Uniqueness of the Cheeger set of a convex body

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Abstract

We prove that if \( C \subset \mathbb{R}^N \) is of class \( C^2 \) and uniformly convex, then the Cheeger set of \( C \) is unique. The Cheeger set of \( C \) is the set which minimizes, inside \( C \), the ratio perimeter over volume.

1 Introduction

Given an nonempty open bounded subset \( \Omega \) of \( \mathbb{R}^N \), we call Cheeger constant of \( \Omega \) the quantity

\[
h_\Omega = \min_{K \subset \Omega} \frac{P(K)}{|K|}
\]

where \( |K| \) denotes the \( N \)-dimensional volume of \( K \) and \( P(K) \) denotes the perimeter of \( K \). The minimum in (1) is taken over all nonempty sets of finite perimeter contained in \( \Omega \). It is well-known that the minimum in (1) is attained at a subset \( G \) of \( \Omega \) such that \( \partial G \) touches \( \partial \Omega \) (otherwise we would diminish the quotient \( P(G)/|G| \) by dilating \( G \)). A Cheeger set of \( \Omega \) is any set \( G \subseteq \Omega \) which minimizes (1). We say that \( \Omega \) is Cheeger in itself if \( \Omega \) minimizes (1).

For any set of finite perimeter \( K \) in \( \mathbb{R}^N \), let us denote

\[
\lambda_K := \frac{P(K)}{|K|}.
\]

Notice that for any Cheeger set \( G \) of \( \Omega \), \( \lambda_G = h_G \). Observe also that \( G \) is a Cheeger set of \( \Omega \) if and only if \( G \) minimizes

\[
\min_{K \subset \Omega} P(K) - \lambda_G |K|.
\]

We say that a set \( \Omega \subset \mathbb{R}^N \) is calibrable if \( \Omega \) minimizes the problem

\[
\min_{K \subset \Omega} P(K) - \lambda_\Omega |K|.
\]

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In particular, if $G$ is a Cheeger set of $\Omega$, then $G$ is calibrable. Thus, $\Omega$ is a Cheeger set of itself if and only if it is calibrable.

Finding the Cheeger sets of a given $\Omega$ is a difficult task. This task is simplified if $\Omega$ is a convex set and $N = 2$. In that case, the Cheeger set in $\Omega$ is unique and is identified with the set $\Omega^R \oplus B(0, R)$ where $\Omega^R := \{x \in \Omega : \text{dist}(x, \partial \Omega) > R\}$ is such that $|\Omega^R| = \pi R^2$ and $A \oplus B := \{a + b : a \in A, b \in B\}$, $A, B \subset \mathbb{R}^2$ [1, 17]. In this case, we see that the Cheeger set of $\Omega$ is convex. Moreover, a convex set $\Omega \subset \mathbb{R}^2$ is Cheeger in itself if and only if $\max_{x \in \partial \Omega} \kappa_\Omega(x) \leq \lambda_\Omega$ where $\kappa_\Omega(x)$ denotes the curvature of $\partial \Omega$ at the point $x$. This has been proved in [13, 8, 17] (see also [1]) though it was stated in terms of calibrability in [8, 1]. The proof in [13] had also a complement result: if $\Omega$ is Cheeger in itself then $\Omega$ is strictly calibrable, that is, for any set $K \subset \Omega$, $K \neq \Omega$, then

$$0 = P(\Omega) - \lambda_\Omega|\Omega| < P(K) - \lambda_\Omega|K|,$$

and this implies that the capillary problem in absence of gravity (with vertical contact angle at the boundary)

$$\begin{align*}
-\text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) &= \lambda_\Omega \quad \text{in } \Omega \\
\frac{Du}{\sqrt{1 + |Du|^2}} \cdot \nu &= 1 \quad \text{in } \partial \Omega
\end{align*}$$

has a solution. Indeed, both problems are equivalent [13].

Our purpose in this paper is to extend the above result to $\mathbb{R}^N$, that is, to prove the uniqueness and convexity of the Cheeger set contained in a set $\Omega \subset \mathbb{R}^N$ which is uniformly convex and of class $C^2$. The characterization of a convex set $\Omega \subset \mathbb{R}^N$ of class $C^{1,1}$ which is Cheeger in itself (also called calibrable) in terms of the mean curvature of its boundary was proved in [2]. The precise result states that such a set $\Omega$ is Cheeger in itself if and only if $\kappa_\Omega(x) \leq \lambda_\Omega$ for any $x \in \partial \Omega$ where $\kappa_\Omega(x)$ denotes the sum of the principal curvatures (or total curvature) of the boundary of $\Omega$. Moreover, in [2], the authors also proved that for any convex set $\Omega \subset \mathbb{R}^N$ there exists a maximal Cheeger set contained in $\Omega$ which is convex. These results were extended to convex sets $\Omega$ satisfying a regularity condition and anisotropic norms in $\mathbb{R}^N$ (including the crystalline case) in [11].

In particular, we obtain that $\Omega \subset \mathbb{R}^N$ is the unique Cheeger set of itself, whenever $\Omega$ is a $C^2$, uniformly convex calibrable set. We point out that, by Theorems 1.1 and 4.2 in [13], this uniqueness result is equivalent to the existence of a solution $u \in W^{1,\infty}_{\text{loc}}(\Omega)$ of the capillary problem (4).

Let us explain the plan of the paper. In Section 2 we collect some definitions and recall some results about the mean curvature operator in (4) and the subdifferential of the total variation. In Section 3 we state and prove the uniqueness result.
2 Preliminaries

2.1 BV functions

Let \( \Omega \) be an open subset of \( \mathbb{R}^N \). A function \( u \in L^1(\Omega) \) whose gradient \( Du \) in the sense of distributions is a (vector valued) Radon measure with finite total variation in \( \Omega \) is called a function of bounded variation. The class of such functions will be denoted by \( BV(\Omega) \). The total variation of \( Du \) on \( \Omega \) turns out to be

\[
\sup \left\{ \int \Omega u \, \operatorname{div} z \, dx : z \in C_0^\infty(\Omega; \mathbb{R}^N), \|z\|_{L^\infty(\Omega)} := \operatorname{ess \ sup}_{x \in \Omega} |z(x)| \leq 1 \right\},
\]

(5)

(where for a vector \( v = (v_1, \ldots, v_N) \in \mathbb{R}^N \) we set \( |v|^2 := \sum_{i=1}^N v_i^2 \) and will be denoted by \( |Du|(\Omega) \) or by \( \int_\Omega |Du| \). The map \( u \to |Du|(\Omega) \) is \( L^1_{\text{loc}}(\Omega) \)-lower semicontinuous. \( BV(\Omega) \) is a Banach space when endowed with the norm \( \int_\Omega |u| \, dx + |Du|(\Omega) \). We recall that \( BV(\mathbb{R}^N) \subseteq L^{N/(N-1)}(\mathbb{R}^N) \).

A measurable set \( E \subseteq \mathbb{R}^N \) is said to be of finite perimeter in \( \mathbb{R}^N \) if (5) is finite when \( u \) is substituted with the characteristic function \( \chi_E \) of \( E \) and \( \Omega = \mathbb{R}^N \). The perimeter of \( E \) is defined as \( P(E) := |D\chi_E|(\mathbb{R}^N) \). For results and informations on functions of bounded variation we refer to [4].

Finally, let us denote by \( \mathcal{H}^{N-1} \) the \( (N-1) \)-dimensional Hausdorff measure. We recall that when \( E \) is a finite-perimeter set with regular boundary (for instance, Lipschitz), its perimeter \( P(E) \) also coincides with the more standard definition \( \mathcal{H}^{N-1}(\partial E) \).

2.2 A generalized Green’s formula

Let \( \Omega \) be an open subset of \( \mathbb{R}^N \). Following [6], let

\[ X_2(\Omega) := \{ z \in L^\infty(\Omega; \mathbb{R}^N) : \operatorname{div} z \in L^2(\Omega) \}. \]

If \( z \in X_2(\Omega) \) and \( w \in L^2(\Omega) \cap BV(\Omega) \) we define the functional \( (z \cdot Dw) : C_0^\infty(\Omega) \to \mathbb{R} \) by the formula

\[
< (z \cdot Dw), \varphi > := - \int_\Omega w \, \varphi \, \operatorname{div} z \, dx - \int_\Omega w \, z \cdot \nabla \varphi \, dx.
\]

Then \( z \cdot Dw \) is a Radon measure in \( \Omega \),

\[
\int_\Omega (z \cdot Dw) = \int_\Omega z \cdot \nabla w \, dx \quad \forall w \in L^2(\Omega) \cap W^{1,1}(\Omega).
\]

Recall that the outer unit normal to a point \( x \in \partial \Omega \) is denoted by \( \nu^\Omega(x) \). We recall the following result proved in [6].

**Theorem 1.** Let \( \Omega \subset \mathbb{R}^N \) be a bounded open set with Lipschitz boundary. Let \( z \in L^\infty(\Omega; \mathbb{R}^N) \) with \( \operatorname{div} z \in L^2(\Omega) \). Then there exists a function \( [z \cdot \nu^\Omega] \in L^\infty(\partial \Omega) \) satisfying \( \| [z \cdot \nu^\Omega] \|_{L^\infty(\partial \Omega)} \leq \|z\|_{L^\infty(\Omega; \mathbb{R}^N)} \), and such that for any \( u \in BV(\Omega) \cap L^2(\Omega) \) we have

\[
\int_\Omega u \, \operatorname{div} z \, dx + \int_\Omega (z \cdot Du) = \int_{\partial \Omega} [z \cdot \nu^\Omega] \, u \, d\mathcal{H}^{N-1}.
\]

Moreover, if \( \varphi \in C^1(\overline{\Omega}) \) then \( (\varphi z) \cdot \nu^\Omega = \varphi [z \cdot \nu^\Omega] \).
This result is complemented with the following result proved by Anzellotti in [7].

**Theorem 2.** Let \( \Omega \subset \mathbb{R}^N \) be a bounded open set with a boundary of class \( C^1 \). Let \( z \in C(\overline{\Omega}, \mathbb{R}^N) \) with \( \text{div} \, z \in L^2(\Omega) \). Then

\[
[z \cdot \nu^\Omega](x) = z(x) \cdot \nu^\Omega(x) \quad \mathcal{H}^{N-1} \text{ a.e. on } \partial \Omega.
\]

### 2.3 Some auxiliary results

Let \( \Omega \) be an open bounded subset of \( \mathbb{R}^N \) with Lipschitz boundary, and let \( \varphi \in L^1(\Omega) \). For all \( \epsilon > 0 \), we let \( \Psi^\epsilon_\varphi : L^2(\Omega) \to (-\infty, +\infty] \) be the functional defined by

\[
\Psi^\epsilon_\varphi(u) := \begin{cases} 
\int_\Omega \sqrt{\epsilon^2 + |Du|^2} + \int_{\partial \Omega} |u - \varphi| & \text{if } u \in L^2(\Omega) \cap BV(\Omega) \\
+\infty & \text{if } u \in L^2(\Omega) \setminus BV(\Omega).
\end{cases}
\]

(6)

As it is proved in [14], if \( f \in W^{1,\infty}(\Omega) \), then the minimum \( u \in BV(\Omega) \) of the functional

\[
\Psi^\epsilon_\varphi(u) + \int_{\Omega} |u(x) - f(x)|^2 \, dx
\]

(7)

belongs to \( u \in C^{2+\alpha}(\Omega) \), for every \( \alpha < 1 \). The minimum \( u \) of (7) is a solution of

\[
\begin{cases}
\frac{1}{\lambda} \text{div} \left( \frac{Du}{\sqrt{\epsilon^2 + |Du|^2}} \right) = f(x) & \text{in } \Omega \\
u = \varphi & \text{on } \partial \Omega
\end{cases}
\]

(8)

where the boundary condition is taken in a generalized sense [18], i.e.,

\[
\left[ \frac{Du}{\sqrt{\epsilon^2 + |Du|^2}}, \nu^\Omega \right] \in \text{sign}(\varphi - u) \quad \mathcal{H}^{N-1} \text{ a.e. on } \partial \Omega.
\]

Observe that (8) can be written as

\[
u + \frac{1}{\lambda} \partial \Psi^\epsilon_\varphi(u) \ni f.
\]

(9)

We are particularly interested in the case where \( \varphi = 0 \). As we shall show below (see also [2]) in the case of interest to us we have \( u > 0 \) on \( \partial \Omega \) and, thus, \( \left[ \frac{Du}{\sqrt{\epsilon^2 + |Du|^2}}, \nu^\Omega \right] = -1 \) \( \mathcal{H}^{N-1} \text{ a.e. on } \partial \Omega \). It follows that \( u \) is a solution of the first equation in (8) with vertical contact angle at the boundary.

As \( \epsilon \to 0^+ \), the solution \( u_\epsilon \) of (8) converges to the solution of

\[
\begin{cases}
\frac{1}{\lambda} \partial \Psi_\varphi(u) = f(x) & \text{in } \Omega \\
u = \varphi & \text{on } \partial \Omega.
\end{cases}
\]

(10)
where $\Psi : L^2(\Omega) \to (-\infty, +\infty]$ is given by

$$
\Psi_\varphi(u) := \begin{cases} 
\int_{\mathbb{R}^N} |Du| + \int_{\partial \Omega} |u - \varphi| & \text{if } u \in L^2(\Omega) \cap BV(\Omega) \\
+\infty & \text{if } u \in L^2(\Omega) \setminus BV(\Omega).
\end{cases}
$$

(11)

In this case $\partial \Psi_\varphi$ represents the operator $-\text{div} (Du/|Du|)$ with the boundary condition $u = \varphi$ in $\partial \Omega$, and this connection is precisely given by the following Lemma (see [5]).

**Lemma 2.1.** The following assertions are equivalent:

(a) $v \in \partial \Psi_\varphi(u)$;

(b) $u \in L^2(\Omega) \cap BV(\Omega)$, $v \in L^2(\Omega)$, and there exists $z \in X_2(\Omega)$ with $\|z\|_\infty \leq 1$, such that

$$
v = -\text{div} z \quad \text{in } D'(\Omega),$$

$$
(z \cdot Du) = |Du|,
$$

and

$$
[z \cdot \nu^\Omega] \in \text{sign}(\varphi - u) \quad \mathcal{H}^{N-1} \text{ a.e. on } \partial \Omega.
$$

Notice that the solution $u \in L^2(\Omega)$ of (10) minimizes the problem

$$
\min_{u \in BV(\Omega)} \int_{\Omega} |Du| + \int_{\partial \Omega} |u(x) - \varphi(x)|d\mathcal{H}^{N-1}(x) + \frac{\lambda}{2} \int_{\Omega} |u(x) - f(x)|^2 dx,
$$

and the two problems are equivalent.

## 3 The uniqueness theorem

We now state our main result.

**Theorem 3.** Let $C$ be a convex body in $\mathbb{R}^N$. Assume that $C$ is uniformly convex, with boundary of class $C^2$. Then the Cheeger set of $C$ is convex and unique.

We do not believe that the $C^2$ assumption is essential for this result, although we could not remove it. Removing the assumption of uniform convexity is probably more tricky. Let us recall the following result proved in [2] (Theorems 6 and 8 and Proposition 4).

**Theorem 4.** Let $C$ be a convex body in $\mathbb{R}^N$ with boundary of class $C^{1,1}$. For any $\lambda, \varepsilon > 0$, there is a unique solution $u_\varepsilon$ of the equation:

$$
\begin{cases} 
\frac{D_{u_\varepsilon}}{\lambda \varepsilon^2 + |Du_\varepsilon|^2} - \lambda \varepsilon^2 = 1 & \text{in } C \\
\lambda \varepsilon^2 = 0 & \text{on } \partial C,
\end{cases}
$$

(13)
such that $0 \leq u_\varepsilon \leq 1$. Moreover, there exist $\lambda_0$ and $\varepsilon_0$, depending only on $\partial C$, such that if $\lambda \geq \lambda_0$ and $\varepsilon \leq \varepsilon_0$, then $u_\varepsilon$ is a concave function such that $u_\varepsilon \geq \alpha > 0$ on $\partial C$ for some $\alpha > 0$. Hence, $u_\varepsilon$ satisfies

$$
\left[ \frac{Du_\varepsilon}{\sqrt{\varepsilon^2 + |Du_\varepsilon|^2}} \cdot \nu^C \right] = \text{sign}(0 - u_\varepsilon) = -1 \quad \text{on } \partial C.
$$

(14)

As $\varepsilon \to 0$, the functions $u_\varepsilon$ converge to the concave function $u$ which minimizes the problem

$$
\min_{u \in BV(C)} \int_C |Du| + \int_{\partial C} |u(x)| d\mathcal{H}^{N-1}(x) + \frac{\lambda}{2} \int_C |u(x) - 1|^2 dx
$$

or, equivalently, if $u$ is extended with zero out of $C$, $u$ minimizes

$$
\int_{\mathbb{R}^N} |Du| + \frac{\lambda}{2} \int_{\mathbb{R}^N} |u - \chi_C|^2 dx.
$$

The function $u$ satisfies $0 \leq u < 1$. Moreover, the level set $\{u \geq t\}$, $t \in (0, 1]$, is contained in $C$ and minimizes the problem

$$
\min_{F \subset C} P(F) - \lambda(1-t)|F|.
$$

(16)

It was proved in [2] (see also [11]) that the set $C^* = \{u = \max_C u\}$ is the maximal Cheeger set contained in $C$, that is, the maximal set that solves (1). Moreover, one has $u = 1 - h_C/\lambda > 0$ in $C^*$ and $h_C = \lambda C^*$.

If we want to consider what happens inside $C^*$ and, in particular, if there are other Cheeger sets, we have to analyze the level sets of $u_\varepsilon$ before passing to the limit as $\varepsilon \to 0^+$. In order to do this, let us introduce the following rescaling of $u_\varepsilon$:

$$
v_\varepsilon = \frac{u_\varepsilon - m_\varepsilon}{\varepsilon} \leq 0,
$$

where $m_\varepsilon = \max_C u_\varepsilon \to 1 - h_C/\lambda$ as $\varepsilon \to 0$. The function $v_\varepsilon$ is a generalized solution of the equation:

$$
\begin{cases}
\varepsilon v_\varepsilon - \frac{1}{\lambda} \text{div} \frac{Dv_\varepsilon}{\sqrt{1 + |Dv_\varepsilon|^2}} = 1 - m_\varepsilon & \text{in } C \\
v_\varepsilon = -m_\varepsilon/\varepsilon & \text{on } \partial C.
\end{cases}
$$

(17)

We let $z_\varepsilon = Du_\varepsilon / \sqrt{\varepsilon^2 + |Du_\varepsilon|^2} = Dv_\varepsilon / \sqrt{1 + |Dv_\varepsilon|^2}$. Notice that $z_\varepsilon$ is a vector field in $L^\infty(C)$, with uniformly bounded divergence, such that $|z_\varepsilon| \leq 1$ a.e. in $C$ and, by (14), $[z_\varepsilon \cdot \nu_C] = -1$ on $\partial C$.

Let us study the limit of $v_\varepsilon$ and $z_\varepsilon$ as $\varepsilon \to 0$. Let us observe that, for each $\varepsilon > 0$ small enough and each $s \in (0, |C|)$, there is a (convex) superlevel set $C^\varepsilon_s$ of $v_\varepsilon$ such that $|C^\varepsilon_s| = s$ for $s \in (0, |C|)$. First we observe that $\{v_\varepsilon = 0\}$ is a null set. Otherwise, since $v_\varepsilon$ is concave, it would be a convex set of positive measure, and it would have a nonempty interior. We would have that $v_\varepsilon = \text{div} z_\varepsilon = 0$, hence $1 - m_\varepsilon = 0$ in the interior of $\{v_\varepsilon = 0\}$. This is a contradiction with Theorem 4 for $\varepsilon > 0$ small enough. Hence we may take $C^\varepsilon_0 := \{v_\varepsilon = 0\}$.
Now, the concavity of $v^\varepsilon$ guarantees the existence of the foliation $C^\varepsilon_s$ made of superlevel sets of $v^\varepsilon$ such that $|C^\varepsilon_s| = s$ for $s \in (0, |C|)$.

We observe that a sequence of uniformly bounded convex sets is compact both for the $L^1$ and Hausdorff topologies. Hence, up to a subsequence, we may assume that $C^\varepsilon_s$ converge to convex sets $C_s$, each of volume $s$, first for any $s \in \mathbb{Q} \cap (0, |C|)$ and then by continuity for any $s$. Possibly extracting a further subsequence, we may assume that there exists $s_* \in [0, |C|]$, such that $v^\varepsilon$ goes to a concave function $v$ in $C_s$ for any $s < s_*$, and to $-\infty$ outside $C_s := C_{s_*}$. We may also assume that $z^\varepsilon \rightharpoonup z$ weakly* in $L^\infty(C)$, for some vector field $z$, satisfying $|z| \leq 1$ a.e. in $C$. From (13) we have in the limit

$$-\text{div } z = \lambda(1-u) \quad \text{in } D'(C).$$

(18)

Moreover, by the results recalled in Section 2, it holds $-\text{div } z \in \partial \Psi_0(u)$. We see from (18) that

$$-\text{div } z = h_C \quad \text{in } C^\varepsilon_s,$$

(19)

while $-\text{div } z > h_C$ a.e. on $C \setminus C^\varepsilon_s$. We let $s^* := |C^\varepsilon_s|$, so that $C^\varepsilon_s = C_{s^*}$. By Theorem 4, for $s \geq s^*$, the set $C_s$ is a minimizer of $P(E) - \mu_E |E|$ among all $E \subseteq C$, for some $\mu_E \geq h_C$ which is equal to the constant value of $-\text{div } z$ on $\partial C_s \cap C$, and is bounded by $P(C)/(|C| - s)$. For $s > s^*$, we have $\mu_s > h_C$ and the set $C_{s^*}$ is the unique minimizer of the variational problem. As a consequence (see [2, 11]) for any $s > s^*$ the set $C_s$ is also the unique minimizer of $P(E)$ among all $E \subseteq C$ of volume $s$.

**Lemma 3.1.** We have $s_* > 0$ and the sets $C_s$ are Cheeger sets in $C$ for any $s \in [s_*, s^*]$.

**Proof.** Let $s_* < s \leq |C|$. If $x \in \partial C^\varepsilon_s \setminus \partial C$, then

$$0 - v^\varepsilon(x) \leq Dv^\varepsilon(x) \cdot (\bar{x}^\varepsilon - x)$$

where $v^\varepsilon(\bar{x}^\varepsilon) = \max_C v^\varepsilon$. Hence, $\lim_{\varepsilon \to 0} \inf_{\partial C^\varepsilon_s \setminus \partial C} |Dv^\varepsilon| = +\infty$. Since $[z^\varepsilon \cdot \nu^\varepsilon] = -1$ on $\partial C$ and $P(C^\varepsilon_s) \to P(C_s)$, we deduce

$$-\int_{\partial C^\varepsilon_s} [z^\varepsilon(x) \cdot \nu^\varepsilon(x)] \, d\mathcal{H}^{N-1}(x)$$

$$= \int_{\partial C^\varepsilon_s \setminus \partial C} \frac{|Dv^\varepsilon(x)|}{\sqrt{1 + |Dv^\varepsilon(x)|^2}} \, d\mathcal{H}^{N-1}(x) + \mathcal{H}^{N-1}(\partial C^\varepsilon_s \cap \partial C) \to P(C_s)$$

as $\varepsilon \to 0^+$. Hence,

$$\int_{\partial C_s} [z \cdot \nu^C] \, d\mathcal{H}^{N-1} = \int_{C_s} \text{div } z = \lim_{\varepsilon \to 0} \int_{C^\varepsilon_s} \text{div } z^\varepsilon$$

$$= \lim_{\varepsilon \to 0} \int_{\partial C^\varepsilon_s} [z^\varepsilon \cdot \nu^\varepsilon] \, d\mathcal{H}^{N-1} = -P(C)$$. 

Since $|z| \leq 1$ a.e. in $C$, we deduce that $[z \cdot \nu^C] = -1$ on $\partial C_s$ for any $s > s_*$ (in particular, we have $|z| = 1$ a.e. in $C \setminus C_s$). Using this and (19), for all $s_* < s \leq s^*$ we have

$$\frac{P(C_s)}{|C_s|} = h_C.$$ 

(20)
This has two consequences. First, from the isoperimetric inequality, we obtain
\[ h_C = \frac{P(C_s)}{|C_s|} \geq \frac{P(B_1)}{|B_1|^{\frac{N-1}{N}} s^{\frac{1}{N}}}, \]
if \( s \in (s_*, s^*) \), so that \( s_* > 0 \). Moreover, \( C_s \) is a Cheeger set for any \( s \in (s_*, s^*) \), and by continuity \( C_s \) is also a Cheeger set.

We point out that, since the sets \( C_s \) are convex minimizers of \( P(E) - \mu_s |E| \) among all \( E \subseteq C \), for \( s \geq s_* \), their boundary is of class \( C^{1,1} \) \([9, 19]\), with curvature less than or equal to \( \mu_s \), and equal to \( \mu_s \) in the interior of \( C \) (note that \( \mu_s = h_C \) for \( s \in [s_*, s^*] \)).

**Remark 3.2.** Observe that we have either \( s_* = s^* \) and therefore \( C_s = C^* \), or \( s_* < s^* \), and we have \( C^* = \bigcup_{s \in (s_*, s^*)} C_s \). In this case, the supremum of the total curvature of \( \partial C^* \) is equal to \( h_C \). Indeed, if it were not the case, by considering \( C' \subseteq \text{int}(C^*) \), with curvature strictly below \( h_C \), and the smallest set \( C_s \), with \( s > s_* \), which contains \( C' \), we would have \( \kappa_{C^*}(x) \geq \kappa_{C_s}(x) = h_C \) at all \( x \in \partial C' \cap \partial C_s \), a contradiction. In particular, if the supremum of the total curvature of \( \partial C \) is strictly less than \( P(C)/|C| \) (which implies \( C = C^* \) by \([2]\)) then \( C = C_s \).

From the strong convergence of \( Dv_{\epsilon} \) to \( Dv \) (in \( L^2(C_s) \) for any \( s < s_* \)), we deduce that \( z = \frac{Dv}{\sqrt{1 + |Dv|^2}} \) in \( C_s \). It follows that \( v \) satisfies the equation
\[ -\text{div} \left( \frac{Dv}{\sqrt{1 + |Dv|^2}} \right) = h_C \quad \text{in } C_s. \quad (21) \]

Integrating both terms of (21) in \( C_s \), we deduce that
\[ \left[ \frac{Dv}{\sqrt{1 + |Dv|^2}}, \nu^C \right] = -1 \quad \text{on } \partial C_s. \]

**Lemma 3.3.** The set \( C_s \) is the minimal Cheeger set of \( C \), i.e., any other Cheeger set of \( C \) must contain \( C_s \).

**Proof.** Let \( K \subseteq C^* \) be a Cheeger set in \( C \). We have
\[ h_C |K| = -\int_K \text{div } z = -\int_{\partial K} [z \cdot \nu^K] dH^{N-1} = P(K) \]
so that \([z \cdot \nu^K] = -1 \) a.e. on \( \partial K \). Let \( \nu^\epsilon \) and \( \nu \) be the vector fields of unit normals to the sets \( C_s^\epsilon \) and \( C_s \), \( s \in [0, |C|] \), respectively. Observe that, by the Hausdorff convergence of \( C_s^\epsilon \) to \( C_s \) as \( \epsilon \to 0^+ \) for any \( s \in [0, |C|] \), we have that \( \nu^\epsilon \to \nu \) a.e. in \( C \). On the other hand, \(|z^\epsilon + \nu^\epsilon| \to 0 \) locally uniformly in \( C \setminus C_s \) because of the definition of \( z^\epsilon \) and the fact that \(|Dv_{\epsilon}| \to \infty \) outside \( C_s \). Both things imply that \( z = -\nu \) a.e. on \( C \setminus C_s \). By modifying \( z \) in a set of null measure, we may assume that \( z = -\nu \) on \( C \setminus C_s \). We recall that the sets \( C_s, s \geq s_* \), are minimizers of variational problems of the form \( \min_{K \subseteq C} P(K) - \mu |K| \), for some values of \( \mu \) (with \( \mu = h_C \) as long as \( s \leq s^* \) and \( \mu > h_C \) continuously increasing with \( s \geq s^* \)). Since these sets are convex, with boundary (locally) uniformly of class \( C^{1,1} \), and
the map \( s \to C_s \) is continuous in the Hausdorff topology, we obtain that the normal \( \nu(x) \) is a continuous function in \( C \setminus \text{int}(C_s) \).

Since \(|z| < 1 \) inside \( C_s \) and \([z \cdot \nu^K] = -1 \) a.e. on \( \partial K \), by [6, Theorem 1]) we have that the boundary of \( K \) must be outside the interior of \( C_s \), hence either \( K \supseteq C_s \) or \( K \cap C_s = \emptyset \) (modulo a null set). Let us prove that the last situation is impossible. Indeed, assume that \( K \cap C_s = \emptyset \) (modulo a null set). Since \( \partial K \) is of class \( C^1 \) out of a closed set of zero \( \mathcal{H}^{N-1} \)-measure (see [15]) and \( z \) is continuous in \( C \setminus \text{int}(C_s) \), by Theorem 2 we have

\[
z(x) \cdot \nu^K(x) = -1 \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial K.
\]

Now, since \( K \cap C_s = \emptyset \) (modulo a null set), then there is some \( s \geq s_s \) and some \( x \in \partial C_s \cap \partial K \) such that \( \nu^K(x) + \nu(x) = 0 \). Fix \( 0 < \epsilon < 2 \). By a slight perturbation, if necessary, we may assume that \( x \in \partial C_s \cap \partial K \) with \( s > s_s \), (22) holds at \( x \) and

\[
|\nu^K(x) + \nu(x)| < \epsilon.
\]

Since by (22) we have \( \nu(x) = -z(x) = \nu^K(x) \) we obtain a contradiction with (23). We deduce that \( K \supseteq C_s \).

Therefore, in order to prove uniqueness of the Cheeger sets of \( C \), it is enough to show that

\[
C_s = C^*.
\]

Recall that the boundary of both \( C_s \) and \( C^* \) is of class \( C^{1,1} \), and the sum of its principal curvatures is less than or equal to \( h_C \), and constantly equal to \( h_C \) in the interior of \( C \). We now show that if \( C_s \neq C^* \) and under additional assumptions, the sum of the principal curvatures of the boundary of \( C^* \) (or of any \( C_s \) for \( s \in (s_s, s^*) \)) must be \( h_C \) out of \( C_s \).

**Lemma 3.4.** Assume that \( C \) has \( C^2 \) boundary. Let \( s \in (s_s, s^*) \) and \( x \in \partial C_s \setminus \partial C_s \). If the sum of the principal curvatures of \( \partial C_s \) at \( x \) is strictly below \( h_C \), then the Gaussian curvature of \( \partial C \) at \( x \) is 0.

**Proof.** Let \( x \in \partial C_s \setminus \partial C_s \) and assume the sum of the principal curvatures of \( \partial C_s \) at \( x \) is strictly below \( h_C \) (assuming \( x \) is a Lebesgue point for the curvature on \( \partial C_s \)). Necessarily, this implies that \( x \in \partial C \). Assume then that the Gauss curvature of \( \partial C \) at \( x \) is positive: by continuity, in a neighborhood of \( x \), \( C \) is uniformly convex and the sum of the principal curvatures is less than \( h_C \). We may assume that near \( x \), \( \partial C \) is the graph of a non-negative, \( C^2 \) and convex function \( f : B \to \mathbb{R} \) where \( B \) is an \( (N-1) \)-dimensional ball centered at \( x \), while \( \partial C_s \) is the graph of \( f_s : B \to \mathbb{R} \), which is \( C^{1,1} \) \([9, 19]\), and also nonnegative and convex. In \( B \), we have \( f_s \geq f \geq 0 \), and

\[
D^2 f \geq \alpha I \quad \text{and} \quad \text{div} \frac{Df}{\sqrt{1 + |Df|^2}} = h
\]

with \( h \in C^0(B) \), \( h < h_C \), \( \alpha > 0 \), while

\[
\text{div} \frac{Df_s}{\sqrt{1 + |Df_s|^2}} = h \chi_{\{f = f_s\}} + h_C \chi_{\{f > f_s\}}
\]

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(where \( \chi_{\{f=f_s\}} \) has positive density at \( x \).

We let \( g = f_s - f \geq 0 \). Introducing now the Lagrangian \( \Psi : \mathbb{R}^{N-1} \to [0, +\infty) \) given by 
\[
\Psi(p) = \sqrt{1 + |p|^2},
\]
we have that for a.e. \( y \in B \)
\[
(h_C - h(y))\chi_{\{y>0\}}(y) = \text{div} \left( D\Psi(Df_s(y)) - D\Psi(Df(y)) \right)
\]
\[
= \text{div} \left( \left( \int_0^1 D^2\Psi(Df(y) + t(Df_s(y) - Df(y))) \, dt \right) Dg(y) \right)
\]
so that, letting \( A(y) := \int_0^1 D^2\Psi(Df(y) + tDg(y)) \, dt \) (which is a positive definite matrix and Lipschitz continuous inside \( B \)), we see that \( g \) is the minimizer of
\[
\int_B A(y)Dg(y) \cdot Dg(y) + (h_C - h(y))g(y) \, dy
\]
under the constraint \( g \geq 0 \) and with boundary condition \( g = f_s - f \) on \( \partial B \). Adapting the results in [10] we get that \( \{f = f_s\} = \{g = 0\} \) is the closure of a nonempty open set with boundary of zero \( \mathcal{H}^{N-1} \)-measure, unless the problem is unconstrained, which would yield \( h = h_C \) a.e., but we have assumed this is not the case.

We therefore have found an open subset \( D \subset \partial C \cap \partial C_s \), disjoint from \( \partial C_s \), on which \( C \) is uniformly convex, with curvature less than \( h_C \). Letting now \( \varphi \) is a smooth, nonnegative function with compact support in \( D \), one easily shows that if \( \varepsilon > 0 \) is small enough, \( \partial C_s - \varepsilon \varphi \nabla C_s \) is a boundary of a set \( C'_\varepsilon \) which is still convex, with \( P(C'_\varepsilon)/|C'_\varepsilon| > P(C_s)/|C_s| = h_C \) (just differentiate the map \( \varepsilon \to P(C'_\varepsilon)/|C'_\varepsilon| \), and the sum of its principal curvatures is less than \( h_C \). This implies that for \( \varepsilon > 0 \) small enough, the set \( C'_\varepsilon := C_s \) is calibrable [2], which in turn implies that \( \min_{K \subset C'} P(K)/|K| = P(C')/|C'|. \) But this contradicts \( C_s \subset C' \), which is true if \( \varepsilon \) was chosen small enough. \( \square \)

**Proof of Theorem 3.** Assume that \( C \) is \( C^2 \) and uniformly convex. Let us prove that its Cheeger set is unique. Assume by contradiction that \( C^* \neq C_s \). From Lemma 3.4 we have that the sum of the principal curvatures of \( \partial C^* \) is \( h_C \) outside of \( C_s \).

Let now \( \bar{x} \in \partial C^* \cap \partial C_s \) be such that \( \partial C^* \cap B_\rho(\bar{x}) \neq \emptyset \) for all \( \rho > 0 \) (\( \partial C^* \cap \partial C_s \neq \emptyset \) since otherwise both \( C^* \) and \( C_s \) would be balls, which is impossible). Letting \( T \) be the tangent hyperplane to \( \partial C^* \) at \( \bar{x} \), we can write \( \partial C^* \) and \( \partial C_s \) as the graph of two positive convex functions \( v^* \) and \( v_s \), respectively, over \( T \cap B_\rho(\bar{x}) \). For \( \rho > 0 \) small enough. Identifying \( T \cap B_\rho(\bar{x}) \) with \( B_\rho \subset \mathbb{R}^{N-1} \), we have that \( v_s, v^* : B_\rho \to \mathbb{R} \) both solve the equation
\[
-\text{div} \frac{Dv}{\sqrt{1 + |Dv|^2}} = f, \tag{25}
\]
for some function \( f \in L^\infty(B_\rho) \), moreover it holds \( v_s \geq v^* \), \( v_s(0) = v^*(0) \) and \( v_s(y) > v^*(y) \) for some \( y \in B_\rho \). Notice that \( f = \lambda_C \) in the (open) set where \( v_s > v^* \), in particular both functions are smooth in this set. Let \( D \) be an open ball such that \( \overline{D} \subset B_\rho \), \( v_s > v^* \) on \( D \) and \( v_s(y) = v^*(y) \) for some \( y \in \partial D \). Notice that, since both \( v^* \) and \( v_s \) belong to \( C^\infty(D) \cap C^1(\overline{D}) \), the fact that \( v_s(y) = v^*(y) \) also implies that \( Dv_s(y) = Dv^*(y) \). In \( D \), both functions solve
(25) with $f = \lambda_C$. Letting now $w = v_s - v^*$, we have that $w(y) = 0$ and $Dw(y) = 0$, while $w > 0$ inside $D$. Recalling the function $\Psi(p) = \sqrt{1 + |p|^2}$, we have that for any $x \in D$

$$0 = \text{div} \left( D\Psi(Dv_s(x)) - D\Psi(Dv^*(x)) \right)$$

$$= \text{div} \left( \left( \int_0^1 D^2\Psi(Dv^*(x)) + t(Dv_s(x) - Dv^*(x)) \right) dt \right) Dw(x)$$

so that $w$ solves a linear, uniformly elliptic equation with smooth coefficients. Then Hopf’s lemma [12] implies that $Dw(y) \cdot \nu_D(y) < 0$, a contradiction. Hence $C_s = C^*$.

**Remark 3.5.** Notice that, as a consequence of Theorem 3 and the results of Giusti [13], we get that if $C$ is of class $C^2$ and uniformly convex, equation (21) has a solution on the whole of $C$, if and only if $C$ is a Cheeger set of itself, i.e. if and only if the sum of the principal curvatures of $\partial C$ is less than or equal to $P(C)/|C|$.

**Remark 3.6.** The results of this paper can be easily extended to the anisotropic setting (see [11]) provided the anisotropy is smooth and uniformly elliptic.

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**References**


