QUANTITATIVE ESTIMATES FOR BENDING ENERGIES AND APPLICATIONS TO NON-LOCAL VARIATIONAL PROBLEMS

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Abstract. We discuss a variational model, given by a weighted sum of perimeter, bending and Riesz interaction energies, that could be considered as a toy model for charged elastic drops. The different contributions have competing preferences for strongly localized and maximally dispersed structures. We investigate the energy landscape in dependence of the size of the 'charge', i.e. the weight of the Riesz interaction energy.

In the two-dimensional case we first prove that for simply connected sets of small elastica energy, the elastica deficit controls the isoperimetric deficit. Building on this result, we show that for small charge the only minimizers of the full variational model are either balls or centered annuli. We complement these statements by a non-existence result for large charge. In three dimensions, we prove area and diameter bounds for configurations with small Willmore energy and show that balls are the unique minimizers of our variational model for sufficiently small charge.

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1. Introduction

In recent years there has been a strong interest in variational models involving a competition between a perimeter type energy and a repulsive term of long-range nature (see for instance the recent review papers [8, 22] and the detailed discussion below). The aim of this paper is to start investigating the effects for this class of problems of higher order interfacial energies such as the Euler elastica in dimension two or the Willmore energy in dimension three. We will consider the simplest possible setting and study volume constrained minimization of functionals defined for sets $E \subset \mathbb{R}^d$ with $d = 2, 3$ as

$$\lambda P(E) + \mu W(E) + Q V_\alpha(E), \quad \lambda, \mu, Q \geq 0. \tag{1.1}$$

The different contributions are given by

- the perimeter P, defined as
$$P(E) = \mathcal{H}^{d-1}(\partial E),$$
- the elastica or Willmore energy W, defined as
$$W(E) = \begin{cases} \displaystyle\int_{\partial E} H^2 \, d\mathcal{H}^1 & \text{for } d = 2, \\ \displaystyle\frac{1}{4}\int_{\partial E} H^2 \, d\mathcal{H}^2 & \text{for } d = 3, \end{cases}$$
where H denotes the mean curvature of ∂E i.e. the curvature in dimension two and the sum of the principal curvatures in dimension three[1],
- the Riesz interaction energy V_α, defined for $\alpha \in (0, d)$ as
$$V_\alpha(E) = \int_{E\times E} \frac{1}{|x-y|^{d-\alpha}} \, dx \, dy.$$

For $\mu = 0$ functional (1.1) is arguably the simplest example of an isoperimetric type problem showing competition between a local attractive term with a non-local repulsive term. In the case of Coulombic interactions, that is $d = 3$ and $\alpha = 2$, this model appears in a variety of contexts. It is for instance the celebrated Gamow liquid drop model for atomic nuclei [19] or the sharp interface limit of the so-called Ohta-Kawasaki model for diblock copolymers [42, 3]. See also [43] for another application of this model. Even though the picture is not complete, it has been shown that minimizers are balls for small Q [28, 29, 12, 25, 15] (actually they are the only stable critical sets [26]) and that no minimizers exist for large Q [28, 29, 37] (see also [16] for a simple proof of non-existence in the three dimensional case). Many more questions related to pattern formation have been investigated for very closely related models, see for instance [2, 1, 9, 46, 5, 23, 20, 30] for a non-exhaustive list. Other examples of functionals presenting this type of competition can be found for instance in shape memory alloys [31, 27], micromagnetics [44] or epitaxial growth [4, 35]. However, the closest model is probably the one for charged liquid drops introduced in [45] where the Riesz interaction energy is replaced by a capacitary term. Surprisingly, it has been shown in [21] that independently of the charge, no minimizers exist for this model (see also [40, 41]). It has been suggested in [22] that a regularization by a Willmore type energy such as the one considered here might restore well-posedness.

[1] We choose to keep the factor $\frac{1}{4}$ in dimension three to stick with the traditional notation.

This paper can be seen as a first step in this direction.
The energy (1.1) could be seen as a toy model for charged elastic vesicles, where the Willmore energy represents a prototype of more general bending energies for fluid membranes and a Coulomb self-interactions refers to the energy of a uniformly charged body. Associated with this interpretation, we refer to the parameter regimes $Q \ll 1$ and $Q \gg 1$ by the terms small and large 'charge'.

Our goal is to understand how the picture changes for (1.1) in the presence of a bending energy i.e. for $\mu > 0$. For $d = 2$ and fixed volume, since an annulus of large radius has arbitrary small elastica energy and also arbitrary small Riesz interaction energy, one needs to either restrict the class of competitors to simply connected sets or to include the perimeter penalization (that is take $\lambda > 0$). In contrast, for $d = 3$, the Willmore functional is scaling invariant and is globally minimized by balls [49]. It seems then natural to consider (1.1) for $\lambda = 0$ and study the stability of the ball. Let us point out that compared to the planar case, configurations with catenoid type parts allow for a much larger variety of structures with low energy. This makes the identification of optimal structures and the distinction between existence and non-existence of minimizers even more challenging.

Setting and main results. Let us set some notation and give our main results. We will always assume that the energy (1.1) contains the bending term and we set without loss of generality $\mu = 1$. For $\lambda, Q \geq 0$, we define

$$\mathcal{F}_{\lambda,Q} = \lambda P + W + QV_\alpha.$$

Regarding the volume constraint, as will be better explained later on, up to a rescaling there is no loss of generality in assuming that $|E| = |B_1|$. For $d = 2, 3$, we define the following classes of admissible sets

$$\begin{aligned}
\mathcal{M} &= \{E \subset \mathbb{R}^d \text{ bounded with } W^{2,2}\text{-regular boundary}\},\\
\mathcal{M}_{sc} &= \{E \in \mathcal{M} : E \text{ simply connected}\},\\
\mathcal{M}(|B_1|) &= \{E \in \mathcal{M} : |E| = |B_1|\},\\
\mathcal{M}_{sc}(|B_1|) &= \{E \in \mathcal{M}_{sc} : |E| = |B_1|\},
\end{aligned}$$

and consider the variational problems

$$\min_{\mathcal{M}_{sc}(|B_1|)} \mathcal{F}_{\lambda,Q}(E) \tag{1.2}$$

and

$$\min_{\mathcal{M}(|B_1|)} \mathcal{F}_{\lambda,Q}(E). \tag{1.3}$$

We start by considering the planar problem $d = 2$ and first focus on the uncharged case $Q = 0$. For $\lambda = 0$ no global minimizer exists in $\mathcal{M}(|B_1|)$, but it has been recently shown in [7, 14] that balls minimize the elastica energy under volume constraint in the class of simply connected sets. Our first result is a quantitative version of this fact in the spirit of the quantitative isoperimetric inequality [18, 10].

Theorem 1.1. *There exists a universal constant $c_0 > 0$ such that for every set $E \in \mathcal{M}_{sc}(|B_1|)$,*

$$W(E) - W(B_1) \geq c_0 \min_x |E\Delta B_1(x)|^2,$$

where $E\Delta F$ denotes the symmetric difference of the sets E and F.
Furthermore, there exist $\delta_0 > 0$ and $c_1 > 0$ such that if $W(E) \leq W(B_1) + \delta_0$, then

$$W(E) - W(B_1) \geq c_1(P(E) - P(B_1)).$$

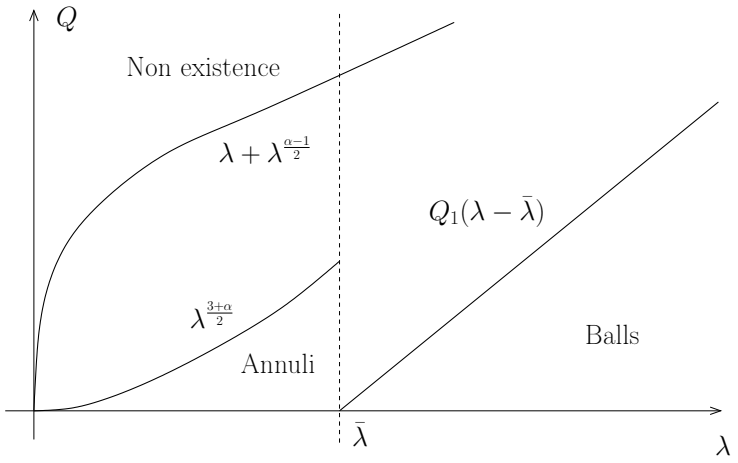

FIGURE 1. The phase diagram.

The proof is based on the idea of [10] for the proof of the quantitative isoperimetric inequality to reduce by a contradiction argument to the case of nearly spherical sets and then compute a Taylor expansion along the lines of [17]. As opposed to [10] which is based on an improved convergence theorem, we obtain the strong convergence to the ball directly from the energy and a delicate refinement of [7] (see Lemma 2.5).

Still in the case $Q = 0$, we then remove the constraint on the sets to be simply connected but consider the minimization problem (1.3) for $\lambda > 0$.

Theorem 1.2. *Let $Q = 0$ and $d = 2$. There exists $\bar{\lambda} > 0$ such that for $\lambda \in (0, \bar{\lambda})$, minimizers of (1.3) are annuli while for $\lambda > \bar{\lambda}$ they are balls.*

Next, we turn to the stability estimates analogous to Theorem 1.1.

Theorem 1.3. *Let $d = 2$ and $\bar{\lambda}$ be given by Theorem 1.2. Then, there exists a universal constant $c_2 > 0$, such that for any $E \in \mathcal{M}(|B_1|)$ and $\lambda > \bar{\lambda}$*

$$\mathcal{F}_{\lambda,0}(E) - \mathcal{F}_{\lambda,0}(B_1) \geq c_2(\lambda - \bar{\lambda}) \min_x |E \Delta B_1(x)|^2,$$

while for any $\lambda_* > 0$ there exists a constant $c(\lambda_*) > 0$ such that for any $\lambda \in [\lambda_*, \bar{\lambda}]$

$$\mathcal{F}_{\lambda,0}(E) - \min_{\mathcal{M}(|B_1|)} \mathcal{F}_{\lambda,0} \geq c(\lambda_*) \min_{\Omega} |E \Delta \Omega|^2,$$

where the minimum is taken among all sets $\Omega \in \mathcal{M}(|B_1|)$ such that Ω is a ball or Ω is an annulus minimizing $\mathcal{F}_{\lambda,0}$ in $\mathcal{M}(|B_1|)$.

Then, we turn to the study of (1.2) and (1.3) for $Q > 0$. Regarding (1.2) we prove the following.

Theorem 1.4. *Let $d = 2$. There exists $Q_0 > 0$ such that for $Q < Q_0$ and all $\lambda \geq 0$, balls are the only minimizers of (1.2).*

The proof is a combination of Theorem 1.1 and [28]. As for (1.3), we obtain a good understanding of part of the phase diagram (see Figure 1).

Theorem 1.5. *Let $d = 2$.*

- *There exists* $Q_1 > 0$ *such that for every* $\lambda > \bar{\lambda}$ *and every* $Q \leq Q_1(\lambda - \bar{\lambda})$*, balls are the only minimizers of* (1.3).
- *There exists* $Q_2 > 0$ *such that for every* $\lambda \in (0, \bar{\lambda}]$ *and every* $Q \leq Q_2 \lambda^{\frac{3+\alpha}{2}}$*, centered annuli are the only minimizers of* (1.3).
- *For every* $\alpha \in (1, 2)$ *there exists* $Q_3(\alpha)$ *such that for every* $\lambda \geq 0$ *and every* $Q \geq Q_3(\alpha)(\lambda + \lambda^{\frac{\alpha-1}{2}})$*, no minimizer exists for* (1.3).

The first part of the theorem is a direct consequence of the minimality of the ball for $\mathcal{F}_{\bar{\lambda},0}$ and for $P + QV_\alpha$ for Q small enough. The second point regarding the minimality of centered annuli is the most delicate part of the theorem. It requires first to argue that sets of small energy are almost annuli and then to use the stability of annuli. The last part of the theorem regarding non-existence is obtained by noticing that if a minimizer exists then we can obtain a lower bound on the energy which is not compatible for large Q with an upper bound obtained by constructing a suitable competitor.

We conclude the paper by studying the three dimensional case where a characterization of the energy landscape is even more difficult. As already pointed out, the Willmore energy is invariant under rescaling and is globally minimized by balls [52, 49]. We can thus focus on the case $\lambda = 0$ where we have competition between the Willmore energy and the Riesz interaction energy. Stability estimates for the Willmore energy have been obtained by De Lellis and Müller [13]. Building on these, on the control of the isoperimetric deficit by the Willmore deficit obtained in [47] and a bound on the perimeter (see Proposition 4.3), we obtain that balls are minimizers of (1.3) for small Q.

Theorem 1.6. *For* $d = 3$ *and* $\lambda = 0$*, there exists* $Q_4 > 0$ *such that for every* $Q \leq Q_4$*, the only minimizers of* (1.3) *are balls.*

Of course, since balls are also minimizers of the isoperimetric problem, a direct consequence of Theorem 1.6 is the minimality of the balls for (1.3) for every $\lambda \geq 0$ and every $Q \leq Q_4$. For the case $\lambda = 0$ we are not able to prove or disprove a non-existence regime in the parameter space. Still, we show that if a minimizer exists for every Q then its isoperimetric quotient must degenerate as $Q \to \infty$ (see Proposition 4.10). This is somewhat reminiscent of earlier results obtained by Schygulla [48].

Finally, we obtain a non-existence result in the case $\lambda > 0$ and $\alpha \in (2, 3)$ in the regime of sufficiently large charge.

Proposition 1.7. *For every* $\alpha \in (2, 3)$*, there exists* $Q_5(\alpha)$ *such that for every* λ, Q *with* $Q \geq Q_5(\lambda^{-\frac{3-\alpha}{2}} + \lambda^{\frac{3+\alpha}{2}})$*, no minimizer of* $\mathcal{F}_{\lambda,Q}$ *in* $\mathcal{M}(|B_1|)$ *exists.*

Outline and notation. In Section 2 we first consider the planar case $d = 2$ in the absence of charge ($Q = 0$) before considering the case $Q > 0$ in Section 3. In the last section we finally consider the three dimensional case. For the reader's convenience, the main theorems given in the introduction are restated in the respective sections and some of them have been extended by more detailed statements. Theorem 1.1, Theorem 1.2, and Theorem 1.3 correspond to Theorem 2.3, Theorem 2.7 and Theorem 2.10, respectively. Theorem 1.4 corresponds to Proposition 3.1, the statements in Theorem 1.5 are collected from Proposition 3.4, Theorem 3.8, Proposition 3.9 and Proposition 3.14. Theorem 1.6 corresponds to Theorem 4.6, and Proposition 1.7 to Proposition 4.11.

For two real numbers A, B the notation $A \gtrsim B$ means that $A \geq cB$ for some $c > 0$ that is universal (unless dependencies are explicitly stated). Correspondingly we use the notation $\lesssim$ and write $A \sim B$ if $A \lesssim B$ and $A \gtrsim B$.

2. The planar case: uncharged drops

We start by investigating the planar case $d = 2$ in the absence of charges i.e. $Q = 0$. Our aim is both to characterize the minimizers and to show that the energy controls the distance to these minimizers.

2.1. **Simply connected sets: controlling the asymmetry index by the elastica deficit.** We first restrict ourselves to simply connected sets. By [7, 14] balls are the unique minimizers of the elastica energy among simply connected sets with prescribed volume. Since they also minimize the perimeter, balls are the unique minimizers of $\mathcal{F}_\lambda = \mathcal{F}_{\lambda,0}$ in this class. Using the quantitative isoperimetric inequality [18], one could directly obtain a quantitative inequality for $\mathcal{F}_\lambda$ which would however degenerate as $\lambda \to 0$. Our aim here is to show that actually the elastica energy $W(E)$ itself controls the distance to balls. This is a quantitative version of [7, 14] which could be of independent interest.

Inspired by a strategy first used in [1, 10] (see for instance also [6, 11, 21] for a few other applications) which was building on ideas from [17], we first restrict ourselves to nearly spherical sets. More precisely, we consider sets E such that ∂E is a graph over ∂B_R for some $R > 0$, i.e.

$$\partial E = \{R(1 + \phi(\theta))e^{i\theta} \ : \ \theta \in [0, 2\pi)\}, \tag{2.1}$$

with $\|\phi\|_{W^{2,2}} \ll 1$. We will need the following estimate on the elastica energy of nearly spherical sets.

Lemma 2.1. *Let $R > 0$ and E be a nearly spherical set. Then,*

$$W(E) - W(B_R) = R^{-1} \int_0^{2\pi} \Big(\ddot{\phi}^2 + \phi^2 + \frac{3}{2}\dot{\phi}^2 - \phi + 4\phi\ddot{\phi} \Big) + o(\|\phi\|^2_{W^{2,2}}). \tag{2.2}$$

Moreover, if $|E| = |B_R|$ and the barycenter of E is equal to zero, then

$$W(E) - W(B_R) \gtrsim R^{-1} \int_0^{2\pi} \Big(\ddot{\phi}^2 + \dot{\phi}^2 + \phi^2 \Big). \tag{2.3}$$

Proof. By scaling, it is enough to prove (2.2), (2.3) for $R = 1$. The elastica energy of E is given by

$$\int_{\partial E} H^2 = \int_0^{2\pi} \frac{(2\dot{\phi}^2 + (1+\phi)^2 - (1+\phi)\ddot{\phi})^2}{(\dot{\phi}^2 + (1+\phi)^2)^{5/2}}.$$

Let us now compute the Taylor expansion of the integrand. Keeping only up to quadratic terms, we get that on the one hand,

$$(2\dot{\phi}^2 + (1+\phi)^2 - (1+\phi)\ddot{\phi})^2 = 1 + 4\phi - 2\ddot{\phi} + 6\phi^2 + \ddot{\phi}^2 + 4\dot{\phi}^2 - 6\phi\ddot{\phi} + (\phi^2 + \dot{\phi}^2 + \ddot{\phi}^2)O(|\phi| + |\dot{\phi}|),$$

and on the other hand,

$$(\dot{\phi}^2 + (1+\phi)^2)^{-5/2} = 1 - 5\phi - \frac{5}{2}\dot{\phi}^2 + 15\phi^2 + o(\phi^2 + \dot{\phi}^2),$$

so that

$$\frac{(2\dot{\phi}^2 + (1+\phi)^2 - (1+\phi)\ddot{\phi})^2}{(\dot{\phi}^2 + (1+\phi)^2)^{5/2}} = 1 - \phi - 2\ddot{\phi} + \ddot{\phi}^2 + \phi^2 + \frac{3}{2}\dot{\phi}^2 + 4\phi\ddot{\phi} + (\phi^2 + \dot{\phi}^2 + \ddot{\phi}^2)O(|\phi| + |\dot{\phi}|).$$

Using that $\int_0^{2\pi} \ddot{\phi} = 0$, we obtain (2.2).
If now $|E| = |B_1|$, using that $2|E| = \int_{\partial E} x \cdot \nu(x)\, d\mathcal{H}^1(x)$, we have

$$\int_0^{2\pi} \Big(\phi + \frac{\phi^2}{2} \Big) = 0, \tag{2.4}$$

while if the barycenter $\bar{x}$ is zero, using that $3|E|\bar{x} = \int_{\partial E} x \cdot \nu(x) x \, d\mathcal{H}^1(x)$,

$$\int_0^{2\pi} [(1+\phi)^3 - 1] e^{i\theta} = 0. \tag{2.5}$$

Using (2.4) in (2.2), we get

$$W(E) - W(B_1) = \int_0^{2\pi} \Big(\ddot{\phi}^2 + \frac{3}{2}\phi^2 + \frac{3}{2}\dot{\phi}^2 + 4\phi\ddot{\phi} \Big) + o(\|\phi\|^2_{W^{2,2}}). \tag{2.6}$$

If $\phi = \sum_{k\in\mathbb{Z}} a_k e^{ik\theta}$ is the Fourier representation of ϕ, then (2.4) and (2.5) imply that

$$|a_0| + |a_{\pm 1}| \lesssim \sum_{|k|\geq 2} |a_k|^2$$

and thus for every $j \in \mathbb{N}$,

$$\left\| \frac{d^j \phi}{d\theta^j} \right\|^2_{L^2} \leq C \sum_{|k|\geq 2} |k|^{2j} |a_k|^2.$$

Since by Parseval's identity,

$$\int_0^{2\pi} \Big(\ddot{\phi}^2 + \frac{3}{2}\phi^2 + \frac{3}{2}\dot{\phi}^2 + 4\phi\ddot{\phi} \Big) = 2\pi \sum_{k\in\mathbb{Z}} \Big(|k|^4 - \frac{5}{2}|k|^2 + \frac{3}{2} \Big) |a_k|^2,$$

and since for $k \in \mathbb{Z}$, the polynomial $k^4 - \frac{5}{2}k^2 + \frac{3}{2}$ is always non-negative and vanishes only for $|k| = 1$, we have for all ϕ with $\|\phi\|_{W^{2,2}}$ sufficiently small

$$\begin{aligned} W(E) - W(B_1) &\gtrsim \sum_{|k|\geq 2} \Big(|k|^4 - \frac{5}{2}|k|^2 + \frac{3}{2} \Big) |a_k|^2 \\ &\gtrsim \int_0^{2\pi} \Big(|\ddot{\phi}|^2 + |\dot{\phi}|^2 + |\phi|^2 \Big), \end{aligned}$$

concluding the proof of (2.3). □

We also recall the following Taylor expansion of the perimeter for nearly spherical sets (see [17]).

Lemma 2.2. *Let $R > 0$ and let E be a nearly spherical set with ∂E represented as in* (2.1)*, then*

$$P(E) - P(B_R) = R \int_0^{2\pi} \Big(\phi + \frac{\dot{\phi}^2}{2} \Big) + o(\|\phi\|^2_{W^{1,2}}). \tag{2.7}$$

We now combine estimate (2.3) and the work of Bucur-Henrot [7], to obtain a quantitative estimate on the elastica deficit.

Theorem 2.3. *There exists a universal constant $c_0 > 0$ such that for every $R > 0$ and every set $E \in \mathcal{M}_{sc}(|B_R|)$,*

$$R\Big(W(E) - W(B_R) \Big) \geq c_0 \left(\min_{x\in\mathbb{R}^2} \frac{|E \Delta B_R(x)|}{|B_R|} \right)^2. \tag{2.8}$$

Furthermore, there exist $\delta_0 > 0$ and $c_1 > 0$ such that if $W(E) \leq R^{-1}(W(B_1) + \delta_0)$, then

$$R\Big(W(E) - W(B_R) \Big) \geq c_1 \frac{P(E) - P(B_R)}{P(B_R)}. \tag{2.9}$$

In order to prove this theorem we will need some auxiliary results. Even for simply connected sets, uniform bounds on the volume and the elastica energy are in general not sufficient to obtain a perimeter or a diameter control (see for example [7, Figure 1]). However, this is the case for sets with elastica energy sufficiently small.

Lemma 2.4. *There exist $\delta_0 > 0$ and $C_0 > 0$ such that for every $E \in \mathcal{M}_{sc}(|B_1|)$ with $W(E) \le W(B_1) + \delta_0$, there holds*

$$P(E) \le C_0. \tag{2.10}$$

Proof. We first prove, by contradiction, a corresponding bound for the diameter, and then deduce the perimeter bound.

Step 1. Assume that there exists a sequence $(E_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_{sc}(|B_1|)$ with

$$W(E_n) \to W(B_1) \quad \text{and} \quad d_n = \operatorname{diam}(E_n) \to \infty.$$

In the following steps, the implicit constants in $\lesssim$ and $\gtrsim$ estimates may depend on the fixed sequence $(E_n)_n$ but are independent of $n \in \mathbb{N}$.

First, up to a rotation, we may assume that for every $\xi \in (0, d_n)$ the vertical section $E^n_\xi = E_n \cap (\{\xi\} \times \mathbb{R}) \neq \emptyset$ is not empty. By Fubini's Theorem, there exists $\xi_n \in (\frac{d_n}{3}, \frac{2d_n}{3})$ such that $|E^n_{\xi_n}| \lesssim d_n^{-1}$. Then, there also exist $x_n \in \partial E_n \cap E^n_{\xi_n}$ and $y_n \in \partial E_n \cap E^n_{\xi_n}$ such that $(x_n, y_n) \subset E_n$ and $|x_n - y_n| \lesssim d_n^{-1}$.

Step 2. Choose an oriented tangent field τ on ∂E_n. We claim that, for some $C_1 > 0$ independent of n, we have

$$\left|\tau(x_n)^\perp \cdot \tau(y_n)\right| \le C_1 d_n^{-1/3}. \tag{2.11}$$

Indeed, assume that (2.11) does not hold so that there exists $\Lambda_n \to \infty$ with

$$\left|\tau(x_n)^\perp \cdot \tau(y_n)\right| \ge \Lambda_n d_n^{-1/3}. \tag{2.12}$$

Without loss of generality, using another translation and rotation, we can assume that $x_n = 0$ and $\tau(x_n) = e_1$ (see Figure 2).

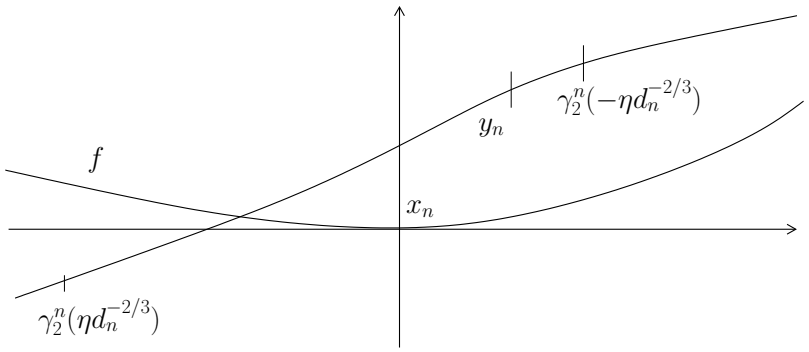

FIGURE 2. Geometry of ∂E_n around x_n under assumption (2.12).

By the bound on the elastica energy we can locally write the part of ∂E_n containing 0 as a graph of the form $\{(\xi, f(\xi)) : |\xi| \le d_n^{-2/3}\}$ with $f(0) = f'(0) = 0$ and uniformly bounded

slope (see [7, Lemma 2.1]). Moreover, $\sup_{|\xi|\le d_n^{-2/3}} |f(\xi)| \le C d_n^{-1}$, since for $0 \le \xi \le d_n^{-2/3}$

$$\begin{aligned} |f(\xi)| &\le \int_0^\xi (\xi - s)|f''(s)|\, ds \\ &\le \Big(\int_0^\xi \frac{(f'')^2}{(1+(f')^2)^{\frac{5}{2}}}\Big)^{1/2}\Big(\int_0^\xi (\xi-s)^2(1+(f')^2)^{\frac{5}{2}}\Big)^{1/2} \\ &\le C\sqrt{W(E_n)}\left(1+\sup_{|\xi|\le d_n^{-2/3}}|f'|^{\frac{5}{2}}\right)\xi^{\frac{3}{2}}, \end{aligned}$$

and similarly for $-d_n^{-2/3} \le \xi \le 0$. Let $\gamma^n = (\gamma_1^n, \gamma_2^n)$ be an arclength parametrization of ∂E_n. We may assume that $y_n = \gamma^n(0)$ with $|\gamma^n(0)| \lesssim d_n^{-1}$. By (2.12), $|\dot\gamma_2^n(0)| = \big|\tau(x_n)^\perp \cdot \tau(y_n)\big| \ge \Lambda_n d_n^{-1/3}$. Let us assume for definiteness that $\dot\gamma_2^n < 0$ (the other case being analogous). We then have by the bound on the elastica energy that for every $|t| \le d_n^{-2/3}$,

$$\dot\gamma_2^n(t) = \dot\gamma_2^n(0) + \int_0^t \ddot\gamma_2^n ds \le -\Lambda_n d_n^{-1/3} + t^{1/2}\left(\int_0^t (\ddot\gamma_2^n)^2\right)^{1/2} ds \le -\frac{\Lambda_n}{2} d_n^{-1/3}.$$

Let η be a small constant chosen so that for $t \in [-\eta d_n^{-2/3}, \eta d_n^{-2/3}]$,

$$|\gamma_1^n(t)| = \left|(y_n)_1 + \int_0^t \dot\gamma_1^n(s) ds\right| \lesssim d_n^{-1} + \eta d_n^{-2/3} \le d_n^{-2/3}.$$

This implies that for $t \in [-\eta d_n^{-2/3}, \eta d_n^{-2/3}]$, $\gamma^n(t)$ stays inside the cylinder $[-d_n^{-2/3}, d_n^{-2/3}] \times \mathbb{R}$. Furthermore,

$$\gamma_2^n(-\eta d_n^{-2/3}) = (y_n)_2 - \int_{-\eta d_n^{-2/3}}^0 \dot\gamma_2^n(s) ds \gtrsim -d_n^{-1} + \Lambda_n d_n^{-1} > \sup_{|\xi|\le d_n^{-2/3}} |f(\xi)|,$$

and

$$\gamma_2^n(\eta d_n^{-2/3}) = (y_n)_2 + \int_0^{\eta d_n^{-2/3}} \dot\gamma_2^n(s) ds \lesssim d_n^{-1} - \Lambda_n d_n^{-1} < -\sup_{|\xi|\le d_n^{-2/3}} |f(\xi)|.$$

Therefore, the graph $\{(\xi, f(\xi)) \,:\, |\xi| \le d_n^{-2/3}\}$ splits the cylinder $[-d_n^{-2/3}, d_n^{-2/3}] \times \mathbb{R}$ into two connected components with $\gamma_2^n(-\eta d_n^{-2/3})$ in one of the components and $\gamma_2^n(\eta d_n^{-2/3})$ in the other (see Figure 2). Hence, γ^n intersects the graph of f which contradicts the fact that

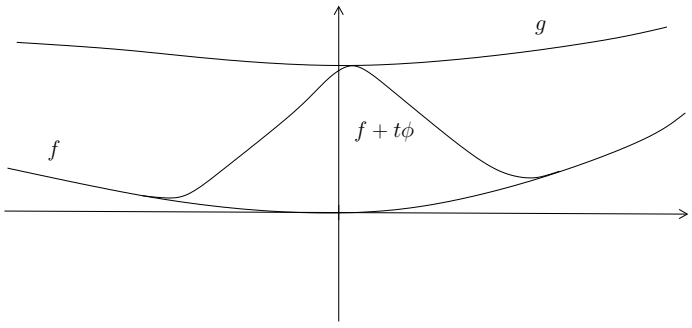

FIGURE 3. Construction of the deformed set.

∂E_n can be locally written as a graph. We have thus shown that (2.11) holds.

Step 3. We recall that we assume (without loss of generality) that $x_n = 0$ and $\tau(x_n) = e_1$. By the bound on the elastica energy and (2.11), for η small enough (but not depending on n), x_n and y_n belong to two different connected components of $\partial E_n \cap Q_\eta$, where $Q_\eta = (-\eta, \eta)^2$. Let Γ_1, respectively Γ_2 be the connected component containing x_n, respectively y_n. By [7, Lemma 2.1] and (2.11), we have $\Gamma_1 \cap Q_\eta = \{(\xi, f(\xi)), |\xi| \le \eta\}$ and $\Gamma_2 \cap Q_\eta = \{(\xi, g(\xi)), |\xi| \le \eta\}$ where without loss of generality, we can assume that $f(\xi) < g(\xi)$ for $|\xi| \le \eta$ (see Figure 3). Let now $\phi \in C^\infty(-\eta, \eta)$ be a non-negative bump function with $\phi(0) = 1$ and let $d_n^{-1} \gtrsim t > 0$ be such that

$$\max_{[-\eta,\eta]} \big((f + t\phi) - g\big) = 0.$$

If we replace in Q_η the component Γ_1 by $\widetilde{\Gamma}_1 = \{(\xi, f(\xi) + t\phi(\xi)) \ : \ |\xi| \le \eta\}$, we obtain a new set $\widetilde{E}_n$ with $|\widetilde{E}_n| \le |E_n|$ (since by construction $\widetilde{E}_n \subset E_n$) and

$$|W(\widetilde{E}_n) - W(E_n)| \le \left| \int_{-\eta}^{\eta} \frac{(f'' + t\phi'')^2}{(1 + (f' + t\phi')^2)^{5/2}} - \frac{f''^2}{(1 + f'^2)^{5/2}} \right| \lesssim t \lesssim d_n^{-1},$$

where we have used that thanks to the energy estimate, f' is uniformly small in $(-\eta, \eta)$ to make the Taylor expansion. The set $\widetilde{E}_n$ is made of two drops E_n^1 and E_n^2 with mass $m_1^n = |E_n^1|$ and $m_2^n = |E_n^2|$ satisfying $m_1^n + m_2^n \le |E_n|$. From [7, Theorem 3.5], for every couple of drops E and F, with $|E| + |F| = |B_{2^{-1/3}}|$,

$$W(F) + W(E) \ge (1 + \delta_*) W(B_{2^{-1/3}}),$$

for some $\delta_* > 0$. By scaling, we deduce that if we choose a ball B such that $|B| = |m_1^n| + |m_2^n| \le |E| = |B_1|$, then

$$W(\widetilde{E}_n) = W(E_n^1) + W(E_n^2) \ge (1 + \delta_*) W(B) \ge (1 + \delta_*) W(B_1),$$

from which we obtain that

$$W(E_n) \ge W(\widetilde{E}_n) - C d_n^{-1} \ge \left(1 + \frac{\delta_*}{2}\right) W(B_1),$$

contradicting the fact that $W(E_n) \to W(B_1)$.
Step 4. By the previous steps we know that there exists $R > 0$ such that $\operatorname{diam}(E) \le 2R$ for all $E \in \mathcal{M}_{sc}(|B_1|)$ with $W(E) \le W(B_1) + \delta_0$. Therefore, after translation $E \subset B_R$. We now choose an arclength parametrization $\gamma : [0, L] \to \mathbb{R}^2$, $L = P(E)$ and obtain as in [7, Lemma 2.5]

$$L \ = \ \int_0^L |\dot\gamma|^2 \, ds \ = \ - \int_0^L \gamma \cdot \ddot\gamma \, ds \ \le \ \Big(\int_0^L |\gamma|^2 \, ds \Big)^{\frac12} \sqrt{W(E)} \ \le \ L^{\frac12} R (2\pi + \delta_0)^{\frac12},$$

from which the perimeter bound follows.

□

Lemma 2.5. *Let $(E_n)_n$ be a sequence in $\mathcal{M}_{sc}(|B_1|)$ with $W(E_n) \to W(B_1)$ as $n \to \infty$. For every $n \in \mathbb{N}$ let $\gamma_n : [0, 2\pi) \to \mathbb{R}^2$ be a constant speed parametrization of ∂E_n. Then, after translation γ_n converges strongly in $W^{2,2}$ to a (unit speed) arclength parametrization of ∂B_1.*

Proof. Consider the sets $\widetilde{E}_n = 2^{-\frac13} E_n$. It follows from [7] and our assumptions that $(\widetilde{E}_n)_n$ is a minimizing sequence of the functional $E \mapsto |E| + \frac12 W(E)$ on $\mathcal{M}$. Moreover, $(\widetilde{E}_n)$ has uniformly bounded perimeter by Lemma 2.4. For such sequences it is proved in [7, Section 4] that, after translation, $\widetilde{E}_n$ must converge to $B_{2^{-\frac13}}$. Hence E_n converges to B_1. This gives

first weak convergence in $W^{2,2}$ which combined with the convergence of the energy gives the strong convergence. □

Proof of Theorem 2.3. Without loss of generality, we may assume that $R = 1$ and $|E| = \pi$.
Step 1. Assume for the sake of contradiction that (2.8) does not hold. Then, there exists a sequence $(E_n)_n$ such that

$$\frac{W(E_n) - W(B_1)}{\min_x |E_n \Delta B_1(x)|^2} \to 0. \tag{2.13}$$

Since $\min_x |E_n \Delta B_1(x)|$ is bounded, this implies that $W(E_n) \to W(B_1)$ as $n \to \infty$. We then obtain from Lemma 2.4 that the diameter d_n of E_n remains bounded. Hence, by Lemma 2.5 E_n must converge up to translation to B_1 strongly in $W^{2,2}$ and thus by Sobolev embedding, also in $C^{1,\alpha}$ for every $\alpha \leq 1/2$. Thus, for n large enough, ∂E_n is a graph over B_1 and ∂E_n is nearly spherical. Since the barycenter of E_n is converging to zero, E_n is also a graph over the ball centered in its barycenter. Up to a translation, this means that we can apply (2.3) and obtain that

$$W(E_n) - W(B_1) \gtrsim \int_0^{2\pi} \phi_n^2 \gtrsim |E_n \Delta B_1|^2, \tag{2.14}$$

which contradicts (2.13).
Step 2. We now turn to (2.9). The additional assumption implies in the rescaled setting that $W(E) \leq 2\pi + \delta_0$. If (2.9) does not hold there exists a sequence $(E_n)_n$ such that

$$\frac{W(E_n) - W(B_1)}{P(E_n) - P(B_1)} \to 0. \tag{2.15}$$

In particular, since by Lemma 2.4, $P(E_n)$ is uniformly bounded, we have as above that $W(E_n) \to W(B_1)$ as $n \to \infty$. Using (2.4) and (2.7) we deduce as in (2.14) that

$$W(E_n) - W(B_1) \gtrsim \int_0^{2\pi} \left(\dot{\phi}_n^2 + \phi_n^2\right) \gtrsim P(E_n) - P(B_1)$$

which is in contradiction with (2.15).

□

2.2. **Minimizers of $\mathcal{F}_\lambda$.** We move on and remove the constraint that sets are simply connected. Since annuli of very large diameter have vanishing elastica energy, in order to have a well-posed problem, we need to consider $\lambda > 0$. Let us recall that the energy is given by

$$\mathcal{F}_\lambda(E) = \lambda P(E) + W(E) = \int_{\partial E} \left(\lambda + H^2\right) d\mathcal{H}^1.$$

Up to a rescaling, we may restrict ourselves to the problem

$$\min_{\mathcal{M}(|B_1|)} \mathcal{F}_\lambda(E). \tag{2.16}$$

We will show below that depending on the value of λ, minimizers are either balls or annuli. Before stating the precise result we compute the energy of an annulus.

Lemma 2.6. *For every $r > 0$, the energy $\mathcal{F}_\lambda$ of an annulus with inner radius $r > 0$ and area equal to π is given by*

$$f_\lambda(r) = 2\pi \left[\lambda(r + (1 + r^2)^{1/2}) + \frac{1}{r} + \frac{1}{(1 + r^2)^{1/2}}\right]. \tag{2.17}$$

The function $f_\lambda : (0,\infty) \to \mathbb{R}$ is strictly convex and $f_\lambda^{(3)} < 0$. Its unique minimizer r_λ is the unique solution of

$$\lambda\left(1+\frac{r}{(1+r^2)^{1/2}}\right)-\frac{1}{r^2}-\frac{r}{(1+r^2)^{3/2}}=0. \tag{2.18}$$

Proof. The strict convexity of f_λ follows from

$$\left[r^{-1}+(1+r^2)^{-1/2}\right]''=2r^{-3}+(1+r^2)^{-5/2}(2r^2-1)\geq 2r^{-3}-(1+r^2)^{-5/2}>0,$$

which can be checked by considering separately the case $r \leq 1$ and the case $r \geq 1$. Since $f_\lambda(r) \to \infty$ as $r \to 0$ and $r \to \infty$ it has a unique minimizer $r_\lambda > 0$ which then satisfies (2.18). We further compute that

$$f_\lambda^{(3)}(r)=-\frac{3\lambda r}{(r^2+1)^{\frac{5}{2}}}-\frac{15r^3}{(r^2+1)^{\frac{7}{2}}}+\frac{9r}{(r^2+1)^{\frac{5}{2}}}-\frac{6}{r^4}\leq 3\frac{-2r^3+3r-10}{(r^2+1)^{\frac{7}{2}}}<0.$$

□

We may now solve the minimization problem (2.16).

Theorem 2.7. *There exists $\bar\lambda > 0$ such that for $\lambda \in (0,\bar\lambda)$, minimizers of (2.16) are annuli of inner radius r_λ (as defined in Lemma 2.6) while for $\lambda > \bar\lambda$ they are balls of radius one. Moreover, for $\lambda = \bar\lambda$ minimizers can be either a ball of radius one or an annulus of inner radius $r_{\bar\lambda}$.*

Proof. For an arbitrary set $E \in \mathcal{M}(|B_1|)$, by translation invariance, we can in addition assume that the connected components of E are far apart from each other so that in particular their convex envelopes do not intersect. In the first two steps we prove that minimizers must be either balls or annuli. In the final step we compare the minimal energy of the optimal annuli with the energy of the ball to conclude the proof.

Step 1. Let E be an admissible set. We first show that the energy of each connected component F of E can be strictly decreased by transforming it into a ball or an annulus. If F is simply connected, by [7, 14] and the isoperimetric inequality

$$\mathcal{F}_\lambda(F) \geq \mathcal{F}_\lambda(B_R),$$

for $R > 0$ with $|B_R| = |F|$. Equality holds if and only if F is a translate of B_R, proving the claim in this case.

If F is not simply connected, then F^c is made of an unbounded connected component and a finite union $G_1,\dots,G_n$ of bounded simply connected sets. For $i = 1,\dots n$, let $m_i = |G_i|$. By the discussion above, among all simply connected sets of mass m_i, $\mathcal{F}_\lambda$ is uniquely minimized by balls B_i of area m_i. For two balls B_i and B_j, if $r > 0$ is such that $|B_r| = |B_i| + |B_j|$, then

$$\mathcal{F}_\lambda(B_i)+\mathcal{F}_\lambda(B_j) > \mathcal{F}_\lambda(B_r)$$

since the inequality is separately true for the perimeter and the elastica parts of the energy. Letting thus $r > 0$ be such that $|B_r| = \sum_{i=1}^n m_i$, we have

$$\mathcal{F}_\lambda(\cup_i G_i)=\sum_i \mathcal{F}_\lambda(G_i)\geq \sum_i \mathcal{F}_\lambda(B_i)\geq \mathcal{F}_\lambda(B_r)$$

with equality if and only if $n = 1$ and G_1 is a translate of B_r. The set $F \cup \big(\cup_{i=1}^n G_i\big)$ given by filling the holes of F is simply connected and letting $R > r$ be such that $|B_R| =$

$|F| + |B_r| = |F \bigcup \cup_{i=1}^n G_i|$, we have as above that

$$\mathcal{F}_\lambda(F \bigcup \cup_{i=1}^n G_i) \geq \mathcal{F}_\lambda(B_R)$$

with equality if and only if $F \bigcup \cup_{i=1}^n G_i$ is a translate of B_R. Putting all this together, we find that

$$\mathcal{F}_\lambda(F) = \mathcal{F}_\lambda(F \bigcup \cup_{i=1}^n G_i) + \mathcal{F}_\lambda(\cup_i G_i) \geq \mathcal{F}_\lambda(B_R) + \mathcal{F}_\lambda(B_r) = \mathcal{F}_\lambda(B_R \backslash B_r)$$

with equality if and only if F is a (not necessarily concentric) annulus of outer radius R and inner radius r, which again proves our claim.

Step 2. We are thus reduced to the class of competitors made of a finite union of balls and annuli. Since the elastica energy (respectively the perimeter) blows up when one of the radii goes to zero (respectively to infinity), existence of a minimizer in this class is easily obtained. Let us prove that a minimizer must be either a single ball or a single annulus. If one of the connected components of E is a ball then E must be the ball. Indeed, by the first part of the proof, it cannot contain two balls. Moreover, the union of a ball and an annulus has energy higher than that of a ball of area the sum of the areas. Indeed, by the isoperimetric inequality the perimeter is better for the ball and since the elastica energy of a ball is a decreasing function of its area, the elastica energy of a single ball is also better than the elastica energy of a ball and an annulus.

We are left to prove that one annulus is better than two. For $i = 1, 2$, let r_i be the internal radii and R_i be the external radii so that $m_i = \pi(R_i^2 - r_i^2)$ are the area of the two annuli. We consider as a competitor the annulus $B_R \backslash B_{r_1}$ with

$$R^2 = R_1^2 + (R_2^2 - r_2^2).$$

Since the elastica part of the energy is smaller, we are left to prove that the perimeter part is smaller, too. That is indeed the case since

$$R = (R_1^2 + R_2^2 - r_2^2)^{1/2} \leq R_1 + R_2 + r_2.$$

Step 3. Let us now prove the existence of the threshold $\bar{\lambda}$. Let us start by noticing that if the ball is a minimizer of (2.16) for some λ then it is also a minimizer of (2.16) for every $\lambda' > \lambda$. Indeed, by the isoperimetric inequality, for every set $E \in \mathcal{M}(|B_1|)$,

$$\mathcal{F}_{\lambda'}(E) = \mathcal{F}_\lambda(E) + (\lambda' - \lambda)P(E) \geq \mathcal{F}_\lambda(B_1) + (\lambda' - \lambda)P(B_1) = \mathcal{F}_{\lambda'}(B_1)$$

with equality if and only if E is a ball.

The energy of an annulus of internal radius r and area equal to $|B_1|$ is larger than the energy of the ball B_1 if and only if

$$2\pi\,(\lambda + 1) < \min_{r>0} f_\lambda(r), \tag{2.19}$$

for f_λ as defined in (2.17). By taking as competitor $r = \lambda^{-1/2}$, we get that

$$\min_r f_\lambda(r) \leq 2\pi\lambda^{1/2}\left(2 + (1+\lambda)^{1/2} + (1+\lambda)^{-1/2}\right),$$

so that for all $\lambda > 0$ sufficiently small (2.19) does not hold and minimizers are annuli.

Using that $\lambda r + r^{-1} \geq 2\lambda^{1/2}$ and $(1 + r^2)^{1/2} \geq 1$, we obtain the lower bound

$$\min_r f_\lambda(r) \geq 2\pi(2\lambda^{1/2} + \lambda),$$

proving that (2.19) holds for $\lambda \geq \frac{\sqrt{2}}{2}$ and concluding the proof.

□

Remark 2.8. Finding the explicit value of $\bar\lambda$ is not straightforward. Indeed, this entails first minimizing $f_\lambda(r)$ and then finding the range of λ such that (2.19) holds. The unique minimizer r_λ of f_λ is determined by (2.18).

By (2.19) we deduce that the ball is the minimizer if and only if

$$g(\lambda) = 2\pi(\lambda+1) - f_\lambda(r_\lambda) < 0.$$

By minimality of f_λ in r_λ we find

$$g'(\lambda) \;=\; 2\pi - 2\pi\big(r_\lambda + (1+r_\lambda^2)^{\frac12}\big) \;<\; 0.$$

Therefore, the threshold $\bar\lambda$ is characterized by the condition

$$g(\bar\lambda) \;=\; 0.$$

Letting $U = (1+r^{-2})^{1/2}$, we observe that finding r_λ is equivalent to find the unique solution $U \geq 1$ of

$$\lambda U^2(1+U) = (U^2-1)(1+U^3).$$

Dividing by $1+U$, we are left with finding a root larger than one of the fourth order polynomial

$$U^4 - U^3 - \lambda U^2 + U - 1.$$

Nevertheless, a simple explicit formula for $\bar\lambda$ seems not to be available.

2.3. **Stability estimates.** We now turn to stability estimates for the functional $\mathcal{F}_\lambda$. As for the proof of Theorem 2.3, we will first need to know that sets with small energy are close (in a non-quantitative way) to minimizers.

Lemma 2.9. *Consider a sequence of positive numbers $(\lambda_n)_n$ with $\lambda_n \to \lambda \in (0,\infty]$ and a sequence $(E_n)_n$ in $\mathcal{M}(|B_1|)$ such that*

$$\mathcal{F}_{\lambda_n}(E_n) - \min_{\mathcal{M}(|B_1|)} \mathcal{F}_{\lambda_n} \to 0 \quad \text{ as } n\to\infty. \tag{2.20}$$

Then, up to translations and passing to a subsequence, E_n converges strongly in $W^{2,2}$ to a minimizer of $\mathcal{F}_\lambda$ in $\mathcal{M}(|B_1|)$.

In particular, for all $n\in\mathbb{N}$ sufficiently large E_n is connected and has topological genus one if $\lambda<\bar\lambda$, zero if $\lambda>\bar\lambda$ and zero or one if $\lambda=\bar\lambda$.

Proof. We may assume without loss of generality that $2\lambda \geq \lambda_n \geq \lambda/2$ for all $n\in\mathbb{N}$.

Let us first prove that for n large enough, E_n must be connected. For the sake of contradiction, assume it is not. Arguing as in the proof of Theorem 2.7, we see that we can replace E_n by a set $\widetilde{E}_n$ made of two connected components each of which is either a ball or an annulus with $\mathcal{F}_{\lambda_n}(\widetilde{E}_n) \leq \mathcal{F}_{\lambda_n}(E_n)$ so that (2.20) still holds for $\widetilde{E}_n$. Inspecting the proof of Theorem 2.7 we see that we reach a contradiction since the minimum of $\mathcal{F}_{\lambda_n}$ in this class is larger than $\min_{\mathcal{M}(|B_1|)}\mathcal{F}_{\lambda_n}$ by a constant not depending on n. If $\lambda<\bar\lambda$ we further obtain that for n large enough, E_n cannot be simply connected since otherwise we would have by Theorem 2.7 for some $\delta(\lambda)$

$$\mathcal{F}_{\lambda_n}(E_n) \geq \mathcal{F}_{\lambda_n}(B_1) \geq \min_{\mathcal{M}(|B_1|)} \mathcal{F}_{\lambda_n} + \delta,$$

contradicting again (2.20). Let us prove that for any λ, E_n is of genus at most one. Otherwise, we can write $E_n = F_n \backslash \cup_{i=1}^{N_n} G_i$ with F_n and G_i simply connected and $N_n \geq 2$. Letting

$V_n = |F_n|$ and $m_i^n = |G_i|$, we have $V_n - \sum_i m_i^n = |B_1|$ and by [7, 14] and the isoperimetric inequality, that

$$\mathcal{F}_{\lambda_n}(E_n) \geq 2\pi^{3/2}\left(V_n^{-1/2} + \sum_i (m_i^n)^{-1/2}\right) + 2\sqrt{\pi}\lambda_n\left(V_n^{1/2} + \sum_i (m_i^n)^{1/2}\right).$$

Since $g_n(x) = \pi^{3/2}x^{-1/2} + \sqrt{\pi}\lambda_n x^{1/2}$ is subadditive, and $N_n \geq 2$,

$$\mathcal{F}_{\lambda_n}(E_n) \geq 2 \min_{V-(m_1+m_2)=|B_1|} g_n(V) + g_n(m_1) + g_n(m_2).$$

Notice that $g_n \to \infty$ as x tends to zero or infinity so that the minimum on the right-hand side is attained for m_1 and m_2 uniformly bounded above and below by a constant depending only on λ. Since for such values of m_i

$$g_n(m_1) + g_n(m_2) \geq g_n(m_1 + m_2) + \delta(\lambda),$$

for some $\delta(\lambda) > 0$, we have by the last two inequalities and Theorem 2.7

$$\mathcal{F}_{\lambda_n}(E_n) \geq 2 \min_{V-m=|B_1|} g_n(V) + g_n(m) + \delta(\lambda) \geq \min_{\mathcal{M}(|B_1|)} \mathcal{F}_{\lambda_n} + \delta(\lambda),$$

contradicting (2.20). Thus, for n large enough, $E_n = F_n \backslash G_n$ with F_n and G_n simply connected (where $G_n = \emptyset$ is possible if and only if $\lambda \geq \bar{\lambda}$).

Consider first the case $G_n \neq \emptyset$ for all n sufficiently large (possibly up to a subsequence). This implies in particular that $\lambda \leq \bar{\lambda}$. Let $A_n = B_{R_1^n} \setminus B_{R_2^n}$ be the centered annulus minimizing $\mathcal{F}_{\lambda_n}$ in $\mathcal{M}(|B_1|)$, hence $R_2^n = r_{\lambda_n}$, $R_1^n = \sqrt{1 + r_{\lambda_n}^2}$, where r_{λ_n} is the unique minimum of f_{λ_n} given in Lemma 2.6. Choosing $\widetilde{R}_1^n$, $\widetilde{R}_2^n$ with $|B_{\widetilde{R}_1^n}| = |F_n|$, $|B_{\widetilde{R}_2^n}| = |G_n|$ we deduce

$$\begin{aligned}
\mathcal{F}_{\lambda_n}(E_n) - \mathcal{F}_{\lambda_n}(A_n) = {} & \Big(\lambda_n P(F_n) + W(F_n) + \lambda_n P(G_n) + W(G_n)\Big) \\
& - \Big[\lambda_n P(B_{R_1^n}) + W(B_{R_1^n}) + \lambda_n P(B_{R_2^n}) + W(B_{R_2^n})\Big] \\
= {} & \Big(\dots\Big) - \Big[\lambda_n P(B_{\widetilde{R}_1^n}) + W(B_{\widetilde{R}_1^n}) + \lambda_n P(B_{\widetilde{R}_2^n}) + W(B_{\widetilde{R}_2^n})\Big] \\
& + f_{\lambda_n}(\widetilde{R}_2^n) - f_{\lambda_n}(R_2^n) \\
\geq {} & \lambda_n\Big(P(F_n) - P(B_{\widetilde{R}_1^n}) + P(G_n) - P(B_{\widetilde{R}_2^n})\Big) \\
& + \Big(W(F_n) - W(B_{\widetilde{R}_1^n}) + W(G_n) - W(B_{\widetilde{R}_2^n})\Big) \\
& + \Big(f_{\lambda_n}(\widetilde{R}_2^n) - f_{\lambda_n}(R_2^n)\Big) \qquad = T_1^n + T_2^n + T_3^n.
\end{aligned}$$

By (2.20) the left-hand side of the inequality vanishes as $n \to \infty$. Since F_n and G_n are simply connected, all three terms on the right-hand side are non-negative and must therefore converge to zero. Since $2\lambda \geq \lambda_n \geq \lambda/2$ we have that R_2^n is uniformly bounded from above and below. Since $T_3^n \to 0$ and R_2^n minimizes f_{λ_n} we deduce by strict convexity of f_{λ_n} and a Taylor expansion that

$$0 = \lim_{n\to\infty} T_3^n \geq c(\lambda) \limsup_{n\to\infty} (\widetilde{R}_2^n - R_2^n)^2,$$

and by the mass constraint, also $\widetilde{R}_1^n - R_1^n \to 0$. Since $T_2^n \to 0$ we deduce that $W(F_n) - W(B_{R_1^n}) \to 0$ and $W(G_n) - W(B_{R_2^n}) \to 0$. After rescaling we can apply Lemma 2.5 and the conclusion follows.

Let us finally consider the case that $\lambda \geq \bar\lambda$ with E_n simply connected. Then,

$$\mathcal{F}_{\lambda_n}(E_n) - \min_{\mathcal{M}(|B_1|)} \mathcal{F}_{\lambda_n} = \lambda_n\big(P(E_n)-P(B_1)\big)+W(E_n)-W(B_1)+\big(\mathcal{F}_{\lambda_n}(B_1)-\min_{\mathcal{M}(|B_1|)}\mathcal{F}_{\lambda_n}\big),$$

and we obtain as in the previous case that $W(E_n) \to W(B_1)$ concluding again by Lemma 2.5. □

We can now prove our stability estimate for $\mathcal{F}_\lambda$.

Theorem 2.10. *Let $\bar\lambda$ be given by Theorem 2.7 and consider an arbitrary set $E \in \mathcal{M}(|B_1|)$. Then, there exists a universal constant $c_2 > 0$, such that for $\lambda > \bar\lambda$*

$$\mathcal{F}_\lambda(E) - \mathcal{F}_\lambda(B_1) \geq c_2(\lambda-\bar\lambda)\min_x |E\Delta B_1(x)|^2, \tag{2.21}$$

while for any $\lambda_* > 0$ there exists a constant $c(\lambda_*) > 0$ such that for any $\lambda \in [\lambda_*, \bar\lambda]$

$$\mathcal{F}_\lambda(E) - \min_{\mathcal{M}(|B_1|)} \mathcal{F}_\lambda \geq c(\lambda_*)\min_\Omega |E\Delta\Omega|^2, \tag{2.22}$$

where the minimum is taken among all sets $\Omega \in \mathcal{M}(|B_1|)$ such that Ω is a ball or Ω is an annulus minimizing $\mathcal{F}_\lambda$ in $\mathcal{M}(|B_1|)$.

Proof. Let us start by proving (2.21). Since $\lambda - \bar\lambda > 0$, using the minimality of B for $\mathcal{F}_{\bar\lambda}$ and the quantitative isoperimetric inequality [18], we can write that

$$\begin{aligned}\mathcal{F}_\lambda(E) - \mathcal{F}_\lambda(B_1) &= \mathcal{F}_{\bar\lambda}(E) - \mathcal{F}_{\bar\lambda}(B_1) + (\lambda-\bar\lambda)\big(P(E)-P(B_1)\big)\\ &\geq (\lambda-\bar\lambda)\big(P(E)-P(B_1)\big)\\ &\geq c_2(\lambda-\bar\lambda)\min_x |E\Delta B_1(x)|^2,\end{aligned}$$

which proves (2.21).

We now turn to the proof of (2.22). As for the proof of (2.8), we assume by contradiction that the inequality does not hold. We thus have sequences $(\lambda_n)_n$ in $[\lambda_*, \bar\lambda]$ and $(E_n)_n$ in $\mathcal{M}(|B_1|)$ with

$$\lim_{n\to\infty} \frac{\mathcal{F}_{\lambda_n}(E_n) - \min_{\mathcal{M}(|B_1|)} \mathcal{F}_{\lambda_n}}{\min_\Omega |E_n\Delta\Omega|^2} = 0. \tag{2.23}$$

Again, implicit constants in $\lesssim$ and $\gtrsim$ estimates may depend on the fixed sequence $(E_n)_n$ but are independent of $n \in \mathbb{N}$.
Since the denominator in (2.23) is uniformly bounded we have $\lim_{n\to\infty} \mathcal{F}_{\lambda_n}(E_n) - \min_{\mathcal{M}(|B_1|)} \mathcal{F}_{\lambda_n} = 0$. Without loss of generality we can also assume $\lambda_n \to \lambda \in [\lambda_*, \bar\lambda]$ as $n \to \infty$.

By Lemma 2.9 E_n converges strongly in $W^{2,2}$ to a minimizer of $\mathcal{F}_\lambda$. For n large enough, it is connected, and has either genus one or zero (the latter being possible only if $\lambda = \bar\lambda$).

Let us first consider the case where up to a translation and up to passing to a subsequence, E_n converges to $B_{R_1}\setminus B_{R_2}(x_2)$ so that for n large enough $E_n = F_n \setminus G_n$ for some simply connected sets $G_n \subset F_n$. Up to a translation, we may assume that $⨍_{F_n} x = 0$ so that by Lemma 2.9, we can write F_n as a graph over the ball of radius $\widetilde R_1^n$ with $|B_{\widetilde R_1^n}| = |F_n|$, and such that $\widetilde R_1^n \to R_1$. At the same time G_n can be written as a graph over $B_{\widetilde R_2^n}(x_n)$ with $|B_{\widetilde R_2^n}| = |G_n|$, $⨍_{G_n} x = x_n$, $\widetilde R_2^n \to R_2$ and $x_n \to x_2$ as $n \to \infty$. Hence,

$$\partial F_n = \{\widetilde R_1^n(1+\phi_n)e^{i\theta},\ \theta \in [0,2\pi)\}, \quad \partial G_n = \{x_n + \widetilde R_2^n(1+\psi_n)e^{i\theta},\ \theta\in[0,2\pi)\}$$

for some functions ϕ_n, ψ_n with small $W^{2,2}$ norm that satisfy (2.4) and (2.5). Let us point out that $B_{\widetilde{R}_1^n}\backslash B_{\widetilde{R}_2^n}$ is in general not an optimal annulus and is thus in particular not admissible for the denominator of (2.23).
By (2.3) and the Sobolev inequality, we obtain that

$$\begin{aligned}\mathcal{F}_{\lambda_n}(E_n) - \mathcal{F}_{\lambda_n}(B_{\widetilde{R}_1^n}\backslash B_{\widetilde{R}_2^n}) &\geq W(F_n) - W(B_{\widetilde{R}_1^n}) + W(G_n) - W(B_{\widetilde{R}_2^n}) \\ &\gtrsim \sup|\phi_n|^2 + \sup|\psi_n|^2 + |F_n\Delta B_{\widetilde{R}_1^n}|^2 + |G_n\Delta B_{\widetilde{R}_2^n}(x_n)|^2.\end{aligned} \tag{2.24}$$

If $B_{\widetilde{R}_2^n}(x_n) \not\subset B_{\widetilde{R}_1^n}$, we need to move x_n inwards to obtain an annulus. To this aim, let

$$\delta_n = \max(|x_n| + \widetilde{R}_2^n - \widetilde{R}_1^n, 0) \tag{2.25}$$

and define $\widetilde{x}_n = x_n - \delta_n \frac{x_n}{|x_n|}$ (notice that if $\delta_n > 0$ then $x_n \neq 0$ so that taking as a convention that $\widetilde{x}_n = 0$ if $x_n = 0$, this quantity is well defined). We then have $B_{\widetilde{R}_2^n}(\widetilde{x}_n) \subset B_{\widetilde{R}_1^n}$ by (2.25). Since

$$B_{\widetilde{R}_2^n(1-\sup|\psi_n|)}(x_n) \subset G_n \subset F_n \subset B_{\widetilde{R}_1^n(1+\sup|\phi_n|)},$$

we must have $|x_n| + \widetilde{R}_2^n(1-\sup|\psi_n|) \leq \widetilde{R}_1^n(1+\sup|\phi_n|)$ that is

$$\delta_n^2 \leq \left(\widetilde{R}_1^n \sup|\phi_n| + \widetilde{R}_2^n \sup|\psi_n|\right)^2 \overset{(2.24)}{\lesssim} \mathcal{F}_{\lambda_n}(E_n) - \mathcal{F}_{\lambda_n}(B_{\widetilde{R}_1^n}\backslash B_{\widetilde{R}_2^n}). \tag{2.26}$$

From this we deduce that

$$|G_n\Delta B_{\widetilde{R}_2^n}(\widetilde{x}_n)|^2 \lesssim |G_n\Delta B_{\widetilde{R}_2^n}(x_n)|^2 + \delta_n^2 \overset{(2.24),(2.26)}{\lesssim} \mathcal{F}_{\lambda_n}(E_n) - \mathcal{F}_{\lambda_n}(B_{\widetilde{R}_1^n}\backslash B_{\widetilde{R}_2^n}). \tag{2.27}$$

We can now estimate

$$\begin{aligned}\mathcal{F}_{\lambda_n}(E_n) - \mathcal{F}_{\lambda_n}(A_n) &= \mathcal{F}_{\lambda_n}(E_n) - \mathcal{F}_{\lambda_n}(B_{\widetilde{R}_1^n}\backslash B_{\widetilde{R}_2^n}) + \mathcal{F}_{\lambda_n}(B_{\widetilde{R}_1^n}\backslash B_{\widetilde{R}_2^n}) - \mathcal{F}_{\lambda_n}(A_n) \\ &\overset{(2.24),(2.27)}{\gtrsim} |F_n\Delta B_{\widetilde{R}_1^n}|^2 + |G_n\Delta B_{\widetilde{R}_2^n(\widetilde{x}_n)}|^2 + \mathcal{F}_{\lambda_n}(B_{\widetilde{R}_1^n}\backslash B_{\widetilde{R}_2^n}) - \mathcal{F}_{\lambda_n}(A_n) \\ &\gtrsim |E_n\Delta(B_{\widetilde{R}_1^n}\backslash B_{\widetilde{R}_2^n}(\widetilde{x}_n))|^2 + \mathcal{F}_{\lambda_n}(B_{\widetilde{R}_1^n}\backslash B_{\widetilde{R}_2^n}) - \mathcal{F}_{\lambda_n}(A_n).\end{aligned}$$

By the minimality of f_{λ_n} at $r = R_2^n$ and by strict convexity of f_{λ_n} (see Lemma 2.6), $f''_{\lambda_n}(R_2^n) > 0$ and thus

$$\mathcal{F}_{\lambda_n}(B_{\widetilde{R}_1^n}\backslash B_{\widetilde{R}_2^n}) - \mathcal{F}_{\lambda_n}(A_n) \gtrsim c(\lambda)(\widetilde{R}_2^n - R_2^n)^2.$$

We therefore conclude that

$$\mathcal{F}_{\lambda_n}(E_n) - \mathcal{F}_{\lambda_n}(A_n) \gtrsim |E_n\Delta(B_{\widetilde{R}_1^n}\backslash B_{\widetilde{R}_2^n}(\widetilde{x}_n))|^2 + |\widetilde{R}_2^n - R_2^n|^2. \tag{2.28}$$

As pointed out above, this is not sufficient to obtain a contradiction with (2.23) since $B_{\widetilde{R}_1^n}\backslash B_{\widetilde{R}_2^n}(\widetilde{x}_n)$ is not an optimal annulus for $\mathcal{F}_{\lambda_n}$. We thus need to prove that there exists $\widehat{x}_n$ close to $\widetilde{x}_n$ such that (2.28) holds with $B_{R_1^n}\backslash B_{R_2^n}(\widehat{x}_n)$ instead of $B_{\widetilde{R}_1^n}\backslash B_{\widetilde{R}_2^n}(\widetilde{x}_n)$.

For $\varepsilon_n \geq 0$, we let $\widehat{x}_n = (1-\varepsilon_n)\widetilde{x}_n$. We want to choose ε_n so that $B_{R_2^n}(\widehat{x}_n) \subset B_{R_1^n}$. If $|\widetilde{x}_n| \ll 1$, we set $\varepsilon_n = 0$ while for $|\widetilde{x}_n| \gtrsim 1$, we claim that we can take

$$\varepsilon_n = C|\widetilde{R}_2^n - R_2^n| \tag{2.29}$$

for some constant $C \lesssim 1$. Indeed, if ε_n satisfies (2.29), then using first that by (2.25) $|\widetilde{x}_n| + \widetilde{R}_2^n \le \widetilde{R}_1^n$ and then that $|\widetilde{R}_1^n - R_1^n| \sim |\widetilde{R}_2^n - R_2^n|$ (since $B_{\widetilde{R}_1^n} \setminus B_{\widetilde{R}_2^n}$ and $B_{R_1^n} \setminus B_{R_2^n}$ have equal mass) we obtain

$$\begin{aligned}
|\widehat{x}_n| + R_2^n &= (1-\varepsilon_n)|\widetilde{x}_n| + \widetilde{R}_2^n + (R_2^n - \widetilde{R}_2^n) \\
&\le \widetilde{R}_1^n - (C|\widetilde{x}_n| - 1)|\widetilde{R}_2^n - R_2^n| \\
&= R_1^n + (\widetilde{R}_1^n - R_1^n) - (C|\widetilde{x}_n| - 1)|\widetilde{R}_2^n - R_2^n| \\
&\le R_1^n - (C|\widetilde{x}_n| - c)|\widetilde{R}_2^n - R_2^n| \\
&\overset{|\widetilde{x}_n| \gtrsim 1}{\le} R_1^n,
\end{aligned}$$

and therefore $B_{R_2^n}(\widehat{x}_n) \subset B_{R_1^n}$. From (2.29) we conclude that

$$|(B_{\widetilde{R}_1^n} \Delta B_{\widetilde{R}_2^n}(\widetilde{x}_n)) \Delta (B_{R_1^n} \setminus B_{R_2^n}(\widehat{x}_n))|^2 \lesssim |\widetilde{R}_1^n - R_1^n|^2 + |\widetilde{R}_2^n - R_2^n|^2 + |\widetilde{x}_n - \widehat{x}_n|^2 \lesssim |\widetilde{R}_2^n - R_2^n|^2$$

so that (2.28) and the fact that $B_{R_1^n} \setminus B_{R_2^n}(\widehat{x}_n)$ is an optimal annulus for $\mathcal{F}_{\lambda_n}$, finally yields

$$\begin{aligned}
&\mathcal{F}_{\lambda_n}(E_n) - \mathcal{F}_{\lambda_n}(A_n) \\
&\qquad \gtrsim |E_n \Delta (B_{\widetilde{R}_1^n} \setminus B_{\widetilde{R}_2^n}(\widetilde{x}_n))|^2 + |(B_{\widetilde{R}_1^n} \setminus B_{\widetilde{R}_2^n}(\widetilde{x}_n)) \Delta (B_{R_1^n} \setminus B_{R_2^n}(\widehat{x}_n))|^2 \\
&\qquad\qquad \gtrsim |E_n \Delta (B_{R_1^n} \setminus B_{R_2^n}(\widehat{x}_n))|^2 \ge \min_{\Omega} |E_n \Delta \Omega|^2,
\end{aligned}$$

which contradicts (2.23).

Let us finally consider the case $\lambda = \bar{\lambda}$ with E_n simply connected. We then have $\mathcal{F}_{\lambda_n}(E_n) \to \inf_{\mathcal{M}(|B_1|)} \mathcal{F}_{\bar{\lambda}} = \mathcal{F}_{\bar{\lambda}}(B_1)$ and $W(E_n) \ge W(B_1)$ by [7, 14] so that the quantitative isoperimetric inequality [18] gives a contradiction to (2.23) (one could also use (2.8)). □

3. The planar case: charged drops

Still considering the planar case $d = 2$, we now turn our attention to the variational problem (1.3) for arbitrary positive parameters m, λ and Q. Our aim is to understand as much as possible the phase diagram of $\mathcal{F}_{\lambda,Q}$ i.e. identify regions of existence/non-existence of minimizers and characterize minimizers when they exist.

By a simple rescaling we have

$$\min_{\mathcal{M}(m)} \mathcal{F}_{\lambda,Q}(E) \;=\; \frac{\sqrt{\pi}}{\sqrt{m}} \min_{\mathcal{M}(|B_1|)} \mathcal{F}_{\lambda(m),Q(m)}(E), \tag{3.1}$$

with $\lambda(m) = \lambda\frac{m}{\pi}$, $Q(m) = Q\left(\frac{m}{\pi}\right)^{\frac{3+\alpha}{2}}$ so that we may assume that $m = |B_1|$. Notice however that, for λ and Q fixed, $\lambda(m)$ and $Q(m)$ tend to zero as m goes to zero. Therefore, if we want to understand the shape of minimizers at small volume, we need to carefully study the phase diagram of $\mathcal{F}_{\lambda,Q}$ close to $(\lambda, Q) = (0,0)$.

3.1. Minimization in the class of simply connected sets.

Let us start by investigating (1.2), i.e. restricting ourselves to simply connected sets.

Proposition 3.1. *There exists $Q_0 > 0$ such that for $Q < Q_0$ and all $\lambda \ge 0$, balls are the only solutions of the minimization problem*

$$\min_{\mathcal{M}_{sc}(|B_1|)} \mathcal{F}_{\lambda,Q}(E).$$

Proof. Consider $\delta_0, c_1 > 0$ from Theorem 2.3. By [28, Proposition 7.1] (see also [15, Theorem 1.3]) there exists Q_1 such that

$$P(E) - P(B_1) \geq Q_1\big(V_\alpha(B_1) - V_\alpha(E)\big) \quad \text{for all } E \in \mathcal{M}(|B_1|). \tag{3.2}$$

Consider now any $E \in \mathcal{M}_{sc}(|B_1|)$ with $\mathcal{F}_{\lambda,Q}(E) \leq \mathcal{F}_{\lambda,Q}(B_1)$ Hence

$$\begin{aligned} W(E) - W(B_1) &\leq Q\big(V_\alpha(B_1) - V_\alpha(E)\big) - \lambda\big(P(E) - P(B_1)\big) \\ &\leq QV_\alpha(B_1) \,<\, \delta_0 \end{aligned} \tag{3.3}$$

for all $Q < \delta_0 V_\alpha(B_1)^{-1}$.Therefore, if $Q < \delta_0 V_\alpha(B_1)^{-1}$, (2.9) applies and

$$c_1 \frac{P(E) - P(B_1)}{P(B_1)} \leq W(E) - W(B_1).$$

Together with (3.3) we get that for all $Q < \delta_0 V_\alpha(B_1)^{-1}$

$$P(E) - P(B_1) \leq Q\Big(\frac{c_1}{P(B_1)} + \lambda\Big)^{-1}\big(V_\alpha(E) - V_\alpha(B_1)\big),$$

which combined with (3.2) gives $P(E) = P(B_1)$ for $Q < Q_1(\frac{c_1}{P(B_1)} + \lambda)$. This implies that $E = B_1$. Choosing $Q_0 := \min\{\delta_0 V_\alpha(B_1)^{-1}, Q_1 \frac{c_1}{P(B_1)}\}$ concludes the proof of the proposition. □

Remark 3.2. Since for $d = 2$ a bound on the perimeter gives a bound on the diameter for simply connected sets, existence of a minimizer for (1.2) holds for every $\lambda > 0$ and $Q > 0$. For $\lambda = 0$, the existence of minimizers for large Q is less clear.

As a direct consequence of (3.1) and Proposition 3.1 we get

Corollary 3.3. *Let Q_0 be given by Proposition 3.1. For any $Q > 0$, $\lambda \geq 0$ and $m \leq \pi\left(\frac{Q_0}{Q}\right)^{\frac{2}{3+\alpha}}$ balls are the only minimizers of*

$$\min_{\mathcal{M}_{sc}(m)} \mathcal{F}_{\lambda,Q}(E).$$

3.2. **Minimization in the class $\mathcal{M}(|B_1|)$.** We now drop the constraint that E is simply connected and study (1.3). We start by focusing on the simplest part of the phase diagram, that is where minimizers are balls. As above, let $Q_1 > 0$ be given by [28, 15] such that balls are the only minimizers of

$$\min_{\mathcal{M}(|B_1|)} P(E) + QV_\alpha(E).$$

Proposition 3.4 (Global minimality of the ball for $\lambda > \bar{\lambda}$)**.** *For every $\lambda > \bar{\lambda}$ and $Q \leq Q_1(\lambda - \bar{\lambda})$ balls are the unique minimizers of $\mathcal{F}_{\lambda,Q}$ in $\mathcal{M}(|B_1|)$.*

Proof. For $\lambda > \bar{\lambda}$ and $Q \leq Q_1(\lambda - \bar{\lambda})$, we have

$$\mathcal{F}_{\lambda,Q}(E) = (\lambda - \bar{\lambda})P(E) + QV_\alpha(E) + \mathcal{F}_{\bar{\lambda}}(E) = (\lambda - \bar{\lambda})\left(P(E) + \frac{Q}{\lambda - \bar{\lambda}}V_\alpha(E)\right) + \mathcal{F}_{\bar{\lambda}}(E).$$

By definition of Q_1, balls are the only minimizers of $P(E) + \frac{Q}{\lambda - \bar{\lambda}}V_\alpha(E)$. Since by Theorem 2.7 they also minimize $\mathcal{F}_{\bar{\lambda}}$, balls are the only minimizers of $\mathcal{F}_{\lambda,Q}$. □

Remark 3.5. Notice that if balls are minimizers of $\mathcal{F}_{\lambda,Q}$ then by the isoperimetric inequality they are also minimizers of $\mathcal{F}_{\lambda',Q}$ for every $\lambda' \geq \lambda$.

We now focus on the most interesting case and show that for $\lambda \le \bar\lambda$ and Q sufficiently small centered annuli are optimal. Our first observation is that among annuli of the form $B_{R_1}\backslash B_{R_2}(x_2)$, the Riesz interaction energy is minimized for the centered annulus i.e. for $x_2 = 0$.

Lemma 3.6. *For every $x_2 \in \mathbb{R}^2$ such that $B_{R_2}(x_2) \subset B_{R_1}$, it holds*

$$V_\alpha(B_{R_1}\backslash B_{R_2}(x_2)) \ge V_\alpha(B_{R_1}\backslash B_{R_2}),$$

with equality if and only if $x_2 = 0$. Moreover $V_\alpha(B_{R_1}\backslash B_{R_2}(x_2)) \le V_\alpha(B_1)$.

Proof. Let $\mathcal{X}_1 = \mathcal{X}_{B_{R_1}}$, $\mathcal{X}_2 = \mathcal{X}_{B_{R_2}}$. By the Riesz rearrangement inequality we deduce

$$\begin{aligned} V_\alpha(B_{R_1}\backslash B_{R_2}(x_2)) &= \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\mathcal{X}_1(x)\mathcal{X}_1(y)}{|x-y|^{2-\alpha}} + \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\mathcal{X}_2(x-x_2)\mathcal{X}_2(y-x_2)}{|x-y|^{2-\alpha}} \\ &\qquad - 2\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\mathcal{X}_1(x)\mathcal{X}_2(y-x_2)}{|x-y|^{2-\alpha}} \\ &\ge \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\mathcal{X}_1(x)\mathcal{X}_1(y)}{|x-y|^{2-\alpha}} + \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\mathcal{X}_2(x)\mathcal{X}_2(y)}{|x-y|^{2-\alpha}} \\ &\qquad - 2\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\mathcal{X}_1(x)\mathcal{X}_2(y)}{|x-y|^{2-\alpha}} \\ &= V_\alpha(B_{R_1}\backslash B_{R_2}). \end{aligned}$$

By [36, Theorem 3.9] equality holds if and only if $\mathcal{X}_1, \mathcal{X}_2$ coincide with their symmetric rearrangement, hence if and only if $x_2 = 0$. The last statement is a direct consequence of the Riesz rearrangement inequality. □

We will also need the following stability lemma.

Lemma 3.7. *For any $\lambda_* \in (0, \bar\lambda]$ there exists $C(\lambda_*) > 0$ such that for all $\lambda \in [\lambda_*, \bar\lambda]$ and all $Q > 0$ the following property holds: Let $E \in \mathcal{M}(|B_1|)$ satisfy*

$$\mathcal{F}_{\lambda,Q}(E) \le \mathcal{F}_{\lambda,Q}(A_\lambda), \tag{3.4}$$

where A_λ is the centered annulus that minimizes $\mathcal{F}_\lambda$ in $\mathcal{M}(|B_1|)$.
Then, there exists $\Omega \in \mathcal{M}(|B_1|)$ which is either a ball or an annulus minimizing $\mathcal{F}_\lambda$, such that

$$|E\Delta\Omega| + \big(V_\alpha(\Omega) - V_\alpha(A_\lambda)\big) \le C(\lambda_*)Q. \tag{3.5}$$

Moreover, for Q small enough Ω must be an annulus minimizing $\mathcal{F}_\lambda$.

Proof. Let $\Omega \in \operatorname{argmin}_{\mathcal{M}(|B_1|)} \mathcal{F}_\lambda \cup \{B_1(x) : x \in \mathbb{R}^2\}$ satisfy

$$|E\Delta\Omega| \le |E\Delta\Omega'| \quad \text{for all } \Omega' \in \operatorname{argmin}_{\mathcal{M}(|B_1|)} \mathcal{F}_\lambda \cup \{B_1(x) : x \in \mathbb{R}^2\}.$$

Thanks to (3.4), (2.22), and the Lipschitz-continuity of the Riesz interaction energy

$$\begin{aligned} c(\lambda_*)|E\Delta\Omega|^2 &\le \mathcal{F}_\lambda(E) - \mathcal{F}_\lambda(A_\lambda) \\ &\le Q\big(V_\alpha(A_\lambda) - V_\alpha(\Omega)\big) + Q\big(V_\alpha(\Omega) - V_\alpha(E)\big) \\ &\le Q\big(V_\alpha(A_\lambda) - V_\alpha(\Omega)\big) + CQ|E\Delta\Omega|. \end{aligned}$$

By Lemma 3.6 the first term on the right-hand side is non-positive and we deduce (3.5). Since $V_\alpha(B_1) - V_\alpha(A_\lambda) \ge c(\lambda_*)$, we also conclude from (3.5) that for Q small enough, Ω cannot be a ball. □

We can now prove the minimality of the centered annulus in this regime.

Theorem 3.8 (Global minimality of the annulus for $\lambda \leq \bar{\lambda}$)**.** *For every $0 < \lambda_* < \bar{\lambda}$ there exists $Q(\lambda_*)$ such that for all $Q < Q(\lambda_*)$ and all $\lambda \in [\lambda_*, \bar{\lambda}]$, the minimizers of $\mathcal{F}_{\lambda,Q}$ in $\mathcal{M}(|B_1|)$ are centered annuli $A_{\lambda,Q}$. Moreover, there exist positive constants $c(\lambda_*), C(\lambda_*)$ depending only on λ_* such that the inner radius $r_{\lambda,Q}$ of $A_{\lambda,Q}$ satisfies*

$$c(\lambda_*)Q \leq |r_{\lambda,Q} - r_\lambda| \leq C(\lambda_*)Q, \tag{3.6}$$

where r_λ is the minimizer of f_λ (see Lemma 2.6).

Proof. Step 1. We minimize first $\mathcal{F}_{\lambda,Q}$ in the class of annuli. By (3.6), this minimum is attained by centered annuli i.e. annuli of the form $A_r = B_R \backslash B_r$ with $R = R(r) = \sqrt{1+r^2}$. We recall that $f_\lambda(r) = \mathcal{F}_\lambda(A_r)$ and let $g(r) = V_\alpha(A_r)$. Hence, we are left with minimizing

$$h_{\lambda,Q}(r) = f_\lambda(r) + Qg(r).$$

Since f_λ is coercive and g is positive (and since both are continuous), there exists at least one minimum $r_{\lambda,Q}$ of $h_{\lambda,Q}$.
Step 2. We claim that $g'(r) < 0$ for all $r > 0$. To prove this, let $v(x) = \int_{B_R \backslash B_r} |x-y|^{-2+\alpha}\, dy$ be the potential created by the annulus $B_R \backslash B_r$ then

$$\begin{aligned} g'(r) &= \frac{d}{dr} \int_{B_{R(r)} \backslash B_r} \int_{B_{R(r)} \backslash B_r} |x-y|^{-2+\alpha}\, dy\, dx \\ &= 2\Big(R'(r) \int_{\partial B_{R(r)}} v(x)\, d\mathcal{H}^1(x) - \int_{\partial B_r} v(x)\, d\mathcal{H}^1(x) \Big) \\ &= 2 \int_{\partial B_r} v\left(\frac{R}{r}x\right) - v(x)\, d\mathcal{H}^1(x). \end{aligned}$$

For $x \in \partial B_r$, let $S_x = B_R((1+\frac{R}{r})x) \cap A_r$ (see Figure 4). By the symmetry of S_x with

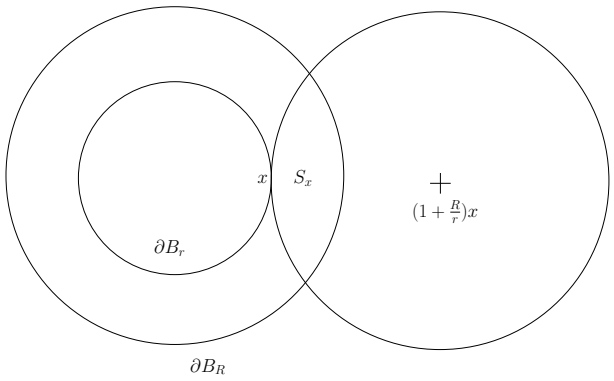

FIGURE 4. The set S_x.

respect to the line $\{\frac{x}{r} \cdot y = \frac{r+R}{2}\}$,

$$\int_{S_x} \frac{1}{|x-y|^{2-\alpha}} = \int_{S_x} \frac{1}{|\frac{R}{r}x - y|^{2-\alpha}}.$$

Moreover, since for $y \in A_r \cap \bar{S}_x^c$, $|x-y| < |\frac{R}{r}x - y|$, we have for $x \in \partial B_r$

$$v\left(\frac{R}{r}x\right) - v(x) = \int_{A_r} \frac{1}{|\frac{R}{r}x - y|^{2-\alpha}} - \int_{A_r} \frac{1}{|x-y|^{2-\alpha}} < 0$$

so that for every $r>0$, $g'(r)<0$.
Step 3. We therefore have $r_{\lambda,Q}>r_\lambda$ and that $r_{\lambda,Q}$ is increasing in Q and decreasing in λ. Since $h_{\lambda,Q}(r_\lambda)\geq h_{\lambda,Q}(r_{\lambda,Q})$ and since f_λ is strictly convex with $f'_\lambda(r_\lambda)=0$ and $f'''_\lambda<0$ we next deduce

$$Q(g(r_\lambda)-g(r_{\lambda,Q}))\geq f_\lambda(r_{\lambda,Q})-f_\lambda(r_\lambda)\geq \frac{1}{2}f''_\lambda(r_{\lambda,Q})(r_{\lambda,Q}-r_\lambda)^2.$$

On the one hand $g(r_\lambda)\leq V_\alpha(B_1)$ and on the other hand $r_{\lambda,Q}\leq C(\lambda_*)$ and $f''_\lambda(r_{\lambda,Q})\geq c(\lambda_*)>0$ for all $\lambda\geq\lambda_*$, $Q<1$. We infer that

$$(r_{\lambda,Q}-r_\lambda)^2\leq C(\lambda_*)Q.$$

Hence $r_{\lambda,Q}\to r_\lambda$ as $Q\to 0$ uniformly for all $\lambda\in[\lambda_*,\bar\lambda]$.
Step 4. By the minimizing property of $r_{\lambda,Q}$, we deduce

$$0=f'(r_{\lambda,Q})+Qg'(r_{\lambda,Q})=f''(\widetilde r_{\lambda,Q})(r_{\lambda,Q}-r_\lambda)+Qg'(r_{\lambda,Q})$$

for some $r_\lambda\leq\widetilde r_{\lambda,Q}\leq r_{\lambda,Q}$. We have $c(\lambda_*)\leq f''(\widetilde r_{\lambda,Q})\leq C(\lambda_*)$ and since $c(\lambda_*)\leq r_{\lambda,Q}\leq C(\lambda_*)$ also $g'(r_{\lambda,Q})$ is uniformly bounded from above and below from which (3.6) follows.
Step 5. Assume for the sake of contradiction that we can find a sequence $\lambda_n\in[\lambda_*,\bar\lambda]$ converging to $\lambda\in[\lambda_*,\bar\lambda]$ and sequences $Q_n\to 0$, $(E_n)_n$ in $\mathcal{M}(|B_1|)$ which are not annuli and such that

$$\mathcal{F}_{\lambda_n,Q_n}(E_n)\leq\mathcal{F}_{\lambda_n,Q_n}(A_n),\tag{3.7}$$

where $A_n=A_{r_{\lambda_n,Q_n}}$ is the optimal annulus (which is centered).

By (3.7) and the fact that $r_{\lambda_n,Q_n}\to r_\lambda$, we see that E_n satisfies (2.20) so that Lemma 2.9 together with Lemma 3.7 imply that for n sufficiently large, up to a translation $E_n=F_n\backslash G_n$ with F_n converging strongly in $W^{2,2}$ to B_{R_λ} (where $R_\lambda=\sqrt{1+r_\lambda^2}$) and G_n converging strongly in $W^{2,2}$ to B_{r_λ}. Let $R_n\to R_\lambda$ and $r_n\to r_\lambda$ be such that

$$|B_{R_n}|=|F_n|\qquad\text{and}\qquad|B_{r_n}|=|G_n|.$$

Since A_n is optimal for $\mathcal{F}_{\lambda_n,Q_n}$ among annuli, we have $\mathcal{F}_{\lambda_n,Q_n}(A_n)\leq\mathcal{F}_{\lambda_n,Q_n}(B_{R_n}\backslash B_{r_n})$ so that (3.7) becomes

$$\begin{aligned}\lambda_nP(F_n)+W(F_n)+\lambda_nP(G_n)+W(G_n)+Q_nV_\alpha(F_n\backslash G_n)\\ \leq\lambda_nP(B_{R_n})+W(B_{R_n})+\lambda_nP(B_{r_n})+W(B_{r_n})+Q_nV_\alpha(B_{R_n}\backslash B_{r_n}).\end{aligned}$$

Using that by [7, 14], $W(F_n)\geq W(B_{R_n})$ and $W(G_n)\geq W(B_{r_n})$, this simplifies to

$$\lambda_nP(F_n)+\lambda_nP(G_n)+Q_nV_\alpha(F_n\backslash G_n)\leq\lambda_nP(B_{R_n})+\lambda_nP(B_{r_n})+Q_nV_\alpha(B_{R_n}\backslash B_{r_n}).\tag{3.8}$$

We now estimate (for simplicity we do not write the kernel in the integrals)

$$\begin{aligned}&V_\alpha(B_{R_n}\backslash B_{r_n})-V_\alpha(F_n\backslash G_n)\\ &\quad=\int_{B_{R_n}}\int_{B_{R_n}}+\int_{B_{r_n}}\int_{B_{r_n}}-2\int_{B_{R_n}}\int_{B_{r_n}}-\int_{G_n}\int_{G_n}-\int_{F_n}\int_{F_n}+2\int_{F_n}\int_{G_n}\\ &\quad=V_\alpha(B_{R_n})-V_\alpha(F_n)+V_\alpha(B_{r_n})-V_\alpha(G_n)+2\left[\int_{F_n}\int_{G_n}-\int_{B_{R_n}}\int_{B_{r_n}}\right]\end{aligned}$$

By the Riesz rearrangement inequality,

$$\int_{F_n}\int_{G_n}-\int_{B_{R_n}}\int_{B_{r_n}}\leq 0$$

from which we obtain

$$V_\alpha(B_{R_n}\backslash B_{r_n}) \le V_\alpha(F_n\backslash G_n) + V_\alpha(B_{R_n}) - V_\alpha(F_n) + V_\alpha(B_{r_n}) - V_\alpha(G_n).$$

Inserting this into (3.8) and dividing by λ_n, we obtain

$$P(F_n)+\frac{Q_n}{\lambda_n}V_\alpha(F_n)+P(G_n)+\frac{Q_n}{\lambda_n}V_\alpha(G_n) \le P(B_{R_n})+\frac{Q_n}{\lambda_n}V_\alpha(B_{R_n})+P(B_{r_n})+\frac{Q_n}{\lambda_n}V_\alpha(B_{r_n}),$$

which implies by [28, Proposition 7.1] that if Q_n/λ_n is small enough then $F_n = B_{R_n}$ and $G_n = B_{r_n}(x_n)$ for some $x_n \in \mathbb{R}^2$. This contradicts our assumption that E_n was not an annulus and concludes the proof.

□

Having in mind the study of (3.1) for λ and Q fixed but m tending to zero, we now focus on the behavior of $Q(\lambda_*)$ for λ_* going to zero.

Proposition 3.9. *There exist $Q_2 > 0$ and $\lambda_0 > 0$ such that if $\lambda \in (0, \lambda_0]$ and $Q \le Q_2\lambda^{\frac{3+\alpha}{2}}$ then every minimizer $E_{\lambda,Q}$ of (3.1) is a centered annulus. If $E_{\lambda,Q} = B_{R_{\lambda,Q}}\backslash B_{r_{\lambda,Q}}$ then $\lambda^{1/2}r_{\lambda,Q} \to 1$ and $\lambda^{1/2}R_{\lambda,Q} \to 1$, as $\lambda \to 0$.*

Proof. Consider sequences $(\lambda_n)_n$, $(Q_n)_n$ with

$$\lambda_n \to 0, \qquad Q_n \to 0, \qquad \limsup_{n\to\infty} Q_n\lambda_n^{-\frac{3+\alpha}{2}} \le Q_2.$$

We start by making the rescaling $E = \lambda_n^{-1/2}\widehat{E}$ so that

$$\mathcal{F}_{\lambda_n,Q_n}(E) = \lambda_n^{1/2}\left(P(\widehat{E}) + W(\widehat{E}) + Q_n\lambda_n^{-\frac{3+\alpha}{2}}V_\alpha(\widehat{E})\right) = \lambda_n^{1/2}\mathcal{F}_{1,\widetilde{Q}_n}(\widehat{E}),$$

with $\widetilde{Q}_n = Q_n\lambda_n^{-\frac{3+\alpha}{2}}$. We are thus left to study the minimization problem

$$\min_{\mathcal{M}(\lambda_n\pi)} \mathcal{F}_{1,\widetilde{Q}_n}(E).$$

Let us prove that if $(E_n)_n$ is a sequence with $\mathcal{F}_{1,\widetilde{Q}_n}(E_n) \le \mathcal{F}_{1,\widetilde{Q}_n}(A_n)$, where A_n is the optimal (centered) annulus, then for n large enough $E_n = F_n\backslash G_n$ with F_n and G_n simply connected and both converging to the unit ball strongly in $W^{2,2}$. Indeed, if we choose $R_n > 0$ with $R_n^2 - 1 = \lambda_n$ and write ∂E_n as a union of simple closed curves $\Gamma_1^n, \dots, \Gamma_{N_n}^n$, we deduce that

$$\sum_{i=1}^{N_n}(P+W)(\Gamma_i^n) \le \mathcal{F}_{1,\widetilde{Q}_n}(E_n) \le \mathcal{F}_{1,\widetilde{Q}_n}(B_{R_n}\setminus B_1) \le 8\pi + C(1+\widetilde{Q}_n)\lambda_n,$$

where in the last inequality we have used that for every set Ω, $V_\alpha(\Omega) \lesssim |\Omega|$. Since ∂B_1 minimizes $P+W$ among all simple closed curves (by [7, 14] and the isoperimetric inequality it is minimized by circles and then a simple optimization on the radius gives the minimality of ∂B_1), and since $(1+\widetilde{Q}_n)\lambda_n \to 0$ as $n \to \infty$ we deduce that for all n sufficiently large we have $N_n \le 2$. By [7, Theorem 1.1] we also have $N_n > 1$ since otherwise $W(E_n)$ would blow up. Hence $N_n = 2$ for all n sufficiently large and $\lim_{n\to\infty} W(\Gamma_i^n) = 2\pi$ for $i = 1, 2$ so that the claim follows from Lemma 2.5.

Arguing then exactly as in the proof of Theorem 3.8 and using that $\widetilde{Q}_n \le Q_2$ we obtain that for n sufficiently large E_n is an annulus if Q_2 is sufficiently small. □

Remark 3.10. We do not expect that the condition $Q \le c_0 \lambda^{\frac{3+\alpha}{2}}$ is sharp. Indeed, looking at the proofs of Proposition 3.9 and Theorem 3.8 we see that we have argued separately that F_n and G_n should be balls without exploiting the fact that the volume of $E_n = F_n \setminus G_n$ is small. One could hope to improve the result by obtaining a better control on $V_\alpha(B_{R_n} \setminus B_{r_n}) - V_\alpha(E_n)$.

As a corollary, we obtain by (3.1) the minimality of annuli for small volumes.

Corollary 3.11. *There exist λ_1 and c_1 such that for $m \le \frac{\lambda_1}{\lambda}$ and $Q^{\frac{2}{3+\alpha}} \le c_1 \lambda$, minimizers of $\mathcal{F}_{\lambda,Q}$ in $\mathcal{M}(m)$ are centered annuli.*

3.3. **Non-existence of minimizers for large charge.** We now prove a non-existence result for $\alpha \in (1,2)$ and Q large enough (depending on λ). The restriction $\alpha \in (1,2)$ comes from the fact that contrarily to what happens for the generalized Ohta-Kawasaki model ((1.1) with $\mu = 0$), we cannot easily use a cutting argument. That procedure has roughly the effect of replacing α by $\alpha + 1$ (see [28, 29, 16]) and thus allows to extend the non-existence result from $\alpha \in (1,2)$ to $\alpha \in (0,2)$.
For any $Q > 0$ we first observe that if a minimizer exists then it must be connected. The following lower bound will thus by useful to prove non-existence results.

Lemma 3.12. *There exists $c_\alpha > 0$ such that, for every $\lambda, Q > 0$ and every connected set $E \in \mathcal{M}(|B_1|)$, there holds*

$$\mathcal{F}_{\lambda,Q}(E) \ge c_\alpha \lambda^{\frac{2-\alpha}{3-\alpha}} Q^{\frac{1}{3-\alpha}}. \tag{3.9}$$

Proof. Let E be a connected set and let $d = diam(E) \ge 2$ be its diameter. If we write $E = F \setminus \cup_{i=1}^n G_i$ with F and G_i simply connected then $diam(E) = diam(F)$ and $W(E) \ge W(F)$. By [7] we have $\mathrm{diam}(F) W(F) \ge 4\pi$ so that $dW(E) \ge 4\pi$. Moreover, $P(E) \ge 2d$ and $V_\alpha(E) \ge d^{-2+\alpha}|E|^2$ and therefore

$$\mathcal{F}_{\lambda,Q}(E) \ge 2\lambda d + \frac{4\pi}{d} + Q d^{-2+\alpha} \pi^2.$$

Since

$$\min_{d \ge 2} \lambda d + Q d^{-2+\alpha} \ge c_\alpha \lambda^{\frac{2-\alpha}{3-\alpha}} Q^{\frac{1}{3-\alpha}},$$

we get (3.9). □

We will also need an estimate for the interaction energy of annuli. Since we will also use it in Section 4 in dimension 3, we state it in arbitrary dimension.

Lemma 3.13. *Consider $0 < \varepsilon \le \frac{1}{2}$ and $\alpha \in (0,d)$. There exists $C_\alpha > 0$ (depending implicitly also on the dimension d) such that*

$$V_\alpha(B_1 \setminus B_{1-\varepsilon}) \le C_\alpha \begin{cases} \varepsilon^2 & \text{if } 1 < \alpha < d, \\ \varepsilon^2 |\ln \varepsilon| & \text{if } \alpha = 1, \\ \varepsilon^{1+\alpha} & \text{if } 0 < \alpha < 1. \end{cases} \tag{3.10}$$

Proof. Let $E = B_1 \setminus B_{1-\varepsilon}$ and consider for an arbitrary $x \in E$ the potential

$$v_\alpha(x) = \int_E \frac{1}{|x-y|^{d-\alpha}}\, dy = \int_0^\infty \mathcal{H}^{d-1}(\partial B_\varrho(x) \cap E) \varrho^{-d+\alpha}\, d\varrho.$$

For any $0 < \varrho \le \varepsilon$ we have $\mathcal{H}^{d-1}(\partial B_\varrho(x) \cap E) \lesssim \varrho^{d-1}$, hence

$$\int_0^\varepsilon \mathcal{H}^{d-1}(\partial B_\varrho(x) \cap E) \varrho^{-d+\alpha}\, d\varrho \lesssim \int_0^\varepsilon \varrho^{\alpha-1} = C_\alpha \varepsilon^\alpha.$$

For $\varepsilon \le \varrho \le 1/2$ we have $\mathcal{H}^{d-1}(\partial B_\varrho(x) \cap E) \le C\varepsilon\rho^{d-2}$, hence

$$\int_\varepsilon^{1/2} \mathcal{H}^{d-1}(\partial B_\varrho(x) \cap E)\varrho^{-d+\alpha}\, d\varrho \le C\varepsilon \int_\varepsilon^{1/2} \varrho^{-2+\alpha}\, d\varrho \le C_\alpha \begin{cases} \varepsilon & \text{for } 1 < \alpha < d, \\ \varepsilon|\ln\varepsilon| & \text{for } \alpha = 1, \\ \varepsilon^\alpha & \text{for } 0 < \alpha < 1. \end{cases}$$

Finally, for $\frac{1}{2} < \varrho < \infty$ we obtain

$$\int_{1/2}^\infty \mathcal{H}^{d-1}(\partial B_\varrho(x) \cap E)\varrho^{-d+\alpha} d\varrho \le 2^{d-\alpha} \int_{1/2}^\infty \mathcal{H}^{d-1}(\partial B_\varrho(x) \cap E)\, d\varrho \le C_\alpha |E| \le C_\alpha \varepsilon.$$

From these inequalities and $V_\alpha(B_1 \setminus B_{1-\varepsilon}) = \int_E v_\alpha(x)\, dx$ the claim follows. □

We next state our main non-existence result.

Proposition 3.14. *For any $\alpha \in (1,2)$, there exists $Q_3(\alpha) > 0$ such that, for all $\lambda, Q > 0$ with $Q \ge Q_3(\alpha)(\lambda + \lambda^{\frac{\alpha-1}{2}})$, the functional $\mathcal{F}_{\lambda,Q}$ does not admit minimizers in $\mathcal{M}(|B_1|)$.*

Proof. Let us start by the case $\lambda \ge 1$. If a minimizer exists then by (3.9),

$$\min_{\mathcal{M}(|B_1|)} \mathcal{F}_{\lambda,Q}(E) \ge c_\alpha \lambda^{\frac{2-\alpha}{3-\alpha}} Q^{\frac{1}{3-\alpha}}. \tag{3.11}$$

Consider as a competitor $N \ge 2$ identical annuli of outer diameter 2. We may assume that they are so far apart that the interaction energy between different annuli becomes negligible. The inner radius is given by $r_N = \left(1 - \frac{1}{N}\right)^{1/2}$, and the elastica energy of the competitor is not larger than $10\pi N \lesssim 4\pi\lambda N$ so that the perimeter is dominant. By Lemma 3.13 the interaction energy of a single annulus is estimated by a constant times N^{-2}. Using this competitor in (3.11), we obtain

$$C_\alpha \left(\lambda N + \frac{Q}{N}\right) \ge \min_{\mathcal{M}(|B_1|)} \mathcal{F}_{\lambda,Q}(E) \ge c_\alpha \lambda^{\frac{2-\alpha}{3-\alpha}} Q^{\frac{1}{3-\alpha}}.$$

Optimizing in N we find $N \sim Q^{1/2}\lambda^{-1/2}$ and

$$\lambda^{1/2} Q^{1/2} \ge c_\alpha \lambda^{\frac{2-\alpha}{3-\alpha}} Q^{\frac{1}{3-\alpha}},$$

which leads to a contradiction if $Q \ge Q_3(\alpha)\lambda$ with $Q_3(\alpha)$ chosen large enough.

We now consider the case $\lambda \le 1$. As above, if a minimizer exists then (3.11) holds. We now consider a competitor $E_{N,R}$ given by $N \ge 2$ identical annuli of outer radius $R \ge 2$, to be optimized. We prescribe

$$R \ge \lambda^{-1/2}, \tag{3.12}$$

so that we are still in the regime where $W(E_{N,R}) \lesssim \lambda P(E_{N,R})$. If $R - \varepsilon$ is the inner radius of the annulus, then $\varepsilon \lesssim \frac{1}{RN}$ and using Lemma 3.13 we deduce that

$$V_\alpha(B_R \setminus B_{R-\varepsilon}) = R^{-2+\alpha} V_\alpha(B_1 \setminus B_{1-\varepsilon/R}) \le C_\alpha R^{-2+\alpha} N^{-2}.$$

Using $E_{N,R}$ in (3.11) we get

$$C_\alpha \left(N\lambda R + \frac{Q}{N} R^{-2+\alpha}\right) \ge c_\alpha \lambda^{\frac{2-\alpha}{3-\alpha}} Q^{\frac{1}{3-\alpha}}.$$

Optimizing the left-hand side in R yields $R = C_\alpha \left(\frac{Q}{N^2\lambda}\right)^{\frac{1}{3-\alpha}}$ and then

$$Q^{\frac{1}{3-\alpha}} N^{\frac{1-\alpha}{3-\alpha}} \lambda^{\frac{2-\alpha}{3-\alpha}} \geq c_\alpha \lambda^{\frac{2-\alpha}{3-\alpha}} Q^{\frac{1}{3-\alpha}},$$

which gives a contradiction for $N \geq N_*(\alpha)$ sufficiently large. It only remains to check that such a choice is compatible with hypothesis (3.12). The latter is equivalent to

$$Q\lambda^{-\frac{\alpha-1}{2}} \geq N_*^2(\alpha),$$

which is satisfied if $Q \geq Q_3(\alpha)\lambda^{\frac{\alpha-1}{2}}$ for $Q_3(\alpha)$ chosen sufficiently large. □

As a consequence, we obtain the following non-existence result for large masses from (3.1).

Proposition 3.15. *For every $\alpha \in (1,2)$, there exists $C_\alpha > 0$ such that if $m \geq C_\alpha \lambda^{-1}$ and $Qm^{\frac{1+\alpha}{2}} \geq C_\alpha \lambda$, there are no minimizers of $\mathcal{F}_{\lambda,Q}$ in $\mathcal{M}(m)$.*

4. The three-dimensional case

In the three dimensional case, the Willmore energy is invariant under dilations. Using Gauss-Bonnet formula it is easy to see that it is minimized by balls [52]. Moreover, sharp quantitative bounds have been obtained in [13], see also [34]. Since the Willmore energy in dimension three does not have the same degeneracy as in dimension two and already exhibits a preference for balls, we mainly study the minimization problem (1.3) without a perimeter penalization. We thus restrict ourselves to the case $\lambda = 0$, i.e. to the functional

$$\mathcal{F}_Q(E) \,=\, W(E) + QV_\alpha(E),$$

and consider the constrained minimization problem $\inf_{\mathcal{M}(m)} \mathcal{F}_Q(E)$. By a rescaling, the above minimization problem can be reduced to

$$\inf_{\mathcal{M}(|B_1|)} \mathcal{F}_Q(E). \tag{4.1}$$

4.1. Upper bound on the energy.

We start by proving a universal upper bound on the energy.

Proposition 4.1. *For any $Q > 0$ we have*

$$\inf_{\mathcal{M}(|B_1|)} \mathcal{F}_Q(E) \,\leq\, 8\pi.$$

Proof. We show that an annulus with very big radius and volume $|B_1|$ has an energy arbitrarily close to 8π. Since the Willmore energy of any annulus is 8π we need to show that the Riesz interaction energy of the annulus vanishes as the radius goes to infinity.

For an arbitrary $R > 0$ consider $\delta = \delta(R)$ such that the annulus $A_{R,R+\delta} = B_{R+\delta}(0) \setminus B_R(0)$ has volume $|B_1|$, that is $\delta = \frac{1}{3R^2} + O(R^{-5})$. Applying (3.10) to $\varepsilon = \frac{\delta}{R+\delta}$ we deduce that for $0 < \alpha < 1$

$$\begin{aligned} V_\alpha(A_{R,R+\delta}) \,&=\, (R+\delta)^{3+\alpha} V_\alpha(B_1 \setminus B_{1-\varepsilon}) \\ &\leq\, C_\alpha R^{3+\alpha}\varepsilon^{1+\alpha} \,\leq\, C_\alpha R^2 \delta^{1+\alpha} \,\leq\, C_\alpha R^{-2\alpha} \overset{R\to\infty}{\to} 0. \end{aligned}$$

We conclude similarly in the case $1 \leq \alpha < 3$. □

Let us prove that the same upper bound can be reached in the class of topological spheres.

Proposition 4.2. *For any $Q > 0$ we have*

$$\inf\left\{\mathcal{F}_Q(E) \,:\, E \in \mathcal{M}(|B_1|), \mathrm{genus}(\partial E) = 0\right\} \,\leq\, 8\pi.$$

Proof. The construction in [39] provides a sequence of sets $E_n \in \mathcal{M}$ with ∂E_n of genus zero and real numbers r_n with $r_n \nearrow 1$ as $n \to \infty$, with the following properties:

$$E_n \subset A_{r_n,1}, \qquad |E_n| > \frac{1}{2}|A_{r_n,1}| \qquad \text{and} \qquad W(E_n) \to 8\pi.$$

Considering the rescaled sets $\widetilde{E}_n = \eta_n E_n$, $\eta_n^3 = \frac{|B_1|}{|E_n|}$ we observe that $|\widetilde{E}_n| = |B_1|$ and $\eta_n \to \infty$, $W(\widetilde{E}_n) \to 8\pi$. Moreover, $\widetilde{E}_n \subset A_{\eta_n r_n, \eta_n}$ with $\eta_n r_n \to \infty$ as $n \to \infty$ and $|A_{\eta_n r_n, \eta_n}| \leq 2|B_1|$. Following the proof of Proposition 4.1 we therefore deduce that

$$V_\alpha(\widetilde{E}_n) \leq V_\alpha(A_{\eta_n r_n, \eta_n}) \overset{n\to\infty}{\to} 0.$$

□

4.2. **Area and diameter bounds.** In this section we prove uniform area and diameter bounds in the class of sets with Willmore energy strictly below 8π.

Proposition 4.3. *Let $E \in \mathcal{M}(|B_1|)$ be such that for some $\delta > 0$, we have*

$$W(E) \,\leq\, 8\pi - \delta.$$

Then, ∂E is connected and there exists a constant C_δ such that

$$\mathcal{H}^2(\partial E) + \mathrm{diam}(E)^2 \,\leq\, C_\delta. \tag{4.2}$$

If one restricts to boundaries of genus zero the proposition follows from [48]. We will prove Proposition 4.3 below and first prepare the compactness argument that we will employ. Therefore, we need to characterize limits of boundaries of sets with vanishing volume and uniformly bounded perimeter and Willmore energy. One natural approach to obtain suitable compactness properties uses the concept of varifolds, which we now quickly introduce. We refer to [49] for a detailed exposition. Since in the following lemma the dimension plays no role, we decided to give the statement in arbitrary dimension d. We let $G(d, d-1)$ be the Grassmannian manifold of unoriented $(d-1)$ planes in $\mathbb{R}^d$ and say that a Radon measure μ on $\mathbb{R}^d \times G(d, d-1)$ is an integer rectifiable varifold[2] if there exist a countably $(d-1)-$rectifiable set Σ, and a function $\theta : \Sigma \to \mathbb{N}$ such that $\mu = \theta\mathcal{H}^{d-1} \llcorner \Sigma \otimes \delta_{T\Sigma}$, i.e., for every $\psi \in C_c^0(\mathbb{R}^d \times G(d, d-1))$ there holds

$$\int_{\mathbb{R}^d \times G(d,d-1)} \psi d\mu = \int_\Sigma \psi(x, T_x\Sigma)\theta(x) d\mathcal{H}^{d-1}.$$

By a slight abuse of notation, we will consider such a μ as a measure on $\mathbb{R}^d$ i.e. we identify it with $\theta\mathcal{H}^{d-1} \llcorner \Sigma$. We say that μ has a *weak mean curvature vector* $H \in L^1_{\mathrm{loc}}(\mu, \mathbb{R}^d)$ if for all $\eta \in C_c^1(\mathbb{R}^d, \mathbb{R}^d)$ the classical first variation formula for smooth surfaces generalizes to

$$\int_{\mathbb{R}^d} \mathrm{div}_\Sigma \eta \, d\mu \,=\, -\int_{\mathbb{R}^d} H \cdot \eta \, d\mu.$$

We extend the Willmore functional to the set of integer rectifiable varifolds with weak mean curvature H, by setting

$$W(\mu) \,=\, \frac{1}{4}\int_{\mathbb{R}^d} |H|^2 \, d\mu.$$

[2]more precisely a $(d-1)$-integer rectifiable varifold, but here we restrict ourselves to the co-dimension one case, and simplify the notation.

We also recall the notion of oriented integral varifolds, introduced by Hutchinson [24]. An oriented integral varifold V^o is a Radon measure on the product $\mathbb{R}^d \times S^{d-1}$ which satisfies for every $\psi \in C_c^0(\mathbb{R}^d \times S^{d-1})$

$$V^o(\psi) \,=\, \int_\Sigma \big(\theta^+(x)\psi(x,\nu(x)) + \theta^-(x)\psi(x,-\nu(x))\, d\mathcal{H}^{d-1}(x),$$

where Σ is a countably $(d-1)$-rectifiable set and $\theta^\pm : \Sigma \to \mathbb{N}$ are such that $\theta^+ + \theta^- > 0$ almost everywhere on Σ, and where ν is a unit normal field on Σ. We can naturally associate to V^o the integral varifold $\mu = \theta\mathcal{H}^{d-1} \llcorner \Sigma$ with $\theta = (\theta^+ + \theta^-)$.

We may now prove the following compactness result (notice that the novel part is the fact that in the limit, the density θ is even).

Lemma 4.4. *Consider a sequence $(E_n)_n$ of open, bounded subsets of $\mathbb{R}^d$ with $W^{2,2}$-regular boundaries and inner unit normal field $\nu_n : \partial E_n \to S^{d-1}$. Assume that $|E_n| \to 0$, $\sup_n \mathcal{H}^{d-1}(\partial E_n) < \infty$ and $\sup_n W(\partial E_n) < \infty$.*
Then, there exists a subsequence and an integer rectifiable varifold μ on $\mathbb{R}^d$ with even density θ such that $\mathcal{H}^{d-1} \llcorner \partial E_n \rightharpoonup \mu$ as Radon measures. Moreover, μ has weak mean curvature $H \in L^2(\mu)$ and satisfies

$$W(\mu) \,\le\, \liminf_{n\to\infty} W(\partial E_n). \tag{4.3}$$

Proof. Consider the associated integer rectifiable varifolds $\mu^n = \mathcal{H}^{d-1} \llcorner \partial E_n$, and the oriented varifolds V_n^o, defined for $\psi \in C_c^0(\mathbb{R}^d \times S^{d-1})$ as

$$V_n^o(\psi) \,=\, \int_{\partial E_n} \psi(x,\nu^n(x))\, d\mathcal{H}^{d-1}(x).$$

By the bounds on $\mathcal{H}^{d-1}(\partial E_n)$ and $W(E_n)$ we know from Allard's compactness theorem [49] and [24, Theorem 3.1] that up to a subsequence, μ_n and V_n^o converge respectively to an integer rectifiable varifold μ with weak mean curvature in $L^2(\mu)$ and to an oriented integral varifold V^0. Moreover, (4.3) is satisfied. If V^o is represented by $(\Sigma, \theta^\pm, \nu)$, then for any $\psi \in C_c^0(\mathbb{R}^d)$

$$\begin{aligned}\mu(\psi) \,=\, \lim_{n\to\infty} \mu_n(\psi) \,&=\, \lim_{n\to\infty} V_n^o(\psi)\\ &=\, \int\limits_{\mathbb{R}^d\times S^{d-1}} \psi(x)\, dV^o(x,p) \,=\, \int_\Sigma (\theta^+ + \theta^-)(x)\psi(x)\, d\mathcal{H}^{d-1}(x).\end{aligned}$$

We therefore conclude that $\mu = (\theta^+ + \theta^-)\mathcal{H}^{d-1} \llcorner \Sigma$.
Now, by definition we have for any $\eta \in C_c^1(\mathbb{R}^d, \mathbb{R}^d)$

$$\int\limits_{\mathbb{R}^d\times S^{d-1}} p\cdot\eta(x)\, dV_n^o(x,p) = \int_{\partial E_n} \nu^n\cdot\eta\, d\mathcal{H}^{d-1} \,=\, -\int_{E_n} \nabla\cdot\eta\, dx.$$

Since $|E_n| \to 0$, passing to the limit in the previous equality yields for $\eta \in C_c^1(\mathbb{R}^d, \mathbb{R}^d)$

$$\int\limits_{\mathbb{R}^d\times S^{d-1}} p\cdot\eta(x)\, dV^o(x,p) = 0,$$

from which we conclude that $\theta^+ = \theta^-$ and therefore that $\theta = \theta^+ + \theta^-$is even. □

Proof of Proposition 4.3. Since the Willmore energy of any compact surface without boundary is at least 4π we deduce that ∂E is connected. It remains to prove the required bounds on the perimeter and diameter. We first prove the uniform area bound. Arguing by contradiction we assume that there exists a sequence $(E_n)_n$ in $\mathcal{M}(|B_1|)$ such that

$$\lim_{n\to\infty} \mathcal{H}^2(\partial E_n) = \infty, \qquad \text{and} \qquad W(E_n) \,\leq\, 8\pi - \delta \quad \text{for all } n \in \mathbb{N}.$$

Set $\widetilde{E}_n = \mathcal{H}^2(\partial \widetilde{E}_n)^{-\frac{1}{2}} E_n$, to obtain a sequence $(\widetilde{E}_n)_n$ such that

$$\mathcal{H}^2(\partial \widetilde{E}_n) \,=\, 1, \qquad W(\widetilde{E}_n) \,\leq\, 8\pi - \delta \quad \text{for every } n \in \mathbb{N}, \qquad \text{and} \qquad \lim_{n\to\infty} |\widetilde{E}_n| = 0.$$

Applying Lemma 4.4 we deduce that there is a non trivial (since $\mu(\mathbb{R}^3) = 1$) limit integer rectifiable varifold μ with even density θ and such that $W(\mu) < 8\pi$. This contradicts Li-Yau inequality (see [32, (A.17)]) and proves the uniform area bound. Finally, by [50, Lemma 1.1] (see [51] for the optimal constant) we obtain the diameter estimates

$$\mathcal{H}^2(\partial E)^{\frac{1}{2}} W(E)^{-\frac{1}{2}} \,\leq\, \operatorname{diam}(E) \,\leq\, \frac{2}{\pi} \mathcal{H}^2(\partial E)^{\frac{1}{2}} W(E)^{\frac{1}{2}}, \tag{4.4}$$

and (4.2) follows. □

Remark 4.5. Let $(E_n)_n$ be a minimizing sequence in $\mathcal{M}(|B_1|)$ with $\mathcal{F}_Q(E_n) \leq 8\pi - \delta$. By Proposition 4.3 we obtain that $(E_n)_n$ has uniformly bounded surface area, volume, Willmore energy, and diameter. After possibly shifting E_n we obtain a subsequence and a bounded set $E \in \mathbb{R}^3$ of finite perimeter such that $|E| = |B_1|$, with $|\nabla \mathcal{X}_E|(\mathbb{R}^3) \leq C_\delta$ and such that $\mathcal{X}_{E_n} \to \mathcal{X}_E$ in $L^1(\mathbb{R}^3)$, which in particular implies $V_\alpha(E_n) \to V_\alpha(E)$. Moreover, building on ideas of [50], one can show as in [48, p.905,p.907] (in particular exploiting the monotonicity formula [48, Theorem 3], see also [33, Section 2.1]) that $\mu = |\nabla \mathcal{X}_E|$ is an integer rectifiable varifold with mean curvature in L^2 and generalized Willmore energy

$$W(\mu) \,\leq\, \liminf_{k\to\infty} W(E_n).$$

In order to obtain existence of a minimizer in $\mathcal{M}(|B_1|)$ it remains to prove further regularity properties of ∂E (for example by adapting the arguments in [48]). This goes however beyond the scope of this work.

4.3. **Minimality of the ball for small charge.** We now prove that for $\alpha \in (1,3)$, minimizers of (4.1) are balls for small Q.

Theorem 4.6. *For any $\alpha \in (1,3)$ there exists $Q_4 > 0$ such that for every $0 < Q < Q_4$ the only minimizers of $\mathcal{F}_Q$ in $\mathcal{M}(|B_1|)$ are balls.*

Proof. For the energy of the unit ball we compute

$$\mathcal{F}_Q(B_1) \,=\, 4\pi + C_\alpha Q \,\leq\, 6\pi \quad \text{for } Q \leq Q_4 = \frac{2\pi}{C_\alpha}.$$

Let $E \in \mathcal{M}(|B_1|)$ be such that

$$\mathcal{F}_Q(E) \leq \mathcal{F}_Q(B_1).$$

This implies in particular that

$$W(E) - W(B_1) \,\leq\, Q\big(V_\alpha(B_1) - V_\alpha(E)\big). \tag{4.5}$$

Since $W(E) \leq \mathcal{F}_Q(B_1) \leq 6\pi < 2\pi^2$ by [38, Theorem A], ∂E is of sphere type. By Proposition 4.3 the diameter of E and the surface area of ∂E are bounded independently of Q. Furthermore, by [13], up to a translation there exists a $W^{2,2}$-parametrization $\Psi : \partial B_r \to \mathbb{R}^3$

of ∂E over ∂B_r, where $r \geq 1$ is chosen such that $\mathcal{H}^2(\partial B_r) = \mathcal{H}^2(\partial E)$. Thanks to the uniform bound on $\mathcal{H}^2(\partial E)$, [13] yields

$$\|\Psi - \mathrm{Id}\,\|_{W^{2,2}} \lesssim \big(W(E) - 4\pi\big)^{\frac{1}{2}} \overset{(4.5)}{\lesssim} \sqrt{Q}\big(V_\alpha(B_1) - V_\alpha(E)\big)^{\frac{1}{2}}.$$

By Sobolev embedding we also have that

$$\omega = \|\Psi - \mathrm{Id}\,\|_{C^0} \lesssim \sqrt{Q}\big(V_\alpha(B_1) - V_\alpha(E)\big)^{\frac{1}{2}} \tag{4.6}$$

becomes arbitrarily small as Q tends to zero. Furthermore, by [47] and $|E| = |B_1|$,

$$\mathcal{H}^2(\partial E) - 4\pi \lesssim \big(W(E) - 4\pi\big),$$

which implies

$$r - 1 \lesssim \big(W(E) - 4\pi\big) \lesssim Q\big(V_\alpha(B_1) - V_\alpha(E)\big). \tag{4.7}$$

Let $v_\alpha(x) = \displaystyle\int_{B_1} \frac{dy}{|x-y|^{3-\alpha}}$ denotes the potential of the unit ball. Applying [29, Lemma 4.5] we get that for every $c \in \mathbb{R}$,

$$V_\alpha(B_1) - V_\alpha(E) \leq 2\int_{B_1\setminus E}(v_\alpha - c) - 2\int_{E\setminus B_1}(v_\alpha - c) = 2\int_{\mathbb{R}^3}(\mathcal{X}_{B_1} - \mathcal{X}_E)(x)(v_\alpha - c)\,.$$

Using that v_α is radially symmetric, we can choose $c = v_\alpha\left(\frac{x}{|x|}\right)$, which by Lipschitz continuity of v_α (see [29, Lemma 4.4]) gives

$$V_\alpha(B_1) - V_\alpha(E) \leq C_\alpha \int_{E\Delta B_1} \big||x| - 1\big| \leq C_\alpha \int_{B_{r+\omega}\setminus B_{1-\omega}} \big||x| - 1\big|\,,$$

where we have used that since $r \geq 1$, $B_{1-\omega} \subset E \subset B_{r+\omega}$. Using (4.6) and (4.7), this yields for Q small

$$V_\alpha(B_1) - V_\alpha(E) \leq C_\alpha(r - 1 + \omega)^2 \leq C_\alpha Q\big(V_\alpha(B_1) - V_\alpha(E)\big),$$

which implies that if Q is small enough then $V_\alpha(E) = V_\alpha(B_1)$ and therefore $E = B_1$. □

Remark 4.7. In light of [15, Theorem 1.3], we expect this result to hold also for $\alpha \in (0, 1]$. However, since our proof relies on the rigidity estimate [13, Theorem 1.1] we must work with sets which are parameterized by a small $W^{2,2}$ function on the sphere rather than with nearly spherical sets. For such sets it is unclear how to obtain a Taylor expansion of V_α analogous to [15, Lemma 5.3]. To overcome this issue, we used that for $\alpha \in (1, 3)$ the potential v_α is Lipschitz continuous. This is not the case for $\alpha \in (0, 1]$ (see [29]). Of course, this problem would be solved if one could prove an improved convergence theorem for minimizers (or almost minimizers) of (4.1).

Remark 4.8. As a consequence of Theorem 4.6 and the isoperimetric inequality, balls are the only minimizers of $\mathcal{F}_{\lambda,Q}$ in $\mathcal{M}(|B_1|)$ for any $0 < Q < Q_4$ and any $\lambda > 0$.

Remark 4.9. By rescaling it follows from Theorem 4.6 that for $Qm^{\frac{3+\alpha}{3}} < Q_4$, the only minimizers of $\mathcal{F}_Q$ in $\mathcal{M}(m)$ are balls.

4.4. **Properties of minimizers for large charge.** For large charge, we are not able to prove or disprove that minimizers of $\mathcal{F}_Q$ exist. In the following proposition we just point out that if minimizers exist, they must have more and more degenerate isoperimetric quotient as the charge increases. This is somewhat reminiscent of [48, Theorem 1].

Proposition 4.10. *Assume that there exists a sequence $Q_n \to \infty$ such that for every $n \in \mathbb{N}$ there exists a set $E_n \in \mathcal{M}(|B_1|)$ with $\mathcal{F}_{Q_n}(E_n) < 8\pi$. Then,*

$$\lim_{n\to\infty} \mathcal{H}^2(\partial E_n) = \infty \quad \text{and} \quad \lim_{n\to\infty} W(E_n) = 8\pi.$$

Proof. We have by the minimality of E_n

$$\begin{aligned} 8\pi - W(E_n) &> Q_n V_\alpha(E_n) \\ &\geq Q_n |B_1|^2 \operatorname{diam}(E_n)^{-3+\alpha} \\ &\overset{(4.4)}{\gtrsim} Q_n \Big(\mathcal{H}^2(\partial E_n)^{\frac{1}{2}} W(E_n)^{\frac{1}{2}} \Big)^{-3+\alpha} \gtrsim Q_n \mathcal{H}^2(\partial E_n)^{\frac{-3+\alpha}{2}} W(E_n)^{\frac{-3+\alpha}{2}} \end{aligned}$$

This yields,

$$8\pi > W(E_n) + C Q_n \mathcal{H}^2(\partial E_n)^{\frac{-3+\alpha}{2}} W(E_n)^{\frac{-3+\alpha}{2}} \geq C_\alpha Q_n^{\frac{2}{5-\alpha}} \mathcal{H}^2(\partial E_n)^{\frac{-3+\alpha}{5-\alpha}},$$

where we have optimized in $W(E_n)$ in the second line. This gives $\mathcal{H}^2(\partial E_n) > C_\alpha Q_n^{\frac{2}{3-\alpha}} \to \infty$. We then conclude by (4.2) that $W(E_n) \to 8\pi$. □

4.5. **A non-existence result.** For the full functional $\mathcal{F}_{\lambda,Q}$, $\lambda > 0$ we can prove non-existence for $\alpha \in (2,3)$ and Q large enough.

Proposition 4.11. *For every $\alpha \in (2,3)$, there exists $Q_5(\alpha)$ such that for every λ, Q with $Q \geq Q_5(\lambda^{-\frac{3-\alpha}{2}} + \lambda^{\frac{3+\alpha}{2}})$, there is no minimizer of $\mathcal{F}_{\lambda,Q}$ in $\mathcal{M}(|B_1|)$.*

Proof. Assume that $Q \gg \lambda^{-\frac{3-\alpha}{2}} + \lambda^{\frac{3+\alpha}{2}}$. If a minimizer E of (1.3) exists then it must be connected. Therefore, there exists one connected component Σ of ∂E such that $\operatorname{diam}(\Sigma) = \operatorname{diam}(E)$. By (4.4), we have

$$\operatorname{diam}(E) = \operatorname{diam}(\Sigma) \lesssim \sqrt{\mathcal{H}^2(\Sigma) W(\Sigma)} \lesssim \sqrt{P(E) W(E)}.$$

Therefore,

$$\begin{aligned} \mathcal{F}_{\lambda,Q}(E) = \lambda P(E) + W(E) + Q V_\alpha(E) &\gtrsim \sqrt{\lambda} \sqrt{P(E) W(E)} + \frac{Q}{\operatorname{diam}(E)^{3-\alpha}} \\ &\gtrsim \sqrt{\lambda} \operatorname{diam}(E) + \frac{Q}{\operatorname{diam}(E)^{3-\alpha}} \gtrsim \lambda^{\frac{3-\alpha}{2(4-\alpha)}} Q^{\frac{1}{4-\alpha}}. \quad (4.8) \end{aligned}$$

For $N \gg 1$ and R to be chosen below, consider a competitor $F_{N,R}$ made of N identical annuli of inner radius R and outer radius $R+h$ with h such that each volume is equal to N^{-1}. As long as $N^{-1} \ll R^3$, we have $h \simeq R^{-2} N^{-1}$ and $h \ll R$. By (3.10), we have

$$\mathcal{F}_{\lambda,Q}(F_{N,R}) \simeq N \lambda R^2 + N + \frac{Q}{N R^{3-\alpha}}.$$

We now choose

$$R = \lambda^{-\frac{1}{2}}, \quad N = \lambda^{\frac{3-\alpha}{4}} Q^{\frac{1}{2}}$$

and observe that $N \gg 1$ since by hypothesis $Q \gg \lambda^{-\frac{3-\alpha}{2}}$ and that $N^{-1} \ll R^3$ since $Q \gg \lambda^{\frac{3+\alpha}{2}}$. We then obtain

$$\mathcal{F}_{\lambda,Q}(F_{N,R}) \simeq \left(\frac{\lambda^{3-\alpha}Q^2}{N^{\alpha-1}}\right)^{\frac{1}{5-\alpha}} + N \simeq \lambda^{\frac{3-\alpha}{4}}Q^{\frac{1}{2}}$$

which gives a contradiction to (4.8) and the fact that $\mathcal{F}_{\lambda,Q}(F_{Q,R}) \geq \mathcal{F}_{\lambda,Q}(E)$ since by hypothesis $Q \gg \lambda^{-\frac{3-\alpha}{2}}$ and $\alpha > 2$. □

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