The Total Variation Flow

Matteo Novaga

Abstract. I consider the gradient flow of the total variation functional, stating general existence and uniqueness results. Particular attention is paid to self-similar solutions, which are partially classified. Finally, as an application, some explicit solutions of the denoising problem are given.

1. Introduction

The aim of this paper is to analyze the following variational parabolic equation

$$u_t - \operatorname{div}\left(\frac{Du}{|Du|}\right) = 0, \qquad u \in L^1_{\mathrm{loc}}(]0,+\infty[\,\times\mathbb{R}^N), \tag{1}$$

coupled with the initial condition

$$u(0,x) = u_0(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^N), \tag{2}$$

illustrating results which have been proved in [5], [6].

This equation corresponds to the L^2-gradient flow for the total variation functional

$$u \longrightarrow \int_{\mathbb{R}^N} |Du|,$$

starting from u_0. Such a flow has many applications to image denoising and reconstruction, and numerical simulations have been performed by several authors [13], [10], [9], [16], [15]. Existence and uniqueness for solutions of (1) has first been established in the case of bounded domains (see [4] and references therein) and then extended to $\mathbb{R}^N$ in [5] (see Theorem 2.4 below). The notion of solution employed in these papers is the so-called *entropy solution*, introduced by Kruzhkov [12] for scalar conservation laws, and first applied to parabolic equations in divergence form by Andreu *et al.* [2], in order to prove uniqueness with initial data in L^1_{loc}.

It has been proved in [3] that the solution of (1) reaches its asymptotic state in finite time, with an extinction profile which solves (up to a rescaling) the following eigenvalue problem

$$-\operatorname{div}\left(\frac{Du}{|Du|}\right) = u, \qquad u \in L^1_{\mathrm{loc}}(\mathbb{R}^N). \tag{3}$$

In [6] (see also Sections 3 and 4) the solutions of (3) are partially characterized, in the case $N = 2$. Particular relevance is given to solutions which are in $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$

(see Proposition 3.4), or which are a finite sum of characteristic functions (see Theorems 4.1, 4.6 and 4.7).

The solutions u which have no jumps, i.e. $(-M) \vee u \wedge M \in W^{1,1}_{\text{loc}}(\mathbb{R}^N)$ for any $M > 0$, can be constructed starting from the "fundamental solutions" $u(x) = \pm\frac{1}{|x|}$ (Proposition 3.4), and can be obtained as limits of "tower solutions".

If a solution u of (3) is positive, then all the connected components of the sets $\{u \geq t\}$ are convex (see Proposition 3.1), and one can construct solutions which look like a tower, i.e. $u = \sum_{i=1}^{p} \alpha_i \chi_{B_i}$, for any choice of balls B_i such that $B_1 \supset B_2 \supset \ldots \supset B_p$ and for suitable constants $\alpha_i > 0$ (see Section 5). More generally, at least in two dimensions, one can construct more complicated towers (which can also oscillate) starting from sets which are not necessarily circles, but which must satisfy a suitable condition on the curvature of the boundary and on the mutual distance (see Theorem 4.6 and Section 5). These towers correspond to solutions of (3) which change sign, and in turn provide new solutions of problem (4) below.

Indeed, a related question is to find solutions of the so-called denoising problem, i.e.

$$\min_{u \in BV(\mathbb{R}^2)} \int_{\mathbb{R}^2} |Du| + \frac{1}{2\lambda} \int_{\mathbb{R}^2} (u - f)^2 \, dx, \tag{4}$$

and, in particular, to determine functions $f \in L^2(\mathbb{R}^2)$ for which the solution u of (4) is constructed by the soft-thresholding rule. In Propositions 6.1 and 6.2 below, it is shown how to construct solutions of problem (4), starting from solutions of (3).

2. Existence and uniqueness of solutions

In the following, I denote by $L^1_w(]0,T[; BV(\mathbb{R}^N))$ the space of functions $w : [0,T] \to BV(\mathbb{R}^N)$ such that $w \in L^1(]0,T[\times\mathbb{R}^N)$, the maps $t \to \int_{\mathbb{R}^N} \phi dDw(t)$ are measurable for any $\phi \in C^1_0(\mathbb{R}^N;\mathbb{R}^N)$ and $\int_0^T |Dw(t)|(\mathbb{R}^N)dt < +\infty$. I denote by $L^1_w(]0,T[; BV_{\text{loc}}(\mathbb{R}^N))$ the space of functions $w : [0,T] \to BV_{\text{loc}}(\mathbb{R}^N)$ such that $w\phi \in L^1_w(]0,T[; BV(\mathbb{R}^N))$ for any $\phi \in C^\infty_0(\mathbb{R}^N)$.

Definition 2.1. *A function $u \in C([0,T]; L^2(\mathbb{R}^N))$ is called a strong solution of (1) if*

$$u \in W^{1,2}_{\text{loc}}(0,T; L^2(\mathbb{R}^N)) \cap L^1_w(]0,T[; BV(\mathbb{R}^N))$$

and there exists $z \in L^\infty\left(]0,T[\times\mathbb{R}^N;\mathbb{R}^N\right)$ with $\|z\|_\infty \leq 1$ such that

$$u_t = \text{div } z \qquad \text{in } \mathcal{D}'\left(]0,T[\times\mathbb{R}^N\right)$$

and

$$\int_{\mathbb{R}^N} (u(t) - w) u_t(t) = \int_{\mathbb{R}^N} (z(t), Dw) - \int_{\mathbb{R}^N} |Du(t)| \quad \forall w \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N), \tag{5}$$

for a.e. $t \in [0,T]$.

Notice that a solution of (1) of class C^1 and such that $Du \neq 0$ is also a strong solution, setting $z := \frac{Du}{|Du|}$.

I recall from [5, Theorem 2] the following result.

Theorem 2.2. *Let* $u_0 \in L^2(\mathbb{R}^N)$. *Then there exists a unique strong solution* u *of (1) and (2) in* $[0,T] \times \mathbb{R}^N$ *for every* $T > 0$. *Moreover, if* u *and* v *are the strong solutions of (1) corresponding to the initial conditions* $u_0, v_0 \in L^2(\mathbb{R}^N)$, *then*

$$\|(u(t) - v(t)) \vee 0\|_2 \leq \|(u_0 - v_0) \vee 0\|_2 \quad \text{for any } t > 0. \tag{6}$$

Since (1) is the gradient flow of a convex functional on $L^2(\mathbb{R}^N)$, which is an Hilbert space, Theorem 2.2 can be proved by means of classical techniques, up to minor modifications (see [7] for a general approach to these problems).

Notice that, putting $w = u(t)$ in (5), it follows

$$\int_{\mathbb{R}^N} (z, Du) = \int_{\mathbb{R}^N} |Du|. \tag{7}$$

Let us give a heuristic explanation of what the vector field z represents. Given a set $E \subseteq \mathbb{R}^N$ of finite perimeter, denote by $\partial^* E$ the reduced boundary of E in the sense of De Giorgi (see [1]). Condition (7) essentially means that z has unit norm and is orthogonal to the level sets of u. In some sense, z is invariant under local contrast changes. To be more precise, observe that if $u = \sum_{i=1}^p c_i \chi_{E_i}$ where E_i are sets of finite perimeter with disjoint closure, $c_i \in \mathbb{R}$ and

$$-\mathrm{div}\left(\frac{Du}{|Du|}\right) = f \in L^2(\mathbb{R}^N), \tag{8}$$

then also $-\mathrm{div}\,(\frac{Dv}{|Dv|}) = f$ for any $v = \sum_{i=1}^p d_i \chi_{E_i}$ where $d_i \in \mathbb{R}$ and $\mathrm{sign}(d_i) = \mathrm{sign}(c_i)$. In an informal way, this means that a local contrast change of a solution of (8) produces a new solution.

2.1. Entropy solutions

In order to generalize the previous result by allowing initial data which are only in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$, I introduce the set $\mathcal{P} \subset W^{1,\infty}(\mathbb{R})$ as follows

$$\mathcal{P} := \{p \in W^{1,\infty}(\mathbb{R}) : p' \geq 0, \ \mathrm{spt}(p') \text{ compact}\}.$$

Definition 2.3. *A function* $u \in C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ *is called an entropy solution of (1) and (2) if* $u(t)$ *converges to* u_0 *in* $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ *as* $t \to 0^+$,

$$p(u) \in L^1_w(]0,T[; BV_{\mathrm{loc}}(\mathbb{R}^N)) \qquad \forall p \in \mathcal{P},$$

and there exists $z \in L^\infty\left(]0,T[\times\mathbb{R}^N;\mathbb{R}^N\right)$ *with* $\|z\|_\infty \leq 1$ *such that*

$$u_t = \mathrm{div}\ z \qquad \text{in } \mathcal{D}'\left(]0,T[\times\mathbb{R}^N\right) \tag{9}$$

and

$$-\int_0^T\int_{\mathbb{R}^N} j(u-l)\eta_t + \int_0^T\int_{\mathbb{R}^N} \eta\, d\left|D\left(p(u-l)\right)\right| + \int_0^T\int_{\mathbb{R}^N} z\cdot\nabla\eta\, p(u-l) \le 0 \tag{10}$$

for all $l \in \mathbb{R}$, *all* $\eta \in C^\infty\left(]0,T[\times\mathbb{R}^N\right)$, *with* $\eta \ge 0$, $\eta(t,x) = \phi(t)\psi(x)$, *being* $\phi \in C_0^\infty\left(]0,T[\right)$, $\psi \in C_0^\infty(\mathbb{R}^N)$, *and all* $p \in \mathcal{P}$, *where* $j(r) := \int_0^r p(s)\, ds$.

Inequality (10) is a weak way to impose equality (5); indeed, substituting u_t with $\mathrm{div} z$ in (5) and integrating by parts, using also $\|z\|_\infty \le 1$ and the fact that η is nonnegative, one gets

$$\begin{aligned}\int_{\mathbb{R}^N} z\cdot\nabla\eta\; p(u-l) &= -\int_{\mathbb{R}^N} j(u-l)_t\eta - \int_{\mathbb{R}^N}\eta\; d(z, D\left(p(u-l)\right)) \\ &\ge -\int_{\mathbb{R}^N} j(u-l)_t\eta - \int_{\mathbb{R}^N}\eta\; d|(z, D\left(p(u-l)\right))|,\end{aligned}$$

which, after integration in time, gives the opposite inequality in (10).

From [5, Theorem 3, Proposition 4] one gets the following results.

Theorem 2.4. *Let* $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. *Then there exists a unique entropy solution of (1) and (2) in* $[0,T]\times\mathbb{R}^N$ *for all* $T>0$. *Moreover, if* $u_0, u_{0k} \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ *are such that* $u_{0k} \to u_0$ *in* $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ *and* u, u_k *denote the corresponding entropy solutions, then* $u_k \to u$ *in* $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ *as* $k\to+\infty$.

Theorem 2.5. *Let* $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ *and* u *be the entropy solution of (1) and (2). Then*

$$p(u)_t \in L^2_{\mathrm{loc}}(0,\infty; L^2_{\mathrm{loc}}(\mathbb{R}^N)), \qquad t^{\frac12}p(u)_t \in L^2_{\mathrm{loc}}([0,\infty[; L^2_{\mathrm{loc}}(\mathbb{R}^N)), \qquad \forall p\in\mathcal{P}.$$

Moreover, if $u_0 \ge -M$ *for some* $M>0$, *it follows*

$$u'(t) \le \frac{u(t)+M}{t} \qquad \text{for a.e. } t>0.$$

Finally,

$$u_t \in L^1_{\mathrm{loc}}\left(]0,T[; L^1_{\mathrm{loc}}(\mathbb{R}^N)\right)$$

for any $T>0$. *A similar statement holds if* $u_0 \le M$ *for some* $M>0$.

Remark 2.6. *If* $u_0 \in L^2(\mathbb{R}^N)$, *then the strong solution of (1) and (2) coincides with the (unique) entropy solution.*

3. Self–similar solutions

The aim of this section is to characterize the solutions of (3), when $N = 2$, i.e. to characterize the solutions of equation (1) which are self-similar. Therefore, I shall consider to the following eigenvalue problem

$$-\mathrm{div}\left(\frac{Du}{|Du|}\right) = u, \qquad u \in L^1_{\mathrm{loc}}(\mathbb{R}^2). \tag{11}$$

An entropy solution of (11) is such that $p(u) \in BV_{\mathrm{loc}}(\mathbb{R}^2)$, for any $p \in \mathcal{P}$. In particular, it follows that $u \in GBV_{\mathrm{loc}}(\mathbb{R}^2)$.

We recall from [6] the following regularity result.

Proposition 3.1. *Let u be a solution of (11). Assume that $u \in L^p_{\mathrm{loc}}(\Omega)$ for some $p > 2$ and for some open set $\Omega \subseteq \mathbb{R}^2$. The following assertions hold.*

(a) *For any $t \in \mathbb{R}$ the sets $\{u > t\}$ and $\{u \geq t\}$ have boundary of class $C^{1,\alpha}$ in Ω, for some $\alpha \in]0, 1[$ ($C^{1,1}$ when $u \in L^\infty_{\mathrm{loc}}(\Omega)$).*

(b) *If $u \geq a$ in Ω (resp. $u \leq a$ in Ω) for some $a \in \mathbb{R}$ then, for any $t \in \mathbb{R}$, the curvatures κ of $\partial\{u > t\} \cap \Omega$ and h of $\partial\{u \geq t\} \cap \Omega$ satisfy $\kappa \geq a$ and $h \geq a$ (resp. $\kappa \leq a$ and $h \leq a$) in the sense of distributions.*

In the sequel, given a function u as in Proposition 3.1 and $t \in \mathbb{R}$, I shall always identify the set $\{u > t\}$ (resp. $\{u < t\}$) with its points of density one, which is an open set, and accordingly define $\{u \geq t\}$ as the complement of $\{u < t\}$.

Proposition 3.2. *Let u be a solution of (11). Assume that $u \in W^{1,1}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ for some open set $\Omega \subseteq \mathbb{R}^2$. Then for any $t \in \mathbb{R}$, $t \neq 0$, every connected component of $\partial\{u > t\} \cap \Omega$ is contained in the boundary of a ball of radius $1/|t|$.*

Proof. Let $t \in \mathbb{R}$, $t \neq 0$, let $\epsilon > 0$ and let Ω_ϵ be the open set defined as $\Omega_\epsilon := \{x \in \Omega : |u(x) - t| < \epsilon\}$. Let also $\gamma := \partial\{u > t\} \cap \Omega$. Since $u \in L^\infty_{\mathrm{loc}}(\Omega)$, by Proposition 3.1 the curve γ and the two curves $\gamma^-_\epsilon := \partial\{u > t - \epsilon\} \cap \Omega$, $\gamma^+_\epsilon := \partial\{u < t + \epsilon\} \cap \Omega$ are of class $C^{1,1}$. Moreover, since $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, the two sets $\Sigma^-_\epsilon := \gamma^-_\epsilon \cap \gamma$ and $\Sigma^+_\epsilon := \gamma^+_\epsilon \cap \gamma$ are closed sets of zero $\mathcal{H}^1$-measure. Then the curve $\gamma \setminus (\Sigma^-_\epsilon \cup \Sigma^+_\epsilon)$ is contained in Ω_ϵ for any $\epsilon > 0$. Since γ is of class $C^{1,1}$, by (b) of Proposition 3.1 it follows that γ has curvature belonging to $(t - \epsilon, t + \epsilon)$ for any $\epsilon > 0$. The thesis follows letting $\epsilon \to 0^+$. $\square$

Note that if u is as in Proposition 3.2 then the set $\{u > t\}$ is a disjoint union of circles of radius $\frac{1}{|t|}$, for any $t \neq 0$ such that the boundary of $\{u > t\}$ is entirely contained in Ω.

The following result shows that there are no nontrivial solutions of (11) which are locally bounded and have no jumps.

Lemma 3.3. *Let $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$ be a solution of (11). Then $u \equiv 0$.*

Proof. Assume by contradiction that $\sup u > 0$ (the case $\inf u < 0$ can be treated in a similar way). From Proposition 3.2 it follows that the set $\{u > t\}$ contains an open ball B_t of radius $\frac{1}{t}$, for any $t \in (0, \sup u)$. Fix $t \in (0, \sup u)$ and let

$t^* := \sup_{B_t} u > t$. Then a connected component of the set $\{u = t^*\}$ is a closed ball $D_{t^*} \subset B_t$ of radius $\frac{1}{t^*}$. From (11) one gets

$$t^* = \frac{(t^*)^2}{\pi} \int_{D_{t^*}} u \, dx = \frac{(t^*)^2}{\pi} \int_{D_{t^*}} u \, dx = -\frac{(t^*)^2}{\pi} \int_{D_{t^*}} \mathrm{div} z \, dx = 2t^*,$$

which is a contradiction. □

From now on, I shall consider solutions u of (11) which satisfy the following property

$$\forall t \in \mathbb{R} \text{ there exists an open set } U_t \supset \partial\{u > t\} \text{ such that } u \in L^\infty_{\mathrm{loc}}(U_t). \qquad (12)$$

Observe that, if u satisfies (12), thanks to Proposition 3.1 all the level sets of u have boundary of class $C^{1,1}$.

The following proposition characterizes the solutions of (11) with have no jumps (see [6, Section 4]).

Proposition 3.4. *Assume that a solution u of (11) satisfies (12) and $(-M) \vee u \wedge M \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$ for any $M > 0$. Then one of the following possibilities holds:*

(i) $u \equiv 0$;
(ii) *u is positive and the set $\{u > t\}$ is a ball of radius $\frac{1}{t}$, for any $t > 0$;*
(iii) *u is negative and the set $\{u < t\}$ is a ball of radius $-\frac{1}{t}$, for any $t < 0$;*
(iv) *u is nonnegative, $\{u > 0\}$ is a halfspace and the set $\{u > t\}$ is a ball of radius $\frac{1}{t}$, for any $t > 0$;*
(v) *u is nonpositive, $\{u < 0\}$ is a halfspace and the set $\{u < t\}$ is a ball of radius $-\frac{1}{t}$, for any $t < 0$;*
(vi) *both $\{u > 0\}$ and $\{u < 0\}$ are halfspaces, the set $\{u > t\}$ is a ball of radius $\frac{1}{t}$, for any $t > 0$, and the set $\{u < \tau\}$ is a ball of radius $-\frac{1}{\tau}$, for any $\tau < 0$.*

Proposition 3.4 essentially means that the solutions of (11) with no jumps can be obtained from the fundamental solutions $u(x) = \pm\frac{1}{|x|}$, by a suitable translation of the level–sets $\{u > t\}$ (or $\{u < t\}$).

4. Properties of the level–sets

In this section I shall consider "tower solutions" of (11), i.e. solutions of the type $u = \sum_{i=1}^p c_i \chi_{E_i}$ for a suitable choice of sets $E_i \subset \mathbb{R}^2$ and constants $c_i \in \mathbb{R}$.

Denote in the following by $C_0, C_1, \dots C_m$ open sets with boundary of class $C^{1,1}$ with the following properties:

$\overline{C_i} \subset C_0$ for any $i \in \{1, \dots, m\}$;
$\overline{C_i} \cap \overline{C_j} = \emptyset$ for any $i, j \in \{1, \dots, m\}$.

Choosing $k \in \{0, \dots, m\}$, define

$$F := C_0 \setminus \bigcup_{i=1}^m \overline{C_i}, \qquad \widehat{F} := F \cup \bigcup_{j=k+1}^m \overline{C_j}.$$

Let also ν^F be the exterior unit normal to ∂F.

I want to identify necessary and sufficient conditions on F in order to have $F = \{u = t\}$ and $\bigcup_{j=k+1}^m C_j = \{u > t\}$, for some $t \in \mathbb{R}$ and for some tower solution u of (11). Using (9) this implies that there exists a vector field $z \in L^\infty(F, \mathbb{R}^2)$ such that

$$\begin{cases} -\operatorname{div} z = t & \text{on } F, \\ \|z\|_\infty \le 1, \\ [z, \nu^F] = -1 & \mathcal{H}^1\text{–a.e. on } \partial C_i, i \in \{1, \dots k\}, \\ [z, \nu^F] = 1 & \mathcal{H}^1\text{–a.e. on } \partial C_j, j \in \{k+1, \dots m\}, \end{cases} \tag{13}$$

where $[z, \nu^F]$ denotes the trace of the scalar product between z and ν^F (see [5] for precise definition and properties).

Notice that a problem closely related to (13) has already been considered in the literature related to capillarity problems, see [11], [8].

On the other hand, given a function $u = \sum_{i=1}^M t_i \chi_{F_i}$, where $F_i \subset \mathbb{R}^2$ are disjoint, with boundary of class $C^{1,1}$ and such that $\bigcup_{i=1}^M \overline{F_i} = \mathbb{R}^2$, then u is a solution of (11) if and only if there exist vector fields $z_i \in L^\infty(F_i, \mathbb{R}^2)$ satisfying (13) for any $i \in \{1, \dots, M\}$ and such that $[z, \nu_i^F] + [z, \nu_j^F] = 0$ on $\partial F_i \cap \partial F_j$.

It follows that the characterization of the tower solutions of (11) reduces to solve problem (13).

4.1. Bounded domains

Let us first consider the case when the set F is bounded. Set

$$\begin{aligned} j_0 \quad &:= \quad \frac{\sum_{i=0}^k P(C_i) - \sum_{j=k+1}^m P(C_j)}{|F|}, \\ \mathcal{F}_F(E) \quad &:= \quad P(E, F) + \sum_{i=0}^k |\partial^* E \cap \partial C_i| - \sum_{j=k+1}^m |\partial^* E \cap \partial C_j| - j_0 |E|. \end{aligned}$$

By the Gauss–Green Formula, problem (13) reduces to

$$\begin{cases} -\operatorname{div} z & = j_0 \quad \text{on } F, \\ \|z\|_\infty & \le 1, \\ [z, \nu^F] & = -1 \quad \text{on } \partial C_i, i \in \{0, \dots k\}, \\ [z, \nu^F] & = 1 \quad \text{on } \partial C_j, j \in \{k+1, \dots m\}. \end{cases} \tag{14}$$

Finally, denote by $\mathcal{A}$ the family of all sets $E \subseteq F$ such that $\partial E \cap \operatorname{int}(F)$ consists of pieces of circumferences of radius $1/j_0$, which meet ∂F tangentially and which span an angle less than or equal to π. I also require that the contact angle is π on $\bigcup_{i=0}^k \partial C_i$ and zero on $\bigcup_{j=k+1}^m \partial C_j$, and that $\{\emptyset, F\} \in \mathcal{A}$.

The following result is proved in [6].

Theorem 4.1. *The following conditions are equivalent.*

(a) *There exists a vector field $z : F \to \mathbb{R}^2$ satisfying (14).*

(b) *It holds*

$$j_0 \int_F w \le \int_F |Dw| + \sum_{i=0}^{k} \int_{\partial C_i} w - \sum_{j=k+1}^{m} \int_{\partial C_j} w \quad \forall w \in BV(F). \tag{15}$$

(c) *For any $E \subseteq F$ of finite perimeter, it holds*

$$\mathcal{F}_F(E) \ge 0. \tag{16}$$

(d) *It holds*

$$\min_{E \in \mathcal{A}} \mathcal{F}_F(E) = 0. \tag{17}$$

Remark 4.2. *Notice that, when $k = 0$, j_0 tends to zero as C_0 tends to $\mathbb{R}^2$; in this case, the minimum problem (17) reduces to the problem considered in* [5, Theorem 6].

Notice that, in most of the cases, the family $\mathcal{A}$ is a finite set, hence condition (d) is often easy to verify. Howerver, Theorem 4.6 below will provide more explicit conditions, involving the curvature of ∂F and the distance between the sets C_i.

I start with the definition of the so-called *ball condition* [8].

Definition 4.3. *Let $\Omega \subseteq \mathbb{R}^2$ be an open set with boundary of class $C^{1,1}$, and let $\rho > 0$. I say that Ω satisfies the ρ-ball condition if a ball of radius ρ can be rotated along $\partial\Omega$ in the interior of Ω such that no antipods of the ball lie on $\partial\Omega$.*

I recall the following result [8, Theorem 4.1].

Lemma 4.4. *Let $\Omega \subseteq \mathbb{R}^2$ be an open set satisfying the ρ-ball condition, for some $\rho > 0$. Then $\sup_{\partial\Omega} \kappa \le \frac{1}{\rho}$. Moreover, given a ball $B \subset \Omega$ of radius ρ and tangent to $\partial\Omega$, the set $\Gamma \cap \partial B$ is connected, for any connected component Γ of $\partial\Omega$, and spans an angle less than π.*

Observe that, in general, the inequality $\sup_{\partial\Omega} \kappa \le \frac{1}{\rho}$ does not imply the ρ-ball condition for the set Ω. Notice also that if Ω is a convex set with boundary of class $C^{1,1}$ such that $\sup_{\partial\Omega} \kappa < \frac{1}{\rho}$, then Ω satisfies the ρ-ball condition.

Let F be as in Theorem 4.1. In the following, when I say that F or $\widehat{F}$ satisfies the ball condition, I shall always mean the $\frac{1}{j_0}$-ball condition.

Remark 4.5. *If C_i is convex for any $i \in \{0, \dots m\}$, $\sup_{\partial C_0} \kappa < j_0$ and*

$$\mathrm{dist}(\partial C_i, \partial C_j) > \frac{2}{j_0} \qquad \forall (i,j) \in \{0, \dots k\},\ i \ne j,$$

then F satisfies the ball condition.

The following result, proved in [6], provides a necessary condition and a sufficient condition for the existence of a vector field $z : F \to \mathbb{R}^2$ satisfying (14).

Theorem 4.6. *Let F be as above and denote by κ the curvature of ∂F. Assume that there exists a vector field $z : F \to \mathbb{R}^2$ satisfying (14). Then*

$$\sup_{\partial C_i} \kappa \leq j_0 \leq \inf_{\partial C_j}(-\kappa), \qquad i \in \{0, \dots, k\},\ j \in \{k+1, \dots, m\}. \tag{18}$$

Conversely, assume that $\widehat{F}$ satisfies the ball condition, the inequalities at the right hand side of (18) hold, and

$$\operatorname{dist}(\partial C_i, \partial C_j) > \frac{2}{j_0} \qquad \forall (i,j) \in \{0, \dots, k\}^2 \cup \{k+1, \dots, m\}^2, i \neq j. \tag{19}$$

Then there exists a vector field $z : F \to \mathbb{R}^2$ satisfying (14).

4.2. Unbounded domains

Let us now consider the case $C_0 = \mathbb{R}^2$ and $C_1, \dots, C_m$ as above, which means the F is unbounded but $\mathbb{R}^2 \setminus F$ is bounded. In this case, problem (13) reduces to the existence of a vector field $z \in L^\infty(F, \mathbb{R}^2)$ such that

$$\begin{cases} -\operatorname{div} z = 0 & \text{on } F, \\ \|z\|_\infty \leq 1, \\ [z, \nu^F] = -1 & \mathcal{H}^1\text{–a.e. on } \partial C_i, i \in \{1, \dots k\}, \\ [z, \nu^F] = 1 & \mathcal{H}^1\text{–a.e. on } \partial C_j, j \in \{k+1, \dots m\}. \end{cases} \tag{20}$$

It holds a result analogous to Theorem 4.1.

Theorem 4.7. *The following conditions are equivalent*

(i) *Problem (20) has a solution.*

(ii) *It holds*

$$0 \leq \int_F |Dw| + \sum_{i=1}^{k} \int_{\partial C_i} w - \sum_{j=k+1}^{m} \int_{\partial C_j} w \quad \forall w \in BV(F). \tag{21}$$

(iii) *For any $X \subseteq F$ of finite perimeter, it holds*

$$P(X, F) \geq \left| \sum_{j=k+1}^{m} \mathcal{H}^1(\partial^* X \cap C_j) - \sum_{i=1}^{k} \mathcal{H}^1(\partial^* X \cap C_i) \right|. \tag{22}$$

(iv) *let E_1 be a solution of the variational problem*

$$\min \left\{ P(E) : \bigcup_{j=k+1}^{m} C_j \subseteq E \subseteq \mathbb{R}^2 \setminus \bigcup_{i=1}^{k} C_i \right\}, \tag{23}$$

then it holds

$$P(E_1) = \sum_{j=k+1}^{m} P(C_j); \tag{24}$$

let E_2 be a solution of the variational problem

$$\min\left\{P(E) : \bigcup_{i=1}^{k} C_i \subseteq E \subseteq \mathbb{R}^2 \setminus \bigcup_{j=k+1}^{m} C_j\right\}, \tag{25}$$

then it holds

$$P(E_2) = \sum_{i=1}^{k} P(C_i). \tag{26}$$

Notice that (iv) implies that each C_i is a convex set. Moreover, since any minimizer of problems (23) and (25) has boundary (lying inside F) made of a finite number of segments which intersect tangentially ∂F (and there are only a finite number of such segments), the number of such minimizers is finite. Finally, conditions (24) and (26) are essentially distance conditions between sets C_i of the same type, for example they are satisfied if $\mathrm{dist}(\partial C_i, \partial C_j) > P(C_l)$, for any $(i,j,l) \in \{1,\dots,k\}^3 \cup \{k+1,\dots,m\}^3$, $i \neq j$.

5. Some examples

In order to clarify the conditions given in Section 4, I shall discuss some explicit examples.

Example 1. Let $u(x,y) = \frac{2x}{x^2+y^2}$; then u is a solution of (11) with no jumps, satisfying condition (vi) of Proposition (3.4). Notice that u is not continuous at $(0,0)$.

Example 2. Let $F \subset \mathbb{R}^2$ be the set in Figure 1. It is easy to check that F satisfies the assumptions of Theorem 4.6, since it is a convex set with $C^{1,1}$ boundary and it holds

$$\sup_{\partial F} \kappa = \frac{1}{r} < \frac{2\pi r + 2L}{\pi r^2 + 2rL} = \frac{P(F)}{|F|}. \tag{27}$$

Moreover, since the inequality in (27) is strict, the solution of (1) starting from $\chi_{F'}$ remains a characteristic function for any convex set F' of class $C^{1,1}$ close enough to F in the $C^{1,1}$-norm.

Example 3. Let $F \subset \mathbb{R}^2$ be the union of two disjoint balls of radius r whose centers are at distance L. In this case, conditions (24), (26) of Theorem 4.7 become

$$L \geq \pi r.$$

It follows that, under this condition, the function $u = \frac{2}{r}\chi_F$ is a solution of (11).

Example 4. I shall now describe a class of radially symmetric tower solutions. Let $0 = R_0 < R_1 < \cdots < R_p < R_{p+1} = +\infty$, and let $\Omega_i := B_{R_i} \setminus \overline{B}_{R_{i-1}}$, $i = 1,\dots,p+1$ (where we set $B_0 = \emptyset$, $B_{+\infty} = \mathbb{R}^2$). Let also $a_1,\dots,a_{p+1}$ be real numbers such that $a_{p+1} = 0$ and $a_i \neq a_{i+1}$, for $i = 1,\dots,p$. Then, there exist real numbers $b_1,\dots,b_{p+1}$, with $b_{p+1} = 0$ and $b_i \neq b_{i+1}$, such that $\mathrm{sign}(b_{i+1}-b_i) = \mathrm{sign}(a_{i+1}-a_i)$ for any $i \in \{1,\dots,p\}$ (i.e. the numbers b_i have the same "qualitative ordering" bean.eps

FIGURE 1. A bean–shaped admissible set

of a_i) and the function $u = \sum_{i=1}^{p} b_i \chi_{\Omega_i}$ is a solution of (11). Indeed, applying Theorem 4.6 to the sets Ω_i, one can check that the function u is a solution of (11) when

$$b_i = -\frac{2}{\operatorname{sign}(a_i - a_{i-1}) R_{i-1} + \operatorname{sign}(a_{i+1} - a_i) R_i}.$$

Notice that, since $R_i > R_{i-1}$, it follows $\operatorname{sign}(b_i) = -\operatorname{sign}(a_{i+1} - a_i)$.

This example has already been discussed in [14].

6. Solutions of the denoising problem

In this last section, I will show how to obtain explicit solutions of the denoising problem (4), starting from solutions of (11), see [5, Proposition 7, Proposition 8].

Proposition 6.1. *Let $\lambda > 0$, $b \in \mathbb{R}$ and $u \in BV(\mathbb{R}^2)$ a solution of (11). Then the function $v = au$ is the solution of the variational problem (4) with $a := \operatorname{sign}(b)[(|b| - \lambda) \vee 0]$ and $f := bu$. Conversely, if $v = au \in BV(\mathbb{R}^2)$ is the solution of (4) with $f = bu$, for some $a, b \in \mathbb{R}$ with $b - a = \pm\lambda$, then the function u is a solution of (11).*

The following proposition, which extends the previous result, clarifies how to construct solutions of (4) from tower solutions of (11).

Proposition 6.2. *Let $C_0, C_1, \dots, C_m \subset \mathbb{R}^2$ be open sets with boundary of class $C^{1,1}$, such that $C_i \cap C_j = \emptyset$ for $i \neq j$, $\bigcup_{i=0}^{m} \overline{C_i} = \mathbb{R}^2$ and C_i is bounded for any $i \in \{1, \dots, m\}$. Assume that the function $u = \sum_{i=1}^{m} c_i \chi_{C_i}$ is a solution of (11), for some $c_i \in \mathbb{R}$. Let now $b_i \in \mathbb{R}$, $i \in \{1, \dots, m\}$, $\lambda > 0$ and $a_i = \operatorname{sign}(b_i)[(|b_i| - \lambda) \vee 0]$. Then the function $u = \sum_{i=1}^{m} a_i c_i \chi_{C_i}$ is the solution of the variational problem (4) with $f = \sum_{i=1}^{m} b_i c_i \chi_{C_i}$.*

References

[1] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford Mathematical Monographs, 2000.

[2] F. Andreu, J.M. Mazón, S. Segura and J. Toledo. Existence and uniqueness for a degenerate parabolic equation with L^1-data. *Trans. Amer. Math. Soc.*, 315:285–306, 1999.

[3] F. Andreu, V. Caselles, J.I. Diaz and J.M. Mazón. Qualitative properties of the total variation flow. *Jour. Funct. Anal.*, 188:516–547, 2002.

[4] F. Andreu, V. Caselles and J.M. Mazón. Existence and uniqueness of solutions for a parabolic quasilinear problem for linear growth functionals with L^1 data. *Math. Ann.*, 322:139–206, 2002.

[5] G. Bellettini, V. Caselles and M. Novaga. The Total Variation Flow in $\mathbb{R}^N$. *Journal of Diff. Eqs.*, 184:475–525, 2002.

[6] G. Bellettini, V. Caselles and M. Novaga. Self–similar solutions of the total variation flow. *In preparation.*

[7] H. Brezis. *Operateurs Maximaux Monotones.* North–Holland, Amsterdam, 1973.

[8] J.T. Chen. On the existence of capillary free surfaces in the absence of gravity. *Pacific J. Math.*, 88:323–361, 1980.

[9] A. Chambolle and P.-L. Lions. Image recovery via total variation minimization and related problems. *Numer. Math.*, 76:167–188, 1997.

[10] S. Durand, F. Malgouyres, and B. Rougé. Image de-blurring, spectrum interpolation and application to satellite imaging. *Control, Optimisation and Calculus of Variation*, 5, 2000.

[11] E. Giusti. On the equation of surfaces of prescribed mean curvature. Existence and uniqueness without boundary conditions. *Invent. Math.*, 46:111–137, 1978.

[12] S.N. Kruzhkov. First order quasilinear equations in several independent variables. *Math. USSR-Sb.*, 10:217–243, 1970.

[13] L. Rudin, S. Osher, and E. Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D*, 60:259–268, 1992.

[14] D.M. Strong and T. F. Chan. Relation of regularization parameter and scale in total variation based image denoising. *CAM Report)*, 1996.

[15] G.H. Golub T. F. Chan and P. Mulet. A nonlinear primal-dual method for total variation based image restoration. *SIAM J. Sci. Computing*, 20:1964–1977, 1999.

[16] C. R. Vogel and M. E. Oman. Fast total variation based image reconstruction. *Proceedings of the 1995 ASME Design Engineering Conferences, Vol. 3*, pages 1009–1015, 1995.

Dipartimento di Matematica,
Università di Pisa,
via Buonarroti 2
56127 Pisa, Italy
E-mail address: novaga@dm.unipi.it