Minimizing movements for hyperbolic obstacle-type problems and applications

Mauro Bonafini, Van Phu Cuong Le, Matteo Novaga and Giandomenico Orlandi

Abstract We survey a number of results obtained in [9, 8, 7] that provide existence of solutions for a wide class of hyperbolic obstacle-type problems, including non local operators as well as vector-valued maps. The main results are obtained through a variational scheme inspired to De Giorgi's minimizing movements. As a first application, a compactness result is derived for energy concentration sets in hyperbolic Ginzburg-Landau models for cosmology. Further applications are given for the description of the dynamics of a string interacting with a rigid substrate through an adhesive layer.

Keywords: minimizing movements, hyperbolic equations, obstacle problem, topological defects, adhesive dynamics

1 Introduction

Obstacle problems in the elliptic and parabolic setting have attracted a lot of attention in the last decade, including the case of non-local operators (see for instance [31, 11, 10, 25, 4] and references therein). In the hyperbolic setting, though, there are still few works on this subject. In this survey note we present a model for the hyperbolic obstacle problem as studied in the series of papers [9, 8, 7]. The problem can be formulated as follows: given an open bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary and a function $g \in C^0(\bar{\Omega})$, $g < 0$ on $\partial\Omega$, consider the system

Mauro Bonafini · Van Phu Cuong Le · Giandomenico Orlandi
Dipartimento di Informatica, Università di Verona, Italy
e-mail: {mauro.bonafini}{vanphucuong.le}{giandomenico.orlandi}@univr.it

Matteo Novaga
Dipartimento di Matematica, Università di Pisa, Italy
e-mail: matteo.novaga@unipi.it

$$\begin{cases} u_{tt} + (-\Delta)^s u + W'(u) \geq 0 & \text{in } (0,T) \times \Omega \\ u(t,\cdot) \geq g & \text{in } [0,T] \times \Omega \\ (u_{tt} + (-\Delta)^s u + W'(u))(u-g) = 0 & \text{in } (0,T) \times \Omega \\ u(t,x) = 0 & \text{in } [0,T] \times (\mathbb{R}^d \setminus \Omega) \\ u(0,x) = u_0(x) & \text{in } \Omega \\ u_t(0,x) = v_0(x) & \text{in } \Omega \end{cases} \tag{1}$$

under the following assumptions:

(i) $u_0 \in \tilde{H}^s(\Omega)$, $v_0 \in L^2(\Omega, \mathbb{R}^m)$, $u_0 \geq g$ a.e. in Ω, where

$$\tilde{H}^s(\Omega) := \Big\{ u \in L^2(\mathbb{R}^d;\, \mathbb{R}^m) \text{ s.t.} \int_{\mathbb{R}^d} (1 + |\xi|^{2s}) |\mathcal{F}u(\xi)|^2 \, d\xi < +\infty,\ \ u = 0 \text{ a.e. in } \mathbb{R}^d \setminus \Omega \Big\}, \tag{2}$$

with $\mathcal{F}$ the Fourier transform (see [15, 21] for more details about the introduction to fractional Sobolev spaces);

(ii) W is a continuous potential with Lipschitz continuous derivative;

(iii) for $s > 0$ the operator $(-\Delta)^s$ stands for the fractional s-Laplacian.

We refer the reader to [8] for main terminology and notations. Concerning problem (1), we investigate both the obstacle-free case and the case where an obstacle is present (in this case, we consider only the scalar case $m = 1$ in the system (1)). In the obstacle case, recall the work of Schatzman and collaborators (see e.g. [28, 29, 30, 26]) where the authors provided an existence and uniqueness result in a suitable setting for problem (1) (the wave equation in the 1-dimensional case) by making use of the validity of the representation formula for solutions of the free wave equation. The approach allows to prescribe how the solution behaves at contact times (e.g. when a string bounces elastically at the contact point). In another direction, by using a time semidiscrete method, the 1-dimensional obstacle problem for the linear wave equation has been treated first in [20] and then adapted in [16]. More recently, similar time semidiscrete methods have also been used to study hyperbolic free boundary problems (see [1]). By following the approach in [20], in [9] a variational time semi-discrete scheme inspired to De Giorgi's minimizing movements is implemented, yielding uniform energy estimates for the approximate solutions of (1) also in the presence of an obstacle, first in the linear case, i.e. when $W = 0$, and subsequently generalized in [8] to the semilinear case $W \neq 0$. Those energy estimates have been proved to be valid also when dealing with non local operators like the fractional Laplacian, and for vector-valued u, yielding existence results to problem (1), at least in the obstacle-free case, also in this more general non local, vector-valued setting.

As a first application we show some compactness results for energy concentration sets in singular limits of hyperbolic Ginzburg-Landau equations, which describe topological defects in cosmological models. Second, we show how the results obtained in [8] are employed in [7] to study the dynamics of a string interacting with a rigid substrate through an adhesive layer, extending the results in [13, 14]. In addition, in a paper in preparation we will also use the results obtained in [8] to study existence results for a class of hyperbolic equations in moving domains. The variational scheme used in [20, 9, 8] relies on De Giorgi's minimizing movements [3] and has been used in many different contexts. In this context it is also known as Morse semi-flow or Rothe's method [27].

The organization of this note is as follows: in Section 2, we introduce the variational scheme and present the existence results for the obstacle-free case as well as its application to singular limits of the hyperbolic Ginzburg-Landau equation. In Section 3, we present the existence results in case the obstacle is present, and finally in Section 4 we discuss the application to adhesive phenomena.

2 Weak solutions for the fractional semilinear wave equations (obstacle-free case)

In this section, we shall introduce the time semidiscrete method to the obstacle-free case, energy estimates, and then we present existence results obtained in [9, 8], which can be also seen as the first step to study the obstacle case. Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain with Lipschitz boundary. For $u = u(t,x) : (0,T) \times \mathbb{R}^d \to \mathbb{R}^m$, let us consider the system

$$\begin{cases} u_{tt} + (-\Delta)^s u + \nabla_u W(u) = 0 & \text{in } (0,T) \times \Omega \\ u(t,x) = 0 & \text{in } [0,T] \times (\mathbb{R}^d \setminus \Omega) \\ u(0,x) = u_0(x) & \text{in } \Omega \\ u_t(0,x) = v_0(x) & \text{in } \Omega \end{cases} \tag{3}$$

with initial data $u_0 \in \tilde{H}^s(\Omega)$ and $v_0 \in L^2(\Omega) := L^2(\Omega; \mathbb{R}^m)$ (we conventionally intend that $v_0 = 0$ in $\mathbb{R}^d \setminus \Omega$), and a non-negative potential $W \in C^1(\mathbb{R}^m; \mathbb{R})$ having Lipschitz continuous derivative with Lipschitz constant $K > 0$, i.e.,

$$|\nabla W(x) - \nabla W(y)| \le K|x - y| \quad \text{for any } x, y \in \mathbb{R}^m. \tag{4}$$

Notice that since we consider also non-local operators, the boundary condition is imposed on the whole complement of Ω.

We define a weak solution of (3) as follows:

Definition 1 Let $T > 0$. We say $u = u(t,x)$ is a weak solution of (3) in $(0,T)$ if

1. $u \in L^\infty(0,T; \tilde{H}^s(\Omega)) \cap W^{1,\infty}(0,T; L^2(\Omega))$ and $u_{tt} \in L^\infty(0,T; H^{-s}(\Omega))$,
2. for all $\varphi \in L^1(0,T; \tilde{H}^s(\Omega))$

$$\int_0^T \langle u_{tt}(t), \varphi(t) \rangle dt + \int_0^T [u(t), \varphi(t)]_s \, dt + \int_0^T \int_\Omega \nabla_u W(u(t)) \cdot \varphi(t) \, dx dt = 0 \tag{5}$$

with

$$u(0,x) = u_0 \quad \text{and} \quad u_t(0,x) = v_0. \tag{6}$$

The energy of u is defined as

$$E(u(t)) = \frac{1}{2}||u_t(t)||^2_{L^2(\Omega)} + \frac{1}{2}[u(t)]^2_s + ||W(u(t))||_{L^1(\Omega)}, \quad t \in [0,T].$$

The main Theorem of this section is the following:

Theorem 1

(i) There exists a weak solution of the fractional semilinear wave equation (3) *such that it satisfies the energy inequality:*

$$E(u(t)) \leq E(u(0)) \quad \textit{for any } t \in [0,T]. \tag{7}$$

(ii) Assume $u_0 \in \tilde{H}^{2s}(\Omega)$ *and* $v_0 \in \tilde{H}^s(\Omega)$*. Then, there exists a solution u of equation* (3) *such that* $u \in W^{1,\infty}(0,T;\tilde{H}^s(\Omega))$, $u_t \in W^{1,\infty}(0,T;L^2(\Omega))$*. Moreover,*

$$E(u(t)) = E(u(0)) \quad \textit{for any } t \in [0,T], \tag{8}$$

i.e. the energy of u is conserved during the evolution.

(iii) The equation (3) *has unique solution in the class:* $X = \{ u \,|\, u \text{ is a weak solution of } (3),\ u_t \in L^\infty(0,T;\tilde{H}^s(\Omega))\}$ *in the sense that if* $v, w \in X$*, then for each* $t \in [0,T]$

$$v(t) = w(t) \text{ in } \tilde{H}^s(\Omega).$$

In particular the solution found in point (ii)*, since it belongs to X, it is unique.*

The solutions in Theorem 1 are constructed by the following scheme.

2.1 Approximating scheme

For $n \in \mathbb{N}$, set $\tau_n = T/n$ and define $t_i^n = i\tau_n$, $0 \leq i \leq n$. Let $u_{-1}^n = u_0 - \tau_n v_0$, $u_0^n = u_0$ and for every $i \geq 1$ let

$$\begin{aligned} J_i^n(u) &= \left[\int_\Omega \frac{|u - 2u_{i-1}^n + u_{i-2}^n|^2}{2\tau_n^2}\, dx + \frac{1}{2}[u]_s^2 + \int_\Omega W(u)dx \right], \\ u_i^n &\in \arg \min_{u \in \tilde{H}^s(\Omega)} J_i^n(u) \end{aligned} \tag{9}$$

By using the direct method of the calculus of variations, each J_i^n admits a minimizer u_i^n in $\tilde{H}^s(\Omega)$ (the uniqueness of minimizers is not guaranteed in the nonlinear case). The Euler's equation of u_i^n:

$$\int_\Omega \left(\frac{u_i^n - 2u_{i-1}^n + u_{i-2}^n}{\tau_n^2} \right) \cdot \varphi\, dx + [u_i^n, \varphi]_s + \int_\Omega \nabla_u W(u_i^n) \cdot \varphi dx = 0 \tag{10}$$

for every $\varphi \in \tilde{H}^s(\Omega)$. Then, we define the piecewise constant and piecewise linear interpolations over $[-\tau_n, T]$ as follows:

- piecewise constant interpolant

$$\bar{u}^n(t,x) = \begin{cases} u^n_{-1}(x) & t = -\tau_n \\ u^n_i(x) & t \in (t^n_{i-1}, t^n_i], \end{cases} \tag{11}$$

- piecewise linear interpolant

$$u^n(t,x) = \begin{cases} u^n_{-1}(x) & t = -\tau_n \\ \dfrac{t - t^n_{i-1}}{\tau_n} u^n_i(x) + \dfrac{t^n_i - t}{\tau_n} u^n_{i-1}(x) & t \in (t^n_{i-1}, t^n_i]. \end{cases} \tag{12}$$

The strategy in proving Theorem 1 is to exploit the Euler's equation of u^n_i to provide an energy estimates on u^n_i, after that passing to the limit as $n \to \infty$ in the Euler's equation and prove that u^n and $\bar{u}^n$ converge to a weak solution of (3) (see [8, Section 3] for more details about the scheme). We have the following energy estimate ([8, Proposition 4]):

Proposition 1 (Key estimate)

The approximate solutions $\bar{u}^n$ and u^n satisfy

$$\frac{1}{2}\left\|u^n_t(t)\right\|^2_{L^2(\Omega)} + \frac{1}{2}[\bar{u}^n(t)]^2_s + ||W(\bar{u}^n(t))||_{L^1(\Omega)} \le E(u(0)) + C\tau_n$$

for all $t \in [0,T]$, with $C = C(E(u(0)), K, T)$ a constant independent of n.

Then, we can derive compactness results of $u^n, \bar{u}^n, W(u^n), W(\bar{u}^n)$, and $\nabla_u W(\bar{u}^n)$.

Proposition 2 (Convergence of u^n and v^n)

There exist a subsequence of steps $\tau_n \to 0$ and a function $u \in L^\infty(0,T;\tilde{H}^s(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega))$, with $u_{tt} \in L^\infty(0,T;H^{-s}(\Omega))$, such that

$$u^n \to u \text{ in } C^0([0,T];L^2(\Omega)), \qquad u^n_t \rightharpoonup^* u_t \text{ in } L^\infty(0,T;L^2(\Omega)),$$
$$u^n(t) \rightharpoonup u(t) \text{ in } \tilde{H}^s(\Omega) \text{ for any } t \in [0,T], \qquad v^n \to u_t \text{ in } C^0([0,T];H^{-s}(\Omega)),$$
$$v^n_t \rightharpoonup^* u_{tt} \text{ in } L^\infty(0,T;H^{-s}(\Omega)).$$

Proposition 3 (Convergence of $\bar{u}^n$, $W(\bar{u}^n)$, and $\nabla_u W(\bar{u}^n)$)

There exist a subsequence of steps $\tau_n \to 0$ and a function $u \in L^\infty(0,T;\tilde{H}^s(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega))$, with $u_{tt} \in L^\infty(0,T;H^{-s}(\Omega))$, such that

$$\bar{u}^n \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;\tilde{H}^s(\Omega)),$$
$$\bar{u}^n(t) \rightharpoonup u(t) \text{ in } \tilde{H}^s(\Omega) \text{ for any } t \in [0,T],$$
$$W(\bar{u}^n) \to W(u) \text{ in } C^0([0,T];L^1(\Omega)),$$
$$\nabla_u W(\bar{u}^n) \overset{*}{\rightharpoonup} \nabla_u W(u) \text{ in } L^\infty(0,T;H^{-s}(\Omega)).$$

Then, by passing to the limit in (10) we obtain Theorem 1. To see the conservative property, we need to prove that if the initial data are more regular, then the limiting solutions also has higher regularity, which in turn allows to obtain energy conservation (see [8, Proposition 10]).

2.2 Singular limits of hyperbolic Ginzburg-Landau equations

In this section, we focus only on the cases $m = 1$, $m = 2$, we consider the hyperbolic Ginzburg-Landau equations:

$$\begin{cases} \varepsilon^2 \left(\dfrac{\partial^2 u_\varepsilon}{\partial t^2} - \Delta u_\varepsilon \right) + \nabla_u W(u_\varepsilon) = 0 & \text{in } (0,T) \times \Omega, \\ u_\varepsilon(0,x) = u^0_\varepsilon(x) & \text{in } \Omega, \\ u_{\varepsilon t}(0,x) = v^0_\varepsilon(x) & \text{in } \Omega, \end{cases} \tag{13}$$

where $\varepsilon > 0$ is a small parameter, Ω is a bounded domain in $\mathbb{R}^d$, for functions

$$u_\varepsilon : (0,T) \times \Omega \to \mathbb{R}^m, \tag{14}$$

W is a non-convex balanced double-well potential of class C^2 and we assume that the potential is given by

$$W(u) = \frac{(1 - |u|^2)^2}{1 + |u|^2}. \tag{15}$$

Under natural bounds on initial energy, we have the following compactness results on the interfaces ($m = 1$) and the vorticity ($m = 2$), which are so-called topological defects (for the relevance of topological defects to cosmology, we refer the reader to [24, 5, 6, 17, 22]).

Proposition 4 *Let* $(u_\varepsilon)_{0<\varepsilon<1}$ *be a sequence of solutions of* (13) *constructed by the approximating scheme in Section* 2 *for each* $0 < \varepsilon < 1$ *fixed such that* $\frac{E(u_\varepsilon(0))}{k_\varepsilon} \leq C$ *where* C *is a constant independent of* ε*,* $k_\varepsilon = \frac{1}{\varepsilon}$ *for* $m = 1$ *and* $k_\varepsilon = |\log \varepsilon|$ *for* $m = 2$*. Then, up to a subsequence* $\varepsilon_n \to 0$*,*

(i) in case $m = 1$*,*

$$u_{\varepsilon_n} \to u \text{ in } L^1((0,T) \times \Omega),$$

where $u(t,x) \in \{-1, 1\}$ *for a.e.* $(t,x) \in (0,T) \times \Omega$*, and* $u \in BV((0,T) \times \Omega)$ *(see [23]),*

(ii) in case $m = 2$*,*

$$Ju_{\varepsilon_n} \rightharpoonup J \text{ in } [C^{0,1}((0,T) \times \Omega)]^*,$$

where $Ju_\varepsilon = du^1_\varepsilon \wedge du^2_\varepsilon$ *is the distributional Jacobian defined on* $(0,T) \times \Omega$ *(see for instance [18, 2]), and* $\frac{1}{\pi} J$ *is a* $d-1$ *dimensional integral current in* $(0,T) \times \Omega$ *(see [19]).*

3 Weak solutions for the obstacle problem for fractional semilinear wave equations

In this section, we consider the obstacle case given by (1) with $m = 1$. We define a weak solution of (1) as follows:

Definition 2 Let $T > 0$. We say $u = u(t,x)$ is a weak solution of (1) in $(0,T)$ if

1. $u \in L^\infty(0,T;\tilde{H}^s(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega))$ and $u(t,x) \geq g(x)$ for a.e. $(t,x) \in (0,T) \times \Omega$;
2. there exist weak left and right derivatives $u_t^\pm$ on $[0,T]$ (with appropriate modifications at endpoints);
3. for all $\varphi \in W^{1,\infty}(0,T;L^2(\Omega)) \cap L^1(0,T;\tilde{H}^s(\Omega))$ with $\varphi \geq 0$, $\operatorname{spt}\varphi \subset [0,T)$, we have
$$-\int_0^T \int_\Omega u_t \varphi_t \, dxdt + \int_0^T [u,\varphi]_s \, dt + \int_0^T \int_\Omega W'(u)\varphi dxdt - \int_\Omega v_0 \, \varphi(0) \, dx \geq 0$$
4. the initial conditions are satisfied in the following sense
$$u(0,\cdot) = u_0, \quad \int_\Omega (u_t^+(0) - v_0)(\varphi - u_0) \, dx \geq 0 \quad \forall \varphi \in \tilde{H}^s(\Omega), \varphi \geq g.$$

By a slightly modification of the approximating scheme in the Section 2.1 and using the same strategy, we can prove the following Theorem:

Theorem 2 *There exists a weak solution u of the obstacle problem for the fractional semilinear wave equation* (1)*, and u satisfies*
$$\begin{aligned}\frac{1}{2}||u_t^\pm(t)||^2_{L^2(\Omega)} + \frac{1}{2}[u(t)]_s^2 + ||W(u(t))||_{L^1(\Omega)} \\ \leq \frac{1}{2}||v_0||^2_{L^2(\Omega)} + \frac{1}{2}[u_0]_s^2 + ||W(u_0)||_{L^1(\Omega)}\end{aligned} \tag{16}$$
for a.e. $t \in [0,T]$.

3.1 Approximating scheme

For $n \in \mathbb{N}$, set $\tau_n = T/n$ and define $t_i^n = i\tau_n$, $0 \leq i \leq n$. Let $u_{-1}^n = u_0 - \tau_n v_0$, $u_0^n = u_0$ and define
$$K_g := \{u \in \tilde{H}^s(\Omega) \mid u \geq g \text{ a.e. in } \Omega\}.$$
For every $0 < i \leq n$, given u_{i-2}^n and u_{i-1}^n, we define u_i^n as
$$u_i^n \in \arg\min_{u \in K_g} J_i^n(u),$$

where J_i^n is defined as in (9). Then, a variational characterization of each minimizer u_i^n can be provided as follows: take $\varphi \in K_g$ and consider the function $(1-\varepsilon)u_i^n + \varepsilon\varphi$, which belongs to K_g for ε small enough. By the minimality of u_i^n, we have

$$\frac{d}{d\varepsilon} J_i^n(u_i^n + \varepsilon(\varphi - u_i^n))|_{\varepsilon=0} \geq 0,$$

which is equivalent to

$$\int_\Omega \frac{u_i^n - 2u_{i-1}^n + u_{i-2}^n}{\tau_n^2}(\varphi - u_i^n)\, dx + [u_i^n, \varphi - u_i^n]_s + \int_\Omega W'(u_i^n)(\varphi - u_i^n) dx \geq 0 \quad (17)$$

for all $\varphi \in K_g$. By choosing the test function $\varphi = u_{i-1}^n$ in (17), and replicating the proof of Proposition 1, we obtain the same energy estimate

$$\frac{1}{2}\left\|u_t^n(t)\right\|^2_{L^2(\Omega)} + \frac{1}{2}[\bar{u}^n(t)]_s^2 + ||W(\bar{u}^n(t))||_{L^1(\Omega)} \leq E(u(0)) + C\tau_n$$

for all $t \in [0, T]$, with $C = C(E(u(0)), K, T)$ a constant independent of n (we refer the reader to [8, Section 4] for more details).

4 Nonlinear waves in adhesive phenomena

In this last section, we shall present results obtained in [7], where the first two authors investigate the dynamic of a string interacting with a rigid substrate through an adhesive layer, which was initially studied in [13, 14] in 1-dimensional setting. We consider the system (3), where the potential W responds for the energetic contribution of the glue layer having the following behavior:

(i) In the case $\nabla_u W$ discontinuously drops to zero:

$$W(y) = \begin{cases} |y|^2 & \text{if } y \in \overline{\mathbf{B}(0,1)} \\ 1 & \text{if } y \notin \mathbf{B}(0,1) \end{cases} \quad (4.18)$$

where $\mathbf{B}(0,1) = \{ y \in \mathbb{R}^m \mid |y| < 1 \}$, $\overline{\mathbf{B}(0,1)} = \{ y \in \mathbb{R}^m \mid |y| \leq 1 \}$, and we define

$$\nabla W(y) = \begin{cases} 2y & \text{if } y \in \overline{\mathbf{B}(0,1)} \\ 0 & \text{if } y \notin \mathbf{B}(0,1) \end{cases} \quad (4.19)$$

In this case, we define the weak solutions as follows:

Definition 3 (Weak solution and energy for the discontinuous case) Let $T > 0$. We say $u = u(t, x)$ is a weak solution of (3) in $(0, T)$ if

a. $u \in L^\infty(0, T; \tilde{H}^s(\Omega)) \cap W^{1,\infty}(0, T; L^2(\Omega))$ and $u_{tt} \in L^\infty(0, T; H^{-s}(\Omega))$,
b. $\nabla_u W(u) \in L^\infty(0, T; H^{-s}(\Omega))$,

c. for all $\varphi \in L^1(0,T;\tilde{H}^s(\Omega))$,

$$\int_0^T \langle u_{tt}(t), \varphi(t)\rangle dt + \int_0^T [u(t), \varphi(t)]_s \, dt + \int_0^T \int_\Omega \nabla_u W(u(t))\varphi(t)\, dxdt = 0 \tag{4.20}$$

with

$$u(0,x) = u_0 \quad \text{and} \quad u_t(0,x) = v_0. \tag{4.21}$$

The energy of u is defined as

$$E(u(t)) = \frac{1}{2}||u_t(t)||^2_{L^2(\Omega)} + \frac{1}{2}[u(t)]^2_s + ||W(u(t))||_{L^1(\Omega)} \quad \text{for } t \in [0,T].$$

we prove the existence of solutions under small conditions on the initial data combined with $2s > d$.

Theorem 4.3 *Consider $2s > d$, W, $\nabla_u W$ as defined in* (4.18)*,* (4.19) *respectively and assume that*

$$||u_0||_{\tilde{H}^s(\Omega)} \leq \varepsilon_1,\; ||v_0||_{L^2(\Omega)} \leq \varepsilon_2 \tag{4.22}$$

for sufficiently small ε_1, ε_2. Then, there exists a weak solution of problem (3) *in the sense of Definition* 3 *with*

$$|u(x,t)| < 1 \quad \textit{for all } (t,x) \in [0,T] \times \Omega \tag{4.23}$$

and

$$E(u(t)) \leq E(u(0)) \quad \textit{for any } t \in [0,T]. \tag{4.24}$$

(ii) In the case the glue layer, namely $\nabla_u W$, continuously decays to zero, we still define weak solutions as in Definition 1. We have the following Theorem:

Theorem 4.4 *Let $W \in C^1(\mathbb{R}^m)$, and W be non-negative. Assume there exists $K > 0$ such that $0 \leq W(y) \leq K$ and $0 \leq |\nabla W(y)| \leq K$ for all $y \in \mathbb{R}^m$, with ∇W uniformly continuous. Then, there exists a weak solution of* (3) *satisfying the energy inequality*

$$E(u(t)) \leq E(u(0)) \quad \textit{for any } t \in [0,T]. \tag{4.25}$$

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