega = P$, $\# P \geq 2$, given $p_- \in P$, the set $\tilde{U}$ defined in (2.1) takes the form

$$\tilde{U} = \{u \in W^{1,2}((T_-^u, T_+^u); \mathbb{R}^n) : -\infty < T_-^u < T_+^u < +\infty, u((T_-^u, T_+^u)) \subset \Omega, u(u(0)) = U_0, \lim_{t \to T_-^u} u(t) = p_-, u(T_+^u) \in P \setminus \{p_-\}\}.$$

In this section we slightly enlarge the set $\tilde{U}$ by allowing $T_\pm^u = \pm \infty$ and consider the admissible set

$$\tilde{U} = \{u \in W^{1,2}_{loc}((T_-^u, T_+^u); \mathbb{R}^n) : -\infty \leq T_-^u < T_+^u \leq +\infty, u((T_-^u, T_+^u)) \subset \Omega, U(u(0)) = U_0, \lim_{t \to T_-^u(t)} u(t) = p_-\} \subset P \setminus \{p_-\}.$$
Proposition 2.7. Assume that $U$ is a non-negative continuous function, which vanishes in a finite set $P$, $\#P \geq 2$, and satisfies
\[
\sqrt{U(x)} \geq \sigma(|x|), \quad x \in \Omega, \quad |x| \geq r_0
\]
for some $r_0 > 0$ and a non-negative function $\sigma : [r_0, +\infty) \to \mathbb{R}$ such that $\int_{r_0}^{+\infty} \sigma(r) dr = +\infty$.

Given $p_- \in P$ there is $p_+ \in P \setminus \{p_-\}$ and a Lipschitz-continuous map $u^* : (T_-, T_+) \to \Omega$ that satisfies (1.4) almost everywhere on $(T_-, T_+)$,
\[
\lim_{t \to T_{\pm}} u^*(t) = p_{\pm},
\]
and minimizes the action functional $A$ on $\tilde{U}$.

Proof. We begin by showing that
\[
a_0 = \inf_{u \in \mathcal{U}} A = \inf_{u \in \mathcal{U}} \tilde{a}_0. \tag{2.41}
\]
Since $\mathcal{U} \subset \tilde{\mathcal{U}}$ we have $a_0 \geq \tilde{a}_0$. On the other hand arguing as in the proof of Lemma 2.2, if $T_+ - T_- = +\infty$, given a small number $\epsilon > 0$, we can construct a map $u_\epsilon \in \mathcal{U}$ that satisfies
\[
a_0 \leq A(u_\epsilon, (T_-^u, T_+^u)) \leq A(u_\epsilon, (T_-^u, T_+^u)) \leq \tilde{a}_0 \tag{2.8}
\]
where $\eta_\epsilon \to 0$ as $\epsilon \to 0$. This implies $a_0 \leq \tilde{a}_0$ and establishes (2.41). It follows that we can proceed as in the proof of Theorem 1.1 and define $u^* \in \mathcal{U}$ as the limit of a minimizing sequence $\{u_j\} \subset \mathcal{U}$. The arguments in the proof of Lemma 2.2 show that (2.8) holds. It remain to show that $u^*$ is Lipschitz-continuous. Looking at the proof of Lemma 2.3 we see that the continuity of $U$ is sufficient for establishing that (1.4) holds almost everywhere on $(T_-, T_+)$, and the Lipschitz character of $u^*$ follows. The proof is complete. \qed

Remark. Without further information on the behavior of $U$ in a neighborhood of $p_{\pm}$ nothing can be said on $T_{\pm}$ being finite or infinite and it is easy to construct examples to show that all possible combinations are possible. As shown in Lemma 2.4 a sufficient condition for $T_{\pm} = \pm\infty$ is that, in a neighborhood of $p = p_{\pm}$, $U(x)$ is bounded by a function of the form $c|x-p|^2$, $c > 0$. $U$ of class $C^1$ is a sufficient condition in order that $u^*$ is of class $C^2$ and satisfies (1.2).

3 Examples

In this section we show a few simple applications of Theorems 1.1 and 1.2. Our first application describes a class of potentials with the property that, in spite of the existence of possibly infinitely many critical values, (1.2) has a nontrivial periodic orbit on any energy level.

Proposition 3.1. Assume that $U : \mathbb{R}^n \to \mathbb{R}$ satisfies
\[
U(-x) = U(x), \quad x \in \mathbb{R}^n,
U(0) = 0, \quad U(x) < 0 \text{ for } x \neq 0,
\lim_{|x| \to \infty} U(x) = -\infty
\]
Assume moreover that each non zero critical point of $U$ is hyperbolic with Morse index $i_m \geq 1$. Then there is a nontrivial periodic orbit of (1.2) on the energy level $\frac{1}{2}|\dot{u}|^2 - U(u) = \alpha$ for each $\alpha > 0$.

Proof. For each $\alpha > 0$ we set $\tilde{U} = U(x) + \alpha$ and let $\Omega \subset \{\tilde{U} > 0\}$ be the connected component that contains the origin. $\Omega$ is open, nonempty and bounded and, from the assumptions on the properties of the critical points of $U$, it follows that $\partial \Omega$ is connected and contains at most a finite number of critical points. Therefore we are under the assumptions of Corollary 1.6 for the case $N = 1$ and the existence of the periodic orbit follows. \qed
An example of potential \( U : \mathbb{R}^2 \to \mathbb{R} \) that satisfies the assumptions in Proposition 3.1 is, in polar coordinates \( r, \theta \),
\[
U(r, \theta) = -r^2 + \frac{1}{2} \tanh^4(r) \cos^2(r^{-1}) \cos^2(2\theta),
\]
where \( k > 0 \) is a sufficiently large number.

Next we give another application of Corollary 1.6. For the potential \( U : \mathbb{R}^2 \to \mathbb{R} \), with
\[
U(x) = \frac{1}{2} (1 - x_1^2)^2 + \frac{1}{2} (1 - 4x_2^2)^2,
\]
the energy level \( \alpha = -\frac{1}{2} \) is critical and corresponds to four hyperbolic critical points \( p_1 = (1, 0) \), \( -p_1 \), \( p_2 = (0, \frac{1}{2}) \) and \( -p_2 \). The connected component \( \Omega \subset \{ U > 0 \} \) that contains the origin is bounded by a simple curve \( \Gamma \) that contains \( \pm p_1 \) and \( \pm p_2 \). In spite of the presence of these critical points, from Theorem 1.2 it follows that there is a minimizer \( u \in \mathcal{U} \), with \( \mathcal{U} \) as in (2.40) and \( u(T^u) \in \Gamma \setminus \{ \pm p_1, \pm p_2 \} \), and Corollary 1.6 implies the existence of a periodic solution \( v^* \). Note that there are also two heteroclinic orbits, solutions of (1.2) and (1.4):
\[
\begin{align*}
  u_1(t) &= (\tanh(t), 0), \\
  u_2(t) &= (0, \frac{1}{2} \tanh(2t)).
\end{align*}
\]
These orbits connect \( p_j \) to \( -p_j \), for \( j = 1, 2 \). By Theorem 1.2 both \( u_1 \) and \( u_2 \) have action greater than \( v^*|_{(-T_+, T_+)} \).

Our last example shows that Theorems 1.1 and 1.2 can be used to derive information on the rich dynamics that (1.2) can exhibit when \( U \) undergoes a small perturbation. We consider a family of potentials \( U : \mathbb{R}^2 \times [0, 1] \to \mathbb{R} \). We assume that \( U(x, 0) = x_1^4 + x_2^2 \) which from various points of view is a structurally unstable potential and, for \( \lambda > 0 \) small, we consider the perturbed potential
\[
U(x, \lambda) = 2\lambda^4 x_1^2 + x_2^2 - 2\lambda^2 x_1 x_2 - 3\lambda^2 x_1^4 + x_6^6.
\]
This potential satisfies \( U(-x, \lambda) = U(x, \lambda) \) and, for \( \lambda > 0 \), has the five critical points \( p_0, \pm p_1 \) and \( \pm p_2 \) defined by
\[
\begin{align*}
  p_0 &= (0, 0), \\
  p_1 &= (\lambda(1 - (\frac{\lambda}{3})^{\frac{1}{2}})^{\frac{1}{2}}, \lambda^3(1 - (\frac{\lambda}{3})^{\frac{1}{2}})^{\frac{1}{2}}), \\
  p_2 &= (\lambda(1 + (\frac{\lambda}{3})^{\frac{1}{2}})^{\frac{1}{2}}, \lambda^3(1 + (\frac{\lambda}{3})^{\frac{1}{2}})^{\frac{1}{2}}),
\end{align*}
\]
which are all hyperbolic.

We have \( U(p_2, \lambda) < 0 = U(p_0, \lambda) < U(p_1, \lambda) \) and \( p_0 \) is a local minimum, \( p_1 \) a saddle and \( p_2 \) a global minimum. Let \( \alpha \) be the energy level. For \( -\alpha < U(p_2, \lambda) \) or \( -\alpha \geq U(p_1, \lambda) \) no information can be
derived from Theorems 1.1 and 1.2 therefore we assume $-\alpha \in [U(p_2, \lambda), U(p_1, \lambda))$. For $-\alpha = U(p_2, \lambda)$ Corollary 1.3 or Corollary 1.6 yields the existence of a heteroclinic connection $u_2$ between $-p_2$ and $p_2$. For $-\alpha \in (U(p_2, \lambda), 0)$ Corollary 1.6 implies the existence of a periodic orbit $u_\alpha$. This periodic orbit converges uniformly in compact intervals to $u_2$ and the period $T_\alpha \to +\infty$ as $-\alpha \to U(p_2, \lambda)^+$. For $\alpha = 0$ Corollary 1.4 implies the existence of two orbits $u_0$ and $-u_0$ homoclinic to $p_0 = 0$. We can assume that $u_0$ satisfies the condition $u_0(-t) = u_0(t)$ and that $u_0(0) = 0$. Then we have that $u_\alpha(\cdot \pm T_\alpha)$ converges uniformly in compact intervals to $\mp u_0$ and $T_\alpha \to +\infty$ as $-\alpha \to 0^-$. For $-\alpha \in (0, U(p_1, \lambda))$, $\partial \Omega$ is the union of three simple curves all of positive diameter: $\Gamma_0$ that includes the origin and $\pm \Gamma_2$ which includes $\pm p_2$ and Corollary 1.5 together with the fact that $U(\cdot, \lambda)$ is symmetric imply the existence of two periodic solutions $\tilde{u}_\alpha$ and $-\tilde{u}_\alpha$ with $\tilde{u}_\alpha$ that oscillates between $\Gamma_0$ and $\Gamma_2$ in each time interval equal to $\frac{T_2}{2}$. Assuming that $\tilde{u}_\alpha(0) \in \Gamma_2$ we have that, as $-\alpha \to 0^+$, $\tilde{u}_\alpha \to u_0$ uniformly in compacts and $T_\alpha \to +\infty$. Finally we observe that, in the limit $-\alpha \to U(p_1, \lambda)^-$, $\tilde{u}_\alpha$ converges uniformly in $\mathbb{R}$ to the constant solution $u \equiv p_1$. 

Figure 4: Bifurcations of dynamics of (1.2) with the $\alpha = 0$, bottom left: $\alpha = 0.05$, bottom right: $\alpha = -U(p_2, 1)$. The shaded regions are not accessible.
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References


