On the jump set of solutions of the Total Variation flow

V. Caselles[*] , K. Jalalzai [†] ,   M. Novaga [‡]

Abstract

We show that the jump set of the solution of the minimizing Total Variation flow decreases with time for any initial condition in $BV(\Omega) \cap L^N(\Omega)$. We prove that the size of the jump also decreases with time.

1   Introduction

The use of total variation as a regularization tool for image denoising and restoration was introduced by L. Rudin, S. Osher and E. Fatemi in [18]. If Ω denotes the image domain, the Total Variation denoising problem corresponds to solving the minumum problem

$$\min_{u \in BV(\Omega)} \left\{ \int_\Omega |Du| + \frac{1}{2\lambda} \int_\Omega (u-f)^2 \, dx, \qquad \lambda > 0 \right\}. \tag{1}$$

One of the main features of (1), confirmed by numerical simulations, is its ability to restore the discontinuities of the image [18, 13, 15]. The a priori assumption is that functions of bounded variation (the BV model [3]) are a reasonable functional setting for many problems in image processing and, in particular, for denoising and restoration. Typically, functions of bounded variation admit a set of discontinuities which is countably rectifiable [3], being continuous (in the measure theoretic sense) away from discontinuities. The discontinuity set corresponds to the edges in the image, and the ability of Total Variation regularization to recover edges is one of the main features which advocates for the use of this model in image processing (its ability to describe textures is less clear, even if some textures can be recovered up to a certain scale of oscillation).

As a support to this idea, in [10] (see also [11, 12]) it has been proved that the jump set of the solution u of the TV denoising problem (1) is contained in the jump set of the datum f, assuming that $f \in BV(\Omega) \cap L^\infty(\Omega)$. Moreover the size of the jump of u at any point x in its jump set is bounded by the size of the jump of f. This result was explicitly stated in [17] in a more general context, even if for the Total Variation it is essentially contained in [10] (see also [12]).

The purpose of this paper is to prove the corresponding result for solutions of the the minimizing Total Variation flow, with Neumann or Dirichlet boundary conditions, and for the Cauchy problem in $\mathbb{R}^N$. That is, if $u(t)$ is the solution of the TV flow and $J_{u(t)}$ denotes the jump set of $u(t)$, we prove that

$$J_{u(t)} \subseteq J_{u(s)} \subseteq J_{u(0)} \quad \mathcal{H}^{N-1} \text{ a.e. for any } t > s > 0. \tag{2}$$

---

[*]Departament de Tecnologia, Universitat Pompeu-Fabra, Barcelona, Spain, `vicent.caselles@upf.edu`

[†]CMAP, CNRS UMR 7641, Ecole Polytechnique, 91128 Palaiseau, France, `khalid.jalalzai@polytechnique.edu`

[‡]Dipartimento di Matematica, Università di Padova, Via Trieste 63, 35121 Padova, Italy, `novaga@math.unipd.it`

Moreover, letting $[u(t)]$ be the jump size of $u(t)$, we also show that

$$[u(t)] \leq [u(s)] \quad \mathcal{H}^{N-1} \text{ a.e. on } J_{u(t)} \text{ for any } t \geq s \geq 0. \tag{3}$$

The inclusions in (2) have already been proved in [10], under the assumption that u_0 belongs to the domain of the operator $-\text{div}\left(\frac{Du}{|Du|}\right)$ in $L^\infty(\Omega)$. Some sufficient conditions for this to happen were given in [10]. By exploting (3), in the present paper we get rid of this condition on $u(0)$.

Let us finally describe the plan of the paper. In Section 2 we recall some basic facts about functions of bounded variation that will be used in the sequel. In Section 3 we review the result of [10] on the jump of the solutions of the denoising problem (1), explicitly proving that the size of the jump of the solution is bounded by the size of the jump of the initial datum. Finally, in Section 4 we extend this result to the Total Variation flow.

2 Notation and preliminaries on BV functions

Let Ω be an open subset of $\mathbb{R}^N$. A function $u \in L^1(\Omega)$ whose gradient Du in the sense of distributions is a (vector valued) Radon measure with finite total variation in Ω is called a function of bounded variation. The class of such functions will be denoted by $BV(\Omega)$. The total variation of Du on Ω turns out to be

$$\sup\left\{\int_\Omega u \, \text{div} \, z \, dx : z \in C_0^\infty(\Omega; \mathbb{R}^N), |z(x)| \leq 1 \; \forall x \in \Omega\right\}, \tag{4}$$

(where for a vector $v = (v_1, \ldots, v_N) \in \mathbb{R}^N$ we set $|v|^2 := \sum_{i=1}^N v_i^2$) and will be denoted by $|Du|(\Omega)$ or by $\int_\Omega |Du|$. The map $u \to |Du|(\Omega)$ is $L^1_{\text{loc}}(\Omega)$-lower semicontinuous. $BV(\Omega)$ is a Banach space when endowed with the norm $\|u\| := \int_\Omega |u| \, dx + |Du|(\Omega)$.

A measurable set $E \subseteq \Omega$ is said to be of finite perimeter in Ω if (4) is finite when u is substituted with the characteristic function χ_E of E. The perimeter of E in Ω is defined as $P(E, \Omega) := |D\chi_E|(\Omega)$. We denote by $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$, respectively, the N-dimensional Lebesgue measure and the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$.

Let $u \in [L^1_{\text{loc}}(\Omega)]^m$. We say that u has approximate limit at $x \in \Omega$ if there exists $z \in \mathbb{R}^m$ such that

$$\lim_{\rho \downarrow 0} \frac{1}{|B(x, \rho)|} \int_{B(x, \rho)} |u(y) - z| dy = 0. \tag{5}$$

The set of points where this does not hold is called the approximate discontinuity set of u, and is denoted by S_u. Using Lebesgue's differentiation theorem, one can show that the approximate limit z exists at $\mathcal{L}^N$-a.e. $x \in \Omega$, and is equal to $u(x)$: in particular, $|S_u| = 0$.

If $x \in \Omega \setminus S_u$, the vector z is uniquely determined by (5) and we denote it by $\tilde{u}(x)$. We say that u is approximately continuous at x if $x \notin S_u$ and $\tilde{u}(x) = u(x)$, that is if x is a Lebesgue point of u with respect to the Lebesgue measure.

For $u \in BV(\Omega)$, the gradient Du is a Radon measure that decomposes into its absolutely continuous and singular parts $Du = D^a u + D^s u$. Then $D^a u = \nabla u \, dx$ where ∇u is the Radon-Nikodym derivative of the measure Du with respect to the Lebesgue measure in $\mathbb{R}^N$. The function u is approximately differentiable $\mathcal{L}^N$ a.e. in Ω and the approximate differential coincides with $\nabla u(x) \, \mathcal{L}^N$ a.e.. The singular

part $D^s u$ can be also split in two parts: the *jump* part $D^j u$ and the *Cantor* part $D^c u$. We say that $x \in \Omega$ is an *approximate jump point of* u if there exist $u^+(x) \neq u^-(x) \in \mathbb{R}$ and $|\nu_u(x)| = 1$ such that

$$\lim_{\rho \downarrow 0} \frac{1}{|B_\rho^+(x, \nu_u(x))|} \int_{B_\rho^+(x, \nu_u(x))} |u(y) - u^+(x)| \, dy = 0$$

$$\lim_{\rho \downarrow 0} \frac{1}{|B_\rho^-(x, \nu_u(x))|} \int_{B_\rho^-(x, \nu_u(x))} |u(y) - u^-(x)| \, dy = 0,$$

where

$$B_\rho^+(x, \nu_u(x)) = \{ y \in B(x, \rho) \ : \ \langle y - x, \nu_u(x) \rangle > 0 \}$$
$$B_\rho^-(x, \nu_u(x)) = \{ y \in B(x, \rho) \ : \ \langle y - x, \nu_u(x) \rangle < 0 \}.$$

We denote by J_u the set of approximate jump points of u. If $u \in BV(\Omega)$, the set S_u is countably $\mathcal{H}^{N-1}$ rectifiable, J_u is a Borel subset of S_u and $\mathcal{H}^{N-1}(S_u \setminus J_u) = 0$ (see [3]). In particular, we have that $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$ is either a point of approximate continuity of $\tilde{u}$, or a jump point with two limits in the above sense. Eventually, we have

$$D^j u = D^s u \llcorner_{J_u} = (u^+ - u^-) \nu_u \mathcal{H}^{N-1} \llcorner_{J_u} \quad \text{and} \quad D^c u = D^s u \llcorner_{(\Omega \setminus S_u)}.$$

For $x \in J_u$ we set $[u(x)] = u^+(x) - u^-(x)$.

For a comprehensive treatment of functions of bounded variation we refer to [3].

3 The jump set of solutions of the TV denoising problem

Given a function $f \in L^2(\Omega)$ and $\lambda > 0$ we consider the minimum problem

$$\min_{u \in BV(\Omega)} \int_\Omega |Du| + \frac{1}{2\lambda} \int_\Omega (u - f)^2 \, dx =: \mathcal{F}_\lambda(u) \, . \tag{6}$$

Notice that problem (6) always admits a unique solution u_λ, since the functional $\mathcal{F}_\lambda$ is strictly convex.

As we mentioned in the Introduction, one of the main reasons to introduce the Total Variation as a regularization term in imaging problems is its ability to recover the discontinuities of the function f. In this section we recall a result proved in [10] showing that the jump set of u_λ is always contained in the jump set of f, that is, the model (6) does not create any new discontinuity besides the existing ones. We refine the proof given in [10] proving that the size of the jump decreases. This fact was observed in [17] where it was proved in the more general context of weighted total variation.

Let us recall the following Proposition, which is proved in [14, 1].

Proposition 3.1. *Let u_λ be the (unique) solution of (6). Then, for any $t \in \mathbb{R}$, $\{u_\lambda > t\}$ (respectively, $\{u_\lambda \geq t\}$) is the minimal (resp., maximal) solution of the minimal surface problem*

$$\min_{E \subseteq \Omega} P(E, \Omega) + \frac{1}{\lambda} \int_E (t - f(x)) \, dx. \tag{7}$$

In particular, for all $t \in \mathbb{R}$ but a countable set, $\{u_\lambda = t\}$ has zero measure and the solution of (7) is unique (up to a negligible set).

A proof that $\{u_\lambda > t\}$ and $\{u_\lambda \geq t\}$ both solve (7) is found in [14, Prop. 2.2]. The proof of Proposition 3.1 then follows from the co-area formula and from the following comparison result for solutions of (7) (see [1, Lemma 4]):

Lemma 3.2. *Let $f, g \in L^1(\Omega)$ and E and F be respectively minimizers of*

$$\min_E P(E, \Omega) - \int_E f(x)\,dx \quad and \quad \min_F P(F, \Omega) - \int_F g(x)\,dx\,.$$

Then, if $f < g$ a.e., $|E \setminus F| = 0$ (in other words, $E \subseteq F$ up to a negligible set).

From Proposition 3.1 and the regularity theory for surfaces of prescribed curvature (see for instance [2]), one has the following regularity result.

Corollary 3.3. *Let $f \in L^p(\Omega)$, with $p > N$. Then, for all $t \in \mathbb{R}$ the super-level set $E_t := \{u_\lambda > t\}$ (respectively, $\{u_\lambda \geq t\}$) has boundary of class $C^{1,\alpha}$, for all $\alpha < (p - N)/p$, out of a closed singular set Σ of Hausdorff dimension at most $N - 8$. Moreover, if $p = \infty$, the boundary of E_t is of class $W^{2,q}$ out of Σ, for all $q < \infty$, and is of class $C^{1,1}$ if $N = 2$.*

Before stating the main result of this section, we recall two simple Lemmata (see).

Lemma 3.4. *Let U be an open set in $\mathbb{R}^N$ and $v \in W^{2,p}(U)$, $p \geq 1$. We have that*

$$\mathrm{div}\left(\frac{\nabla v}{\sqrt{1 + |\nabla v|^2}}\right) = \mathrm{trace}\left(A(\nabla v)D^2 v\right) \qquad \text{a.e. in } U,$$

where

$$A(\xi)_{ij} := \frac{1}{(1 + |\xi|^2)^{\frac{1}{2}}}\left(\delta_{ij} - \frac{\xi_i \xi_j}{(1 + |\xi|^2)}\right) \qquad \xi \in \mathbb{R}^N.$$

Lemma 3.5. *Let U be an open set in $\mathbb{R}^N$ and $v \in W^{2,1}(U)$. Assume that u has a minimum at $y_0 \in U$ and*

$$\lim_{\rho \to 0+} \frac{1}{|B(y_0, \rho)|} \int_{B(y_0, \rho)} \frac{\left|u(y) - u(y_0) - \nabla u(y_0) \cdot (y - y_0) - \frac{1}{2}\langle D^2 v(y_0)(y - y_0), y - y_0\rangle\right|}{\rho^2}\,dy = 0\,, \quad (8)$$

then $D^2 v(y_0) \geq 0$.

Recall that, if $v \in W^{2,1}(U)$, then (8) holds a.e. in U [19, Th. 3.4.2].

Theorem 1. *Let $f \in BV(\Omega) \cap L^\infty(\Omega)$. Then, for all $\lambda > 0$,*

$$J_{u_\lambda} \subseteq J_f \tag{9}$$

up to a set of zero $\mathcal{H}^{N-1}$-measure. Moreover

$$[u_\lambda(x)] \leq [f(x)] \qquad \mathcal{H}^{N-1}\text{-a.e. on } J_{u_\lambda}. \tag{10}$$

Proof. Notice that (10) implies (9), so that it is enough to prove (10).

Let us first recall some consequences of Corollary 3.3. Let $E_t := \{u_\lambda > t\}$, $t \in \mathbb{R}$, and let Σ_t be its singular set given by Corollary 3.3. Since $f \in L^\infty(\Omega)$, around each point $x \in \partial E_t \setminus \Sigma_t$, $t \in \mathbb{R}$, ∂E_t is locally the graph of a function in $W^{2,p}$ for all $p \in [1, \infty)$ (hence in $C^{1,\alpha}$ for any $\alpha \in (0, 1)$). Let $\mathbb{Q}$

4

be a countable dense set in $\mathbb{R}$ such that $\{u_\lambda > t\}$ is a set of finite perimeter for any $t \in \mathbb{Q}$. If we let $\mathcal{N} := \bigcup_{t \in \mathbb{Q}} \Sigma_t$, we then have $\mathcal{H}^{N-1}(\mathcal{N}) = 0$.

Since we can write (as in [3])

$$J_{u_\lambda} = \bigcup_{t_1, t_2 \in \mathbb{Q}, t_1 < t_2} \partial E_{t_1} \cap \partial E_{t_2},$$

in order to prove (10) it suffices to show

$$t_2 - t_1 \le [f(x)] \qquad \mathcal{H}^{N-1}\text{-a.e. on } \partial E_{t_1} \cap \partial E_{t_2}, \tag{11}$$

for all $t_1, t_2 \in \mathbb{Q}$ with $t_1 < t_2$.

Fix $t_1, t_2 \in \mathbb{Q}$ with $t_1 < t_2$, and let B_R' be the ball of radius $R > 0$ in $\mathbb{R}^{N-1}$ centered at 0. Let $C_R := B_R' \times (-R, R)$, and denote $x \in C_R$ as $x = (y, z) \in B_R' \times (-R, R)$. Given $x \in \partial E_{t_1} \cap \partial E_{t_2} \setminus \mathcal{N}$, by Corollary 3.3 there is $R > 0$ such that, after a suitable change of coordinates, we can write the set $\partial E_{t_i} \cap C_R$, $i = 1, 2$, as the graph of a function $v_i \in W^{2,p}(B_R')$, $p \in [1, \infty)$, such that $x = (0, v_i(0)) \in C_R \subseteq \Omega$ and $\nabla v_i(0) = 0$. Without loss of generality, we assume that $v_i > 0$ in B_R', and that E_{t_i} is the supergraph of v_i, . From $t_1 < t_2$ and Lemma 3.2, it follows $E_{t_2} \subseteq E_{t_1}$, which gives in turn $v_2 \ge v_1$ in B_R'. We may also assume that

$$\mathcal{H}^{N-1}\left(\{y \in B_R' : v_1(y) = v_2(y)\}\right) > 0. \tag{12}$$

Notice that, since ∂E_{t_i} is of finite $\mathcal{H}^{N-1}$-measure, we may cover $\partial E_{t_1} \cap \partial E_{t_2} \setminus \mathcal{N}$ by a countable set of such cylinders. By [3, Th. 3.108], for $\mathcal{H}^{N-1}$-a.e. $y \in B_R'$ the function $f(y, \cdot)$ belongs to $BV((-R, R))$, and the jumps of $f(y, \cdot)$ are the points z such that $(y, z) \in J_f$. Recalling that v_i is a local minimizer of

$$\min_v \int_{B_R'} \sqrt{1 + |\nabla v|^2} \, dy - \frac{1}{\lambda} \int_{B_R'} \int_0^{v(y)} (t_i - f(y, z)) \, dz \, dy =: \mathcal{E}_i(v),$$

by taking a positive smooth test function $\psi(y)$ witn compact support in B_R', and computing

$$\lim_{\epsilon \to 0^+} \frac{\mathcal{E}_i(v + \epsilon \psi) - \mathcal{E}_i(v)}{\epsilon} \ge 0,$$

we deduce that

$$\operatorname{div}\left(\frac{\nabla v_i(y)}{\sqrt{1 + |\nabla v_i(y)|^2}}\right) + \frac{1}{\lambda}\left(t_i - f^+(y, v_i(y))\right) \le 0, \qquad \mathcal{H}^{N-1}\text{-a.e. in } B_R'. \tag{13}$$

In a similar way, we get

$$\operatorname{div}\left(\frac{\nabla v_i(y)}{\sqrt{1 + |\nabla v_i(y)|^2}}\right) + \frac{1}{\lambda}\left(t_i - f^-(y, v_i(y))\right) \ge 0, \qquad \mathcal{H}^{N-1}\text{-a.e. in } B_R'. \tag{14}$$

Finally we observe that, since $v_1, v_2 \in W^{2,p}(B_R')$ for any $p \in [1, \infty)$ and $v_2 \ge v_1$ in B_R', by Lemma 3.5 we have that $\nabla v_1(y) = \nabla v_2(y)$ and $D^2(v_1 - v_2)(y) \le 0$ $\mathcal{H}^{N-1}$-a.e. on $\{y \in B_R' : v_1(y) = v_2(y)\}$. Using both inequalities (13) and (14) and Lemma 3.4, it follows that

$$
\begin{aligned}
0 < t_2 - t_1 &\le \lambda \left(\operatorname{trace}(A(\nabla v_1(y)) D^2 v_1(y)) - \operatorname{trace}(A(\nabla v_2(y)) D^2 v_2(y))\right) \\
&+ \; f^+(y, v_2(y)) - f^-(y, v_1(y)) \\
&\le \; f^+(y, v_2(y)) - f^-(y, v_1(y)),
\end{aligned}
$$

$\mathcal{H}^{N-1}$-a.e. on $\{y \in B_R' : v_1(y) = v_2(y)\}$, which gives (11) and concludes the proof. $\qquad \square$

4 The jump set of solutions of the Total Variation flow

Let Ω be an open bounded set with Lipschitz boundary. We consider the Total Variation flow

$$\frac{\partial u}{\partial t} = \text{div}\left(\frac{Du}{|Du|}\right) \qquad \text{in } Q_T = (0,T) \times \Omega,$$

$$\frac{Du}{|Du|} \cdot \nu^\Omega = 0 \qquad \text{in } Q_T = (0,T) \times \partial\Omega,$$

(15)

with the initial condition

$$u(0,x) = f(x), \qquad x \in \Omega. \tag{16}$$

Let us recall that (15) it is the L^2-gradient flow of the total variation as defined in [9]. In the general case we shall follow [4, 8]. The purpose of this Section is to prove the following result.

Theorem 2. *Let $f \in L^N(\Omega) \cap BV(\Omega)$. Let $u(t)$ be the solution of (15) with initial condition (16). Then $u(t) \in L^\infty(\Omega) \cap BV(\Omega)$ for any $t > 0$, and*

$$J_{u(t)} \subseteq J_{u(s)} \subseteq J_f \qquad \forall t > s > 0. \tag{17}$$

Moreover

$$[u(t)] \le [u(s)] \le [f] \qquad \mathcal{H}^{N-1} \text{ a.e. } \forall t > s > 0. \tag{18}$$

The proof of Theorem 2 is based on the approach in [10] and uses the estimate (10). Let us first recall some basic facts about the operator $-\text{div}\left(\frac{Du}{|Du|}\right)$ in L^p spaces. Since it suffices for our purposes, we shall only consider the case $p \in [\frac{N}{N-1}, \infty]$. For any $p \in [1, \infty]$, let us define the space

$$X(\Omega)_p := \left\{ z \in L^\infty(\Omega, \mathbb{R}^N) \ : \ \text{div}(z) \in L^p(\Omega) \right\}.$$

If $z \in X(\Omega)_p$ and $w \in BV(\Omega) \cap L^q(\Omega)$, $p^{-1} + q^{-1} = 1$, we define the functional $(z \cdot Dw) : C_0^\infty(\Omega) \to \mathbb{R}$ by the formula

$$\langle (z \cdot Dw), \varphi \rangle := -\int_\Omega w\,\varphi\,\text{div }z\,dx - \int_\Omega w\,z \cdot \nabla\varphi\,dx.$$

Then $(z \cdot Dw)$ is a Radon measure in Ω, and $(z \cdot Dw) = z \cdot \nabla w$ if $w \in W^{1,1}(\Omega) \cap L^q(\Omega)$.

Following [7], we observe that for any $z \in X(\Omega)_p$ there exists a function $[z \cdot \nu^\Omega] \in L^\infty(\partial\Omega)$ satisfying $\|[z \cdot \nu^\Omega]\|_{L^\infty(\partial\Omega)} \le \|z\|_{L^\infty(\Omega;\mathbb{R}^N)}$, and such that for any $u \in BV(\Omega) \cap L^q(\Omega)$ we have

$$\int_\Omega u\,\text{div }z\,dx + \int_\Omega (z \cdot Du) = \int_{\partial\Omega} [z \cdot \nu^\Omega] u\,d\mathcal{H}^{N-1}.$$

Definition 4.1. *We define the operator $\mathcal{A}_p \subseteq L^p(\Omega) \times L^p(\Omega)$, with $\frac{N}{N-1} \le p \le \infty$, by:*
$(u,v) \in \mathcal{A}_p$ if and only if $u,v \in L^p(\Omega)$, $u \in BV(\Omega)$ and there exists $z \in X(\Omega)_p$ with $\|z\|_\infty \le 1$, such that $(z \cdot Du) = |Du|$, $[z \cdot \nu^\Omega] = 0$ and

$$v = -\text{div}(z) \qquad \text{in } \mathcal{D}'(\Omega).$$

By $v \in \mathcal{A}_p u$ we mean that $(u,v) \in \mathcal{A}_p$. By $L_w^1((0,T); BV(\Omega))$ we denote the space of weakly measurable functions $w : [0,T] \to BV(\Omega)$ (i.e., the map $t \in [0,T] \to \langle w(t), \phi \rangle$ is measurable for any $\phi \in BV(\Omega)^*$ where $BV(\Omega)^*$ denotes the dual of $BV(\Omega)$) such that $\int_0^T \|w(t)\|\,dt < \infty$.

6

Definition 4.2. *A function $u \in C([0,T]; L^p(\Omega))$ is a strong solution of (15) if $u \in W^{1,1}_{loc}(0,T; L^p(\Omega)) \cap L^1_w((0,T); BV(\Omega))$ and there exists $z \in L^\infty\left((0,T) \times \Omega; \mathbb{R}^N\right)$ with $\|z\|_\infty \leq 1$ such that*

$$\int_\Omega (z(t) \cdot Du(t)) = \int_\Omega |Du(t)| \quad \text{for a.e. } t > 0. \tag{19}$$

$$[z(t) \cdot \nu^\Omega] = 0 \qquad \text{in } \partial\Omega \text{ for a.e. } t > 0. \tag{20}$$

and

$$u_t = \text{div } z \qquad \text{in } \mathcal{D}'\left((0,T) \times \Omega\right).$$

Proposition 4.3. *The operator $\mathcal{A}_p$ is m-accretive in $L^p(\Omega)$, that is for any $f \in L^p(\Omega)$ and any $\lambda > 0$ there is a unique solution $u \in L^p(\Omega)$ of the problem*

$$u + \lambda \mathcal{A}_p u \ni f. \tag{21}$$

Moreover, if $u_1, u_2 \in L^p(\Omega)$ are the solutions of (21) corresponding to the right hand sides $f_1, f_2 \in L^p(\Omega)$, then

$$\|u_1 - u_2\|_p \leq \|f_1 - f_2\|_p.$$

Moreover the domain of $\mathcal{A}_p$ is dense in $L^p(\Omega)$ when $p < \infty$.

We denote by $J_\lambda f$ the solution of (21).

We also recall the notion of strong solution for nonlinear semigroups generated by accretive operators.

Definition 4.4. *A function u is called a strong solution in the sense of semigroups of*

$$\frac{du}{dt} + \mathcal{A}_p u \ni 0$$

with $u(0) = f$ if

$$\begin{cases} u \in C([0,T]; L^p(\Omega)) \cap W^{1,1}_{loc}((0,T); L^p(\Omega)) \\ \\ u(t) \in \text{Dom}(\mathcal{A}_p) \quad \text{a.e. in } t > 0 \text{ and } u' + \mathcal{A}_p u(t) \ni 0 \text{ a.e. } t \in (0,T) \\ \\ u(0) = f. \end{cases} \tag{22}$$

By Crandall-Ligget's semigroup generation theorem [16], using Proposition 4.3 as in [6, 10], one obtains the following result:

Theorem 3. *Let $f \in L^p(\Omega)$ if $\frac{N}{N-1} \leq p < \infty$, or $f \in \overline{\text{Dom}(\mathcal{A}_\infty)}$ if $p = \infty$. Then there is a unique strong solution in the sense of semigroups $u(t) = S(t)f := \lim_{\lambda \downarrow 0, k\lambda \to t} J^k_\lambda f \in C([0,T], L^p(\Omega))$ of the problem*

$$\frac{du}{dt} + \mathcal{A}_p u \ni 0, \qquad u(0) = f. \tag{23}$$

Moreover, the semigroup solution is a strong solution of (15) and conversely, any strong solution of (15) is a strong solution in the sense of semigroups of (23).

7

Remark 4.5. Notice that given $p \in [\frac{N}{N-1}, \infty]$ the limit $\lim_{\lambda \downarrow 0, k\lambda \to t} J_\lambda^k f$ is taken in $L^p(\Omega)$.

To prove Theorem 2, we need the following Lemma.

Lemma 4.6. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of functions in $BV(\Omega) \cap L^\infty(\Omega)$. Assume that $J_{u_n} \subseteq J_{u_0}$, for all $n \in \mathbb{N}$, and $u_n \to u$ strongly in $L^\infty(\Omega)$. Then, $\mathcal{H}^{N-1}$-almost every point of $\Omega \backslash J_{u_0}$ is a Lebesgue point for u. In particular, if $u \in BV(\Omega)$, then $J_u \subseteq J_{u_0}$. Moreover, if all the functions u_n are continuous at $x \in \Omega$, then also u is continuous at x.*

Proof. The thesis follows observing that if $x \in \Omega$ is a Lebesgue point for all the functions u_n, then it is also a Lebesgue point for u, and the same is true for a continuity point. $\square$

Proof of Theorem 2. We divide the proof into three steps.

Step 1. Assume that $f \in \overline{\mathrm{Dom}(\mathcal{A}_\infty)} \cap BV(\Omega)$. In this case we know from [16] that $J_\lambda^k f \to u(t)$ when $\lambda \to 0^+$ and $k\lambda \to t$. Then the result follows from Theorem 1, Remark 4.5 and Lemma 4.6.

Step 2. Let $f \in L^N(\Omega)$. By Theorem 3, $u \in C([0,T], L^N(\Omega))$. We also know from [6] that $u(t) = S(t)f \in C((0,T]; L^\infty(\Omega))$ and $u(t) \in BV(\Omega)$ for any $t > 0$. Moreover, the following estimate is a consequence of the 0-homogeneity of the operator $\mathcal{A}_\infty$ [4, 6]:

$$\left\| \frac{d}{dt} S(t) f_\lambda \right\|_\infty \leq 2 \frac{\|f_\lambda\|_\infty}{t} \leq 2 \frac{\|f\|_\infty}{t} \qquad \text{for any } t > 0. \tag{24}$$

This implies that $u(t) \in \overline{\mathrm{Dom}(\mathcal{A}_\infty)}$ for any $t > 0$. Notice that by Step 1 and Theorem 1, we know that

$$J_{u(t)} \subseteq J_{u(s)} \qquad \text{for all } t > s > 0. \tag{25}$$

Moreover, by (10) we have

$$[u(t)] \leq [u(s)] \qquad \mathcal{H}^{N-1}\text{-a.e. on } J_{u(t)}, \text{ for all } t > s > 0. \tag{26}$$

Step 3. Let $f \in L^N(\Omega) \cap BV(\Omega)$. We shall prove that $J_{u(t)} \subseteq J_f$ (modulo an $\mathcal{H}^{N-1}$ null set) for any $t > 0$.

Let $Z \subset J_{u(t)}$ be such that $[u(t, \cdot)] \geq \epsilon > 0$ on Z and let x_0 be a point of $\mathcal{H}^{N-1}$-density 1 in Z. Let us consider $t_n \downarrow 0^+$, $t_n < t$. By (25) we have $J_{u(t_n)} \subseteq J_{u(t_{n-1})}$ and $Z \subseteq J_{u(t_n)}$ for all $n \in \mathbb{N}$ (modulo a $\mathcal{H}^{N-1}$-null set). By (26) we also have

$$[u(t_n, x)] \geq [u(t, x)] \geq \epsilon \qquad \mathcal{H}^{N-1} - \text{a.e. on } Z.$$

Now, observe that

$$\int_\Omega |Du(t)| \leq \int_\Omega |Df|. \tag{27}$$

Since

$$\limsup_{t\to 0^+} \int_\Omega |Du(t)| \leq \int_\Omega |Df| \leq \liminf_{t\to 0^+} \int_\Omega |Du(t)|.$$

we have that

$$\lim_{t\to 0^+} \int_\Omega |Du(t)| = \int_\Omega |Df|. \tag{28}$$

8

By Step 2 we know that $u(t) \to f$ in $L^N(\Omega)$. Then $Du(t) \to Df$ as $t \to 0+$ weakly* as vector measures in Ω. Since (28) holds, we also have that $|Du(t)| \to |Df|$ as $t \to 0+$ weakly* as measures in Ω ([3], Proposition 1.62).

Let $r > 0$ be such that $|Df|(\partial B(x_0, r)) = 0$. By [3, Proposition 1.62] we get

$$|Df|(B(x_0, r)) = \lim_{n \to \infty} |Du(t_n)|(B(x_0, r)) \geq \epsilon \mathcal{H}^{N-1}(Z \cap B(x_0, r)).$$

It follows that, for all points x_0 of $\mathcal{H}^{N-1}$ density 1 in Z, we have

$$\liminf_{r \to 0^+} \frac{|Df|(B(x_0, r))}{\omega_{N-1} r^{N-1}} \geq \epsilon,$$

where ω_{N-1} is the area of the $(N-1)$-dimensional ball. This implies that $Z \subset J_f$ (modulo an $\mathcal{H}^{N-1}$ null set). Thus, $J_{u(t)} \subseteq J_f$ (modulo an $\mathcal{H}^{N-1}$ null set) for any $t > 0$. □

Remark 4.7. Theorem 2 still holds, with analogous proof, in the case of zero Dirichlet boundary conditions or in $\mathbb{R}^N$ [5, 8, 6]. Moreover, as in [10] (see also [17]), it also holds for the anisotropic Total Variation flow, when the anisotropy is smooth and elliptic.

References

[1] F. Alter, V. Caselles, and A. Chambolle. A characterization of convex calibrable sets in $\mathbb{R}^N$. *Math. Ann.*, 332(2):329–366, 2005.

[2] L. Ambrosio. Corso introduttivo alla teoria geometrica della misura ed alle superfici minime. Scuola Normale Superiore, Pisa, 1997.

[3] L. Ambrosio, N. Fusco, and D. Pallara. Functions of bounded variation and free discontinuity problems. Oxford Mathematical Monographs XVIII, Clarendon Press, 2000.

[4] F. Andreu, C. Ballester, V. Caselles, and J. M. Mazón. Minimizing total variation flow. *Differential Integral Equations*, 14(3):321–360, 2001.

[5] F. Andreu, C. Ballester, V. Caselles, and J. M. Mazón. The Dirichlet problem for the total variation flow. *Journal Functional Analysis*, 180: 347-403 (2001).

[6] F. Andreu, V. Caselles, and J. M. Mazón. Parabolic Quasilinear Equations Minimizing Linear Growth Functionals. Birkhaüser Verlag, 2004.

[7] G. Anzellotti. Pairings between measures and bounded functions and compensated compactness. *Ann. Mat. Pura Appl.*, 135:293–318, 1983.

[8] G. Bellettini, V. Caselles, and M. Novaga. The total variation flow in $\mathbb{R}^N$. *J. Differential Equations*, 184(2):475–525, 2002.

[9] H. Brézis. Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert. North-Holland Publishing Co., Amsterdam, 1973.

[10] V. Caselles, A. Chambolle, and M. Novaga. The discontinuity set of solutions of the TV denoising problem and some extensions. *Multiscale modeling & simulation*, 6(3):879–894, 2007.

[11] V. Caselles, A. Chambolle, and M. Novaga. Regularity for solutions of the total variation denoising problem. *Rev. Mat. Iberoamericana*, 27(1):233–252, 2011.

[12] A. Chambolle, V. Caselles, D. Cremers, M. Novaga and T. Pock. An introduction to Total Variation for Image Analysis. In *Theoretical Foundations and Numerical Methods for Sparse Recovery*, De Gruyter, Radon Series Comp. Appl. Math., vol. 9, pp. 263-340, 2010.

[13] A. Chambolle and P.-L. Lions. Image recovery via total variation minimization and related problems. *Numer. Math.*, 76:167–188, 1997.

[14] A. Chambolle. An algorithm for mean curvature motion. *Interfaces Free Bound.*, 6(2):195–218, 2004.

[15] T.F. Chan, G.H. Golub and P. Mulet. A nonlinear primal-dual method for total variation based image restoration. *SIAM J. Sci. Computing*, 20:1964–1977, 1999.

[16] M.G. Crandall and T.M. Liggett, *Generation of Semigroups of Nonlinear Transformations on General Banach Spaces*, *Amer. J. Math.*, 93:265–298, 1971.

[17] K. Jalalzai. Regularization of inverse problems in image processing. *PhD Thesis, École Polytechnique*, Palaiseau, Mars 2012.

[18] L. Rudin, S. Osher and E. Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D*, 60:259–268, 1992.

[19] W. P. Ziemer. *Weakly Differentiable Functions*, GTM 120, Springer Verlag, 1989.