Approximation of the anisotropic mean curvature flow

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Abstract

In this note, we provide simple proofs of consistency for two well known algorithms for mean curvature motion, Almgren-Taylor-Wang’s [1] variational approach, and Merriman-Bence-Osher’s algorithm [24]. Our techniques, based on the same notion of strict sub- and superflows, also work in the (smooth) anisotropic case.

1 Introduction

The Mean curvature flow refers to the motion of a hypersurface $\Gamma(t) \subset \mathbb{R}^N$ whose normal velocity, at each point, is equal to (minus) its mean curvature. We will consider only compact hypersurfaces $\Gamma(t)$, that are the boundary of some evolving set $E(t)$ (bounded or unbounded). In this case, the motion is also known as the “area-diminishing” flow, and is in some sense the gradient flow of the perimeter of $E(t)$. It is well-known that this motion can be characterized in terms of the distance function to $\Gamma = \partial E$ [18, 2]. More precisely, if we define $d(x,t)$ as

$$d(x,t) := \text{dist}(x,E(t)) - \text{dist}(x,\mathbb{R}^N \setminus E(t))$$

(the signed distance function to $\partial E(t)$), then the exterior normal to $E$ is given by $\nabla d$ whereas the curvature is $\Delta d$. On the other hand, the normal velocity of a point of the boundary is given, at each time, by $-\partial d/\partial t$, so that the evolution is characterized by

$$\frac{\partial d}{\partial t}(x,t) = \Delta d(x,t)$$

(1)

at any $x \in \partial E(t)$ (i.e., $(x,t)$ such that $d(x,t) = 0$).

It is well known that the Mean curvature flow enjoys a comparison principle: if $E$, $F$ are two (smooth) evolutions such that $E(t) \subseteq F(t)$ at some time $t$, then $E(s) \subseteq F(s)$ at any subsequent time $s > t$ as long as the flows are defined. This key property allows to define a generalized flow for nonsmooth surfaces, by comparison with smooth flows: basically, a generalized flow will be a flow such that any smooth

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flow starting inside remains inside while any smooth flow starting outside remains outside. The formal theory that provides such a generalization is known as the barrier theory and is initially due to De Giorgi [16, 8, 5]. The theory of viscosity solutions (which is also based on the comparison principle) defines the generalized flow as the zero sub- or superlevel set of a function \( u \) that solves an appropriate degenerate parabolic equation, and yields the same generalized flows as the barrier theory [6]. The generalized flow starting from a set \( E \) is usually unique, except when the “fattening” phenomenon occurs, which corresponds to the fattening of the level line \( \{ u = 0 \} \) of the corresponding viscosity solution.

It is shown in [5] that a barrier solution can be characterized by comparison with appropriate sub- and superflow: in this case, a generalized flow will be characterized by the property that any smooth flow starting inside and evolving (strictly) faster than the Mean curvature flow remains inside, while a smooth flow starting outside and evolving (strictly) slower than the Mean curvature flow remains outside. The definition of a strict super- or subflow of (1) is the following: \( E(t) \) will be a strict superflow (on a small time interval \([t_0, t_1]\)) iff its signed distance function satisfies

\[
\frac{\partial d}{\partial t}(x, t) > \Delta d(x, t)
\]

in a neighborhood of \( \{d = 0\} \), while a strict subflow is defined with the reverse inequality.

We show in this note that such a definition (which will be slightly adapted to cover non-isotropic cases) makes very easy the proof of convergence for two well-known approximation schemes for the Mean curvature flow, namely, the Almgren-Taylor-Wang [1] approach and the Merriman-Bence-Osher [24] approach. In both schemes, a time step \( h > 0 \) is fixed and a discrete-in-time evolution is defined, by providing a simple evolution operator \( E \mapsto T_h E \) that approximates the evolution of a initial set \( E \) over a time interval of duration \( h \). Given \( E_0 \), the discrete evolution \( E_h(t) \) is simply \( T_h^{[t/h]}(E_0) \) where \([\cdot]\) denotes the integer part. One then wants to know whether \( E_h(t) \rightarrow E(t) \) as \( h \to 0 \), where \( E(t) \) is the generalized evolution starting from \( E_0 \). The key to prove this convergence are the two properties of monotonicity and consistency. The operator \( T_h \) will be monotone if given any \( E, F \) with \( E \subseteq F \), one has \( T_h E \subseteq T_h F \). The notion of consistency we will use is based on our notion of strict super- and subflow: \( T_h \) will be consistent if, given any superflow \( E \) on \([t_0, t_1]\) and given \( h > 0 \) small enough, one has \( E(t + h) \subseteq T_h E(t) \) for any \( t \in [t_0, t_1 - h] \), while given any subflow, the same holds with the reverse inclusion. It follows from the theory of barriers that if \( T_h \) is monotone and consistent in the above-defined sense, then \( \partial E_h(t) \) converges to \( \partial E(t) \) as \( h \to 0 \) (in the Hausdorff sense), at any time, as long as the generalized flow \( \partial E(t) \) is uniquely defined (i.e., no fattening occurs).

In our cases, the set \( T_h E(t) \) will be defined as a level set of some function \( u \) (depending on \( h \) and \( E(t) \)), satisfying some elliptic or parabolic equation, and it
will be quite easy to build from a function \( d \) satisfying (2) a sub- or supersolution \( v \) of the same equation that will be compared to \( u \), yielding a comparison of the level sets.

This note is organized as follows: in Section 2 we introduce the anisotropic curvature flow and we give a rigorous definition of the corresponding super and subflows. Then, in Section 3 we introduce the Merriman-Bence-Osher’s scheme and we prove its consistency. In Section 4 we do the same for the Almgren-Taylor-Wang’s algorithm. We observe that in this case, a result of consistency with smooth flows is already found in [1], however, its proof is by far more complicated than ours.

## 2 Anisotropic curvature flow

We follow the definitions and notation in [7, 9]. Let us consider \((\phi, \phi^o)\) a pair of mutually polar, convex, one-homogeneous functions in \(\mathbb{R}^N\) (i.e., \(\phi^o(\xi) = \sup_{\eta \leq 1} \xi \cdot \eta, \phi(\eta) = \sup_{\phi^o(\xi) \leq 1} \xi \cdot \eta\), see [25]). These are assumed to be locally finite, and, to simplify, even. The pair \((\phi, \phi^o)\) is referred as the anisotropy (the isotropic case corresponds to \(\phi = \phi^o = |\cdot|\)). The local finiteness implies that there is a constant \(c > 1\) such that

\[
c^{-1}|\eta| \leq \phi(\eta) \leq c|\eta| \quad \text{and} \quad c^{-1}|\xi| \leq \phi^o(\xi) \leq c|\xi|
\]

for any \(\eta\) and \(\xi\) in \(\mathbb{R}^N\). We refer to [7, 9] for the main properties of \(\phi\) and \(\phi^o\).

Being convex and 1-homogeneous, \(\phi^o\) (and \(\phi\)) is also subadditive, so that the function \((x, y) \mapsto \phi(x - y)\) defines a distance, the “\(\phi\)-distance”. For \(E \subset \mathbb{R}^N \) and \(x \in \mathbb{R}^N\), we denote by \(\operatorname{dist}^\phi(x, E) := \inf_{y \in E} \phi(x - y)\) the \(\phi\)-distance of \(x\) to the set \(E\), and by

\[
d^\phi_E(x) := \operatorname{dist}^\phi(x, E) - \operatorname{dist}^\phi(x, \mathbb{R}^N \setminus E)
\]

the signed \(\phi\)-distance to \(\partial E\), negative in the interior of \(E\) and positive outside its closure. One easily checks that

\[
|d^\phi_E(x) - d^\phi_E(y)| \leq \phi(x - y) \leq c|x - y|
\]

for any \(x, y \in \mathbb{R}^N\), so that (by Rademacher’s theorem) \(d^\phi_E\) is differentiable a.e. in \(\mathbb{R}^N\). The former inequality shows moreover that \(\nabla d^\phi_E(x) \cdot h \leq \phi(h)\) for any \(h \in \mathbb{R}^N\), if \(x\) is a point of differentiability: hence \(\phi^o(\nabla d^\phi_E(x)) \leq 1\). In this note we will always assume that \(\phi\) and \(\phi^o\) are at least in \(C^2(\mathbb{R}^N \setminus \{0\})\). In this case, one shows quite easily that \(d^\phi_E\) is differentiable at each point \(x\) which has a unique \(\phi\)-projection \(y \in \partial E\) (solving \(\min_{y \in \partial E} \phi(x - y)\)). Then, \(\nabla d^\phi_E(x)\) is given by \(\nabla \phi((x - y)/d^\phi_E(x))\), so that \(\phi^o(\nabla d^\phi_E(x)) = 1\). See [7, 9] for details.

The Cahn-Hoffman vector field \(n_\phi\) is a vector field on \(\partial E\) given by \(n_\phi(x) = \nabla \phi^o(\nu_E(x)) = \nabla \phi^o(\nabla d^\phi_E(x))\) a.e. on \(\partial E\). Here, \(\nu_E\) is the (Euclidean) exterior normal to \(\partial E\). If \(E\) is smooth enough, then \(\nabla d^\phi_E\) does not vanish near \(\partial E\) so that one can define \(n_\phi(x) = \nabla \phi^o(\nabla d^\phi_E(x))\) in a neighborhood of \(\partial E\).
Then, we define the $\phi$-curvature of $\partial E$ by $\kappa_{\phi} = \text{div} \, n_{\phi}$. The $\phi$-curvature flow is an evolution $E(t)$ such that at each time, the velocity of $\partial E(t)$ is given by

$$V = -\kappa_{\phi} n_{\phi},$$

where $n_{\phi}$ is the Cahn-Hoffman vector field and $\kappa_{\phi}$ is the $\phi$-curvature. It is shown that, in some sense, it is the fastest way to diminish the anisotropic perimeter $\int_{\partial E} \phi^\circ(\nu_{E}) \, dH^{N-1}$. If $\phi$, $\phi^\circ$ are merely Lipschitz (when, for instance, the Wulff shape $\{\phi \leq 1\}$ is a convex polytope), then $n_{\phi}$ can be nonunique and the anisotropy is called crystalline [28, 7]. We refer to [14] for a proof of convergence of Merriman-Bence-Osher’s scheme in the crystalline case.

The anisotropic variant of (1) is the following characterization of the anisotropic mean curvature flow: letting $d(x, t) = d^\phi_{E(t)}(x)$, the smooth set $E(t)$ evolves by anisotropic curvature if

$$\frac{\partial d}{\partial t}(x, t) = \text{div} \nabla \phi^\circ(\nabla d(x, t)),$$

for any $(x, t)$ with $d(x, t) = 0$. One therefore introduces the following definition of (strict) super- and subflows, which is simplified from [15]:

**Definition 2.1** Let $E(t) \subset \mathbb{R}^N$, $t \in [t_0, t_1]$. We say that $E(t)$ is a superflow of (4), if there exists a bounded open set $A \subset \mathbb{R}^N$, with $\bigcup_{t_0 \leq t \leq t_1} \partial E(t) \times \{t\} \subset A \times [t_0, t_1]$, and $\delta > 0$, such that $d(x, t) = d^\phi_{E(t)}(x) \in C^1([t_0, t_1]; C^2(A))$, and

$$\frac{\partial d}{\partial t}(x, t) \geq \text{div} \nabla \phi^\circ(\nabla d)(x, t) + \delta,$$

for any $x \in A$ and $t \in [t_0, t_1]$. We say that $E(t)$ is a subflow whenever $\delta < 0$ and the reverse inequality holds in (5).

Considering now a time discrete evolution scheme $E \mapsto T_h E$ ($T_h E$ needs not be defined for all sets $E$, in our applications, it will be sufficient to define it for closed sets with compact boundary), parametrized by the time step $h > 0$, we introduce the following definition of consistency:

**Definition 2.2** The scheme $T_h$ is consistent if and only if for any superflow $E(t)$, $t_0 \leq t \leq t_1$, in the sense of Definition 2.1, there exists $h_0$ such that if $h \leq h_0$, then $T_h E(t) \supseteq E(t + h)$ for any $t \in [t_0, t_1 - h]$, while for any subflow, the same holds with the reverse inclusion.

This definition means that given a superflow, it will also go faster than the discretized evolutions, as soon as $h$ is small enough. The following results follows from the theory of barriers, see [5, 6, 8, 15].

**Proposition 2.3** Assume $T_h$ is a consistent scheme, in the sense of Definition 2.2 above, which is also monotone: for any $E, F \subset \mathbb{R}^N$, $E \subseteq F \Rightarrow T_h E \subseteq T_h F$. Let $E_0 \subset \mathbb{R}^N$ be a closed set with compact boundary such that the generalized anisotropic
curvature flow $E(t)$ starting from $E_0$ is uniquely defined (no fattening). For any $t \geq 0$ let $E_h(t) := T^{1/h}[E_0]$. Then, for any $t$ as long as $E(t)$ is not empty, $\partial E_h(t) \rightarrow \partial E(t)$ in the Hausdorff sense.

In the next sections, we prove consistency (and monotonicity), first for the (anisotropic) Merriman-Bence-Osher scheme, then for the Alvarez-Taylor-Wang scheme, yielding, by Proposition 2.3, convergence to the generalized solution, when unique.

3 The Merriman-Bence-Osher algorithm

More than ten years ago, Merriman, Bence and Osher [24] proposed the following algorithm for the computation of the motion by mean curvature of a surface. Given a closed set $E \subset \mathbb{R}^N$, they let $T_h E = \{ u(\cdot, h) \geq 1/2 \}$, where $u$ solves the heat equation with initial data $u(\cdot, 0) = \chi_E$, the characteristic function of $E$. They then conjectured that $E_h(t) := T^{1/h}_k E$ would converge to $E(t)$, where $E(t)$ is the (generalized) evolution by mean curvature starting from $E$.

The proof of convergence of this scheme was established by Evans [17], Barles and Georgelin [3]. Other proofs were given by H. Ishii [19] and Cao [11], where the heat equation was replaced by the convolution of $\chi_E$ with a more general symmetric kernel. Extensions and variants are found in [20, 27, 26, 29, 22].

As easily shown by formal asymptotic expansion, the natural anisotropic generalization of the Merriman-Bence-Osher algorithm is as follows. Given $E$ a closed set with compact boundary in $\mathbb{R}^N$, we let $T_h(E) = \{ x : u(x, h) \geq 1/2 \}$ where $u(x, t)$ is the solution of

$$
\begin{aligned}
\frac{\partial u}{\partial t}(x, t) &\in \text{div} \left( \phi^\circ(\nabla u) \partial \partial^\circ(\nabla u) \right)(x, t) & t > 0, x \in \mathbb{R}^N, \\
u(\cdot, 0) &= \chi_E & (t = 0).
\end{aligned}
$$

The function $u(x, t)$ is well defined and unique by classical results on contraction semigroups [10]: if $E$ is compact, it corresponds to the flow in $L^2(\mathbb{R}^N)$ of the subdifferential of the functional $u \mapsto \int_{\mathbb{R}^N} \phi^\circ(\nabla u)^2/2 \, dx$ if $u \in H^1(\mathbb{R}^N)$, and $+\infty$ otherwise. On the other hand, if $\mathbb{R}^N \setminus E$ is compact, one defines $u$ by simply letting $u = 1 + v$ where $v$ solves the same equation with initial data $\chi_E - 1$.

We first observe that the monotonicity of this scheme is obvious. Indeed, it follows from the comparison principle for equation (6)). Let us now prove the following:

**Proposition 3.1** $T_h$, defined as above, is consistent in the sense of Definition 2.2.

**Proof.** Let $E$ be a superflow on $[t_0, t_1]$, in the sense of Definition 2.1, and let $A$ be the associated neighborhood of $\partial E(t), t \in [t_0, t_1]$.
We introduce the function $\gamma : \mathbb{R} \times [0, +\infty) \rightarrow [0, 1]$ that solves the following (1D) heat equation
\[
\begin{align*}
\frac{\partial \gamma}{\partial \tau}(\xi, \tau) &= \frac{\partial^2 \gamma}{\partial \xi^2}(\xi, \tau), \quad \xi \in \mathbb{R}, \ \tau > 0, \\
\gamma(\xi, 0) &= Y(\xi), \quad \xi \in \mathbb{R}, \ (\tau = 0),
\end{align*}
\]
where $Y = \chi_{[0, +\infty)}$ is the Heavyside function. It is well known that $\gamma$ is given by
\[
\gamma(\xi, \tau) = \frac{1}{2\sqrt{\pi \tau}} \int_{-\infty}^\xi e^{-\frac{t^2}{4\tau}} \, dt.
\]
In particular, one readily sees that it is self-similar: indeed, the change of variables $s' = s/\sqrt{\tau}$ yields
\[
\gamma(\xi, \tau) = \frac{1}{2\sqrt{\pi \tau}} \int_{-\infty}^{\frac{\xi}{\sqrt{\tau}}} e^{-\frac{s'^2}{2}} \, ds' = \gamma \left( \frac{\xi}{\sqrt{\tau}} \right).
\]

Fix $t < t_0$. The simplest idea would be to introduce the function $v(x, \tau) := \gamma(-d(x, t + \tau), t)$, defined in $A$ for small $\tau$. It satisfies $\{v(\cdot, \tau) \geq 1/2\} = E(t + \tau)$ and one has (using (5))
\[
\frac{\partial v}{\partial \tau} = -\frac{\partial \gamma}{\partial \xi} \frac{\partial d}{\partial \xi} - \frac{\partial \gamma}{\partial \tau} \leq -\frac{\partial \gamma}{\partial \xi} (\text{div} \, \nabla \phi^\circ(\nabla d) + \delta) - \frac{\partial \gamma}{\partial \tau}.
\]
Also: $\nabla v = -(\partial \gamma / \partial \xi) \nabla d$, so that $\phi^\circ(\nabla v) = (\partial \gamma / \partial \xi)$ and $\nabla \phi^\circ(\nabla v) = -\nabla \phi^\circ(\nabla d)$, hence
\[
\text{div} \, \phi^\circ(\nabla v) \nabla \phi^\circ(\nabla v) = -\text{div} \, \frac{\partial \gamma}{\partial \xi} \nabla \phi^\circ(\nabla d) = -\frac{\partial \gamma}{\partial \xi} \text{div} \, \nabla \phi^\circ(\nabla d) - \frac{\partial^2 \gamma}{\partial \xi^2}.
\]
Here, we have used the fact that $\phi^\circ$ is even and one-homogeneous, $\nabla \phi^\circ$ is odd and zero-homogeneous, $\phi^\circ(\nabla d) = 1$, and $\nabla d \cdot \nabla \phi^\circ(\nabla d) = \phi^\circ(\nabla d) = 1$ (by Euler’s identity). Using $\partial \gamma / \partial \tau = \partial^2 \gamma / \partial \xi^2$, we find:
\[
\frac{\partial v}{\partial \tau} \leq \text{div} \, \phi^\circ(\nabla v) \nabla \phi^\circ(\nabla v) - \delta \frac{\partial \gamma}{\partial \xi}.
\]
Hence, $v$ is a good candidate to be a subsolution of (6), with initial data $v(x, 0) = \chi_{E(t)}(x)$. If this were the case, we would get that $v \leq u$ (where $u$ solves (6) with initial data $\chi_{E(t)}$), so that $\{v(\cdot, h) \geq 1/2\} \subseteq \{u(\cdot, h) \geq 1/2\}$, in other words, $E(t + h) \subseteq T_h E(t)$, which is our consistency. However, we cannot show that this $u$ is less than $u$ at the boundary of $A$ (for instance), for $t \leq t + \tau \leq t + h$. This is why we define $v$ in a slightly more complicated way: we let $v(x, \tau) := \gamma(-d(x, t + \tau) + \delta \tau, \tau) - \eta h$, where $\eta < \delta / \sqrt{2\pi} \eta$ is fixed. Since now $\partial v / \partial \tau$ differs from the previous time derivative by $\delta \partial \gamma / \partial \xi$, one still has
\[
\frac{\partial v}{\partial \tau} \leq \text{div} \, \phi^\circ(\nabla v) \nabla \phi^\circ(\nabla v).
\]
at any $(x, \tau) \in A \times [0, h]$; hence $v$ is a subsolution of (6). At $\tau = 0$, $v(x, 0) = \chi_{E(t)}(x) - \eta h < \chi_{E(t)}(x)$.

Let $u$ solve (6) with initial data $\chi_{E(t)}$. First of all, we observe that since $d \in C^1([t_0, t_1]; C^2(A))$, $\partial E(t)$ is a $C^2$ compact hypersurface, continuous in time. Hence
there exists $\rho > 0$, independent of $t$, such that each point $x \in \partial E(t)$, $E(t)$ satisfies an interior and exterior Wulff shape condition of radius $\rho$: there exist $z \in E(t)$ and $z' \not\in E(t)$ with $\{\phi(\cdot - z) \leq \rho\} \subset E(t)$ and $\{\phi(\cdot - z') < \rho\} \cap E(t) = \emptyset$, while $\phi(x - z) = \phi(x - z') = \rho$. One may always assume that $\{|d(\cdot, s)| \leq \rho\} \subset A$ for all $s \in [t_0, t_1]$. Let $B = \{|d(\cdot, t)| < \rho\}$. If $h$ is small enough (independently of $t$), one also may assume that $|d(x, t + \tau) - d(x, t)| \leq \rho/2$ in $B$ for any $\tau \in [0, h]$, so that $\text{dist}^2(\partial E(t + \tau), \partial B) \geq \rho/2$. We assume $h \leq \rho/(4\delta)$. Let $x \in \partial B$ with $d(x, t) = \rho$: then $d(x, t + \tau) \geq \rho/2$ for any $\tau \in [0, h]$, so that $-d(x, t + \tau) + \delta \tau \leq \delta h - \rho/2 \leq -\rho/4$, and $v(x, \tau) \leq \gamma(-\rho/4, \tau) - \eta h$ for any $\tau \in [0, h]$. Hence $v(x, \tau) \leq \gamma_1(-\rho/(4\sqrt{h})) - \eta h \leq \gamma_1(-\rho/(4\sqrt{h})) - \eta h$ which is negative if $h$ is small enough. This shows that if $h$ is small enough, $v(x, \tau) < 0 \leq u(x, \tau)$ for any $\tau \leq h$ and $x \in \partial B \cap \{d(\cdot, \tau) = \rho\}$.

If now $x \in \partial B$ with $d(x, t) = -\rho$, we use the fact that $u \geq w$, where $w$ solves (6) with initial data $w_0 = \chi_{\{\phi(\cdot - z) \leq \rho\}}$. One shows that $w(y, \tau) = U(\phi(y - x)/\rho, \tau/\rho^2)$ where $U(|x|, \tau) = \bar{U}(x, \tau)$ and $\bar{U}$ is the (radial) solution of the heat equation $\partial \bar{U} / \partial t = \Delta \bar{U}$ with initial datum $\chi_{B_1}$, the characteristic function of the unit ball. It is well-known that

$$\bar{U}(y, \tau) = \frac{1}{\sqrt{4\pi \tau}} \int_{|z| \leq 1} \exp \left( -\frac{|y - z|^2}{4\tau} \right) \, dz$$

so that

$$U(0, \tau) = 1 - \frac{1}{\sqrt{4\pi \tau}} \int_{|z| \geq 1/\sqrt{\tau}} \exp \left( -\frac{z^2}{4} \right) \, dz.$$ 

Hence, $u(x, \tau) \geq 1 - (1/\sqrt{4\pi}) \int_{|z| \geq 1/\sqrt{\tau}} \exp(-z^2/4) \, dz \geq 1 - c \exp(-\rho/(4\sqrt{h}))$ for some constant $c > 0$, and any $\tau \in [0, h]$. Hence, for $x \in \partial B \cap \{d(\cdot, \tau) = \rho\}$, $v(x, \tau) - u(x, \tau) \leq c \exp(-\rho/(4\sqrt{h})) - \eta h$: clearly, this is negative if $h$ is small enough (depending only on $\rho$). We have shown that $v$ is below $u$ on $\partial B \times [0, h]$, if $h$ is small enough (uniformly in $t$).

By standard results on parabolic equations, we find that $v \leq u$ on $B \times [0, h]$ and in particular $v(\cdot, h) \leq u(\cdot, h)$ in $B$. Hence, $\{v(\cdot, h) \geq 1/2\} \subseteq \{u(\cdot, h) \geq 1/2\}$. Observe that $v(x, h) \geq 1/2$ iff $-d(x, t + \delta h) \geq (\gamma(\cdot, h))^{-1}(1/2 + \eta h) = \sqrt{2\pi \eta} h + o(h)$, that is, $d(x, t + \delta h) \leq (\sqrt{2\pi \eta} - \delta) h + o(h) =: \sigma_h$. If $h$ is small enough, $\sigma_h > 0$, so that $x \in E(t + h) \Rightarrow d(x, t + h) \leq \sigma_h \Rightarrow v(x, h) \geq 1/2$: we deduce $E(t + h) \subseteq T_h E(t)$, which was our claim. The proof of consistency with subflows is identical. □

See [14] for a proof of consistency and convergence which works in more general situations (namely, the crystalline case). See also K. Ishii [21]'s recent paper on an optimal estimate on the rate of convergence of Merriman-Bence-Osher’s algorithm, in the isotropic case, where the proof of convergence is very close to ours.
4 The Almgren-Taylor-Wang algorithm

In Almgren, Taylor and Wang’s paper [1], the transformation $T_h E$ is defined as a solution of
\[
\min_{F \subseteq \mathbb{R}^N} P_\phi(F) + \frac{1}{h} \int_{F \Delta E} |d_E^\phi(x)| \, dx ,
\]
where now, $F \Delta E$ is the symmetric difference of the two sets $F$ and $E$ and $P_\phi(F)$ is the anisotropic perimeter. This is rigorously defined by $\int_{\mathbb{R}^N} \phi^\circ(D\chi_F)$, where the anisotropic total variation is given by
\[
\int_{\mathbb{R}^N} \phi^\circ(Dv) := \sup \left\{ \int_{\mathbb{R}^N} v(x) \text{div} \psi(x) \, dx : \psi \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N), \phi(\psi(x)) \leq 1 \forall x \in \mathbb{R}^N \right\}.
\]
The same approach to curvature motion has also been proposed by Luckhaus and Sturzenhecker [23], in the isotropic case.

It is shown in [13, 12, 4] that a monotone selection of $T_h E$ can be built in the following way: one fixes a bounded open set $\Omega \supseteq E$, and one lets $w$ be the (unique) minimizer of
\[
\int_\Omega \phi^\circ(Dw) + \frac{1}{2h} (w(x) - d_E^\phi(x))^2 \, dx ,
\]
then, $F = \{ w \leq 0 \}$ is a solution of (9), as soon as the domain $\Omega$ is large enough. Clearly, letting $T_h E$ be this solution defines a monotone operator, since $E \subseteq E' \Rightarrow d_E^\phi \geq d_{E'}^\phi$, so that $w \geq w'$ (being $w'$ the solution of (10) with $E$ replaced with $E'$), and $T_h E \subset T_h E'$. On the other hand, it is also shown in [13, 12, 4] that this choice gives the largest solution, whereas $\{ w < 0 \}$ would be the smallest (yielding uniqueness, up to a negligible set, whenever $\{ w = 0 \} = 0$, which is "generically" true in some sense). The proof of consistency we will next give would also work with this second choice, yielding convergence of any selection of Almgren-Taylor-Wang’s scheme to the generalized solution, when unique. We now show:

**Proposition 4.1** $T_h$, defined as above, is consistent in the sense of Definition 2.2.

**Proof.** Let $E$ be a superflow on $[t_0, t_1]$, in the sense of Definition 2.1, and let $A$ be the associated neighborhood of $\partial E(t), t \in [t_0, t_1]$.

Observe that as in the previous section, there exists $\rho > 0$ such that $\{ d(\cdot, t) \leq \rho \} \subset A$ at any time $t \in [t_0, t_1]$, and $\partial E(t)$ satisfies both an interior and exterior Wulff shape condition of radius $\rho$.

We fix $t \in [t_0, t_1]$, and let $B = \{ d(\cdot, t) < \rho \}$. Consider $\psi : \mathbb{R} \to \mathbb{R}$ a smooth increasing function with $\psi(s) \geq s$ and $\psi(s) = s$ for $|s| \leq \varepsilon/2$. We set, for $x \in B$, $v(x) := \psi(d(x, t + h))$. Then, from (5), it follows
\[
\frac{v(x) - d_{E(t)}(x)}{h} \geq \frac{d(x, t + h) - d(x, t)}{h} = \frac{1}{h} \int_0^h \frac{\partial d}{\partial t}(x, t + \tau) \, d\tau
\]
\[
\geq \frac{1}{h} \int_t^{t+h} \text{div} \nabla \phi(\nabla d)(x, t + \tau) \, d\tau + \delta.
\]
Let now $\omega$ be a modulus of continuity for $\text{div} \nabla \phi^c(\nabla d)$ in $\{|d| \leq \rho\}$: we find
\[
\frac{v(x) - d_{E(t)}(x)}{h} \geq \text{div} \nabla \phi^c(\nabla d)(x, t + h) + \delta - \omega(h).
\]
Observe that for any $x \in B$ it holds $\nabla v(x) = \psi'(d(x, t + h))\nabla d(x, t + h)$, so that (recall that $\nabla \phi^c$ 0-homogeneous), $
abla \phi^c(\nabla v(x)) = \nabla \phi^c(\nabla d(x, t + h))$ hence $\text{div} \nabla \phi^c(\nabla d)(x, t + h) = \text{div} \nabla \phi^c(\nabla v)(x)$. Therefore, if $h$ is small enough so that $\omega(h) \leq \delta$, we get
\[
\frac{v(x) - d_{E(t)}(x)}{h} \geq \text{div} \nabla \phi^c(\nabla v)(x).
\]
Let $w$ solve (10), with $E = E(t)$. We will show that we may choose $\psi$ in order to have $v \geq w$ on $\partial B$, so that $v$ is a supersolution for the problem
\[
\min \left\{ \int_B \phi^c(Du) + \frac{1}{2h} \int_B (u(x) - d_{E(t)}(x))^2 \, dx : u = w \text{ on } \partial B \right\}
\]
(11)
(which is solved by $w$). We will deduce that $v \geq w$ in $B$, so that $\{w \leq 0\} \supseteq \{v \leq 0\} = \{d(-, t+h) \leq 0\}$, that is, $T_h(E(t)) \supseteq E(t+h)$.

First of all, $d$ is uniformly continuous in time, so that if $h$ is small enough, one has $d(x, t+h) \geq 3\rho/4$ if $d(x, t) = \rho$. If $M > \text{diam } \Omega$, then one shows that $M \geq w$ in $\Omega$. We may choose a function $\psi$ with $\psi(3\rho/4) \geq M$, so that $v(x) \geq M \geq w(x)$ if $d(x, t) = \rho$.

On the other hand, since $E(t)$ satisfies an interior Wulff shape condition of radius $\rho$, one has $d^0_E \leq \phi(-x) - \rho$ at any point $x \in \partial B$ with $d(x, t) = -\rho$. The analysis in [12, 15] shows that the solution of (10) with $d^0_E$ replaced with $\phi$ takes the value $2N\sqrt{\rho}/\sqrt{N+1}$ at the origin. We deduce that $w(x) \leq 2N\sqrt{\rho}/\sqrt{N+1} - \rho$: hence, if $h$ is small enough, we get $w(x) \leq -3\rho/4$. We can choose $\psi$ such that $\psi(s) \geq -3\rho/4$ for any $s$, so that $v(x) \geq w(x)$ if $d(x, t) = -\rho$. We conclude that $v \geq w$ on $\partial B$. Hence $v$ is a supersolution for (11), which implies $T_{t,t+h}(E(t)) \supseteq E(t+h)$.

If $E(t)$ is a subflow, we can reproduce the same proof to show that $T_{t,t+h}(E(t)) \subseteq E(t+h)$.

While a (much more difficult) proof of consistency with smooth flows is already found in Almgren, Taylor and Wang’s paper [1], our proof is more easily adapted to other situations: in [15], we consider the case of a flow driven by anisotropic curvature with an additional time-dependent forcing term, possibly discontinuous.

References


