FRACTIONAL MEAN CURVATURE FLOW OF LIPSCHITZ GRAPHS

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Abstract. We consider the fractional mean curvature flow of entire Lipschitz graphs. We provide regularity results, and we study the long time asymptotics of the flow. In particular we show that in a suitable rescaled framework, if the initial graph is a sublinear perturbation of a cone, the evolution asymptotically approaches an expanding self-similar solution. We also discuss some results in the unrescaled case.

Contents

1. Introduction

Given a set $E \subseteq \mathbb{R}^{n+1}$, we define the fractional mean curvature flow E_t starting from E as the flow obtained by the following geometric evolution law: The velocity at a point $p \in \partial E_t$ is given by

$$(1.1) \quad \partial_t p \cdot \nu(p) = -H_s(p, E_t) := -\lim_{\varepsilon \to 0} \int_{\mathbb{R}^{n+1} \setminus B_\varepsilon(p)} \left(\chi_{\mathbb{R}^{n+1} \setminus E_t}(y) - \chi_{E_t}(y) \right) \frac{1}{|p-y|^{n+1+s}} \, dy,$$

where $s \in (0,1)$ is a fixed parameter and $\nu(p)$ is the outer normal at ∂E_t in p.

The fractional mean curvature flow can be interpreted as the fractional analogue of the classical mean curvature flow: Indeed, as the mean curvature flow is the L^2 gradient flow of the perimeter, the fractional mean curvature flow is the L^2 gradient flow of the so-called fractional perimeter, see [3], which can be interpreted as an interpolation norm (the Gagliardo fractional seminorm) of the characteristic function of a measurable set, which interpolates between the BV norm, which is the standard perimeter, and the L^1 norm, which is the volume.

Therefore, the fractional mean curvature flow presents some analogies with the classical mean curvature flow. Recently a local existence result for smooth solutions starting from compact $C^{1,1}$ initial sets was provided in [17] (see also [4,8] for an analog of the BMO scheme), moreover existence and uniqueness of the level set flow for general nonlocal evolution equations, including (1.1), has been developed in [7] (see also [16]), using the maximum principle and the monotonicity of the curvature with respect to inclusions. On the other hand, the fractional flow presents some different features with respect to the classical mean curvature flow, since nonlocal effects come into play. For instance, as a basic example one can consider a planar strip, which is stationary for the curvature flow, and shrinking for the fractional flow.

An important issue in the study of the fractional flow, as for the classical one, is the investigation of the long time behavior of solutions and the analysis of the formation of singularities. In the local case, one of the most important tool in this analysis is the monotonicity formula established by Huisken in [15]. The analog of such formula in the fractional setting is still an open problem. As a consequence, it is still missing a systematic approach for the study of long time asymptotics of the flow and the classification of the possible singularities which may appear. Nevertheless some results have been recently obtained, we recall for instance the analysis of the formation of neckpinch singularities in [9], and of the fattening phenomenon for the evolution of curves with cross-type and cusp-type singularities in [5]. Moreover, in [10] it has been proved that smooth convex sets evolving under the volume preserving fractional mean curvature flow approach round spheres.

Here we analyze the evolution (1.1) under the additional assumption that the boundary of the initial datum E_0 can be written as a Lipschitz graph, that is, there exists $e \in \mathbb{R}^{n+1}$, such that $\nu(p) \cdot e > 0$ for every $p \in \partial E_0$. By monotonicity of the flow it is possible to show that the evolution E_t maintains this property for all positive times $t > 0$, that is, $\nu(p) \cdot e > 0$ for every $p \in \partial E_t$, see Section 2. Up to a rotation of coordinates, we will assume that $e = e_{n+1}$.

For the local case, the analysis of the flow of entire Lipschitz graphs goes back to the work by Ecker and Huisken [12,13]. In particular, in [12] it is proved that the evolution admits a smooth solution for all times, which approaches a self-similar solution as $t \to +\infty$, provided that the initial graph is "straight" at infinity, in the sense that is a sublinear perturbation of a cone, see assumption (5.1) below. In this paper we provide analogous results in the fractional setting. In particular, in Section 3 we prove the $C^{1,\alpha}$ regularity of the flow starting from a Lipschitz graph, which can be strengthened if the initial graph enjoys more regularity. These results are based on the fact that the fractional curvature is an elliptic operator, and so we may apply the regularity results for nonlinear fractional parabolic problems obtained in [20, 21]. Actually we do not recover the C^∞ regularity obtained in the local case, expect in the case of self-similar solutions (see Theorem 4.1), and this is due to the fact that a parabolic bootstrap regularity argument is still missing for quasilinear fractional operators. Moreover, in Section 5 we provide the convergence of the rescaled solution to a self-similar expanding solution, under the assumption that the initial graph is straight at infinity in the sense of Ecker and Huisken. We recall here what we mean for self-similar expanding or contracting solution to 1.1.

Definition 1.1. An expanding homothetic solution is a solution to (1.1) such that $E_t = \lambda(t)E_1$ where $\lambda(1) = 1$ and $\lambda'(t) \geqslant 0$ for $t > 1$. This is equivalent to assume that E_1 is a solution to

$$(1.2) \qquad\qquad c(p \cdot \nu) = -H_s(p, E_1)$$

for some $c \geqslant 0$. Observe that necessarily $\lambda(t) = [c(s+1)(t-1) + 1]^{\frac{1}{s+1}}$.

A shrinking homothetic solution to (1.1) is a solution to (1.1) such that $E_t = \lambda(t)E_1$ where $\lambda(1) = 1$ and $\lambda'(t) \leqslant 0$ for $t > 1$. This is equivalent to assume that E_1 is a solution to

$$(1.3) \qquad\qquad c(p \cdot \nu) = H_s(p, E_1)$$

for some $c \geqslant 0$.

In Section 4 we study the main properties of the expanding selsimilar solutions to (1.1), whereas in Section 7 we show that the only graphical shrinking self-similar solutions to (1.1) are actually stationary solutions. In the local setting, this result has been obtained for entire graphs without growth condition at infinity in [22]. In the fractional setting we obtain the result for entire Lipschitz graphs as a byproduct of a Liouville theorem for ancient solutions of parabolic nonlinear equations obtained in [20]. We also recall that a preliminary analysis of existence and stability of fractional symmetric shrinkers has been developed in [6].

Finally Section 6 contains convergence results in the unrescaled setting. In particular we provide the stability of hyperplanes, when we start the evolution from graphs which are asymptotically flat. In the local setting this stability can be proved by using comparison with large balls and area decay estimates, see [11, 13, 18], whereas in the fractional setting we use comparison with large balls and an argument based on construction of appropriate periodic barriers. We show also stability of convex cones and, in some particular cases, of mean convex cones, in the unrescaled setting. Analogous results in the local setting were obtained in [11].

2. LEVEL SET FORMULATION

The level set flow associated to (1.1) can be defined as follows. Given an initial set $E_0 \subset \mathbb{R}^n$ we choose a bounded Lipschitz continuous function $U_0 : \mathbb{R}^{n+1} \to \mathbb{R}$ such that

$$\partial E_0 = \{x \in \mathbb{R}^n \text{ s.t. } U_0(x) = 0\} = \partial\{x \in \mathbb{R}^n \text{ s.t. } U_0(x) \geqslant 0\}$$
$$\text{and} \qquad E_0 = \{x \in \mathbb{R}^n \text{ s.t. } U_0(x) \geqslant 0\}.$$

Let also $U(x, t)$ be the viscosity solution of the following nonlocal parabolic problem

$$(2.1) \qquad \begin{cases} U_t + |DU|H_s((x, y, t), \{(x', y', t) \,|\, U(x', z', t) \geqslant U(x, z, t)\}) = 0 \\ U(x, 0) = U_0(x). \end{cases}$$

Then the level set flow of ∂E_0 is given by

$$(2.2) \qquad\qquad \Sigma_E(t) := \{x \in \mathbb{R}^n \text{ s.t. } U(x, t) = 0\}.$$

We associate to this level set the outer and inner flows defined as follows:

$$(2.3) \quad E^+(t) := \{x \in \mathbb{R}^n \text{ s.t. } U(x, t) \geqslant 0\} \qquad \text{and} \qquad E^-(t) := \{x \in \mathbb{R}^n \text{ s.t. } U(x, t) > 0\}.$$

We observe that the equation in (2.4) is geometric, so if we replace the initial condition with any function V_0 with the same level sets $\{U_0 \geqslant 0\}$ and $\{U_0 > 0\}$, the evolutions $E^+(t)$ and $E^-(t)$ remain the same. Existence and uniqueness of viscosity solutions to the level set formulation of (1.1) has been provided in [7, 16], and qualitative properties of smooth solutions have been studied in [19].

In this paper we consider the particular case in which the initial set E_0 can be written- up to suitable rotation of coordinates- as the subgraph of a function $u_0 : \mathbb{R}^n \to \mathbb{R}$. So, it is possible to define the evolution E_t at time t as the subgraph of the solution $u(\cdot, t)$ to the following nonlocal quasilinear system

$$(2.4) \qquad \begin{cases} u_t + \sqrt{1 + |Du|^2} H_s((x, u(x,t)), \{(x', z) \mid z \leqslant u(x', t)\}) = 0 \\ u(x, 0) = u_0(x). \end{cases}$$

Theorem 2.1. *Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be a uniformly continuous function.*
Let $v, w \in C(\mathbb{R}^n \times [0, +\infty))$ respectively a viscosity subsolution and a viscosity supersolution to (2.4) such that $v(x, 0) \leqslant u_0(x) \leqslant w(x, 0)$. Then $v(x, t) \leqslant w(x, t)$ for all $(x, t) \in \mathbb{R}^n \times (0, +\infty)$.
In particular (2.4) admits a unique viscosity solution $u(x, t) \in C(\mathbb{R}^n \times [0, +\infty))$ with $u(x, 0) = u_0(x)$. Moreover, if u_0 is Lipschitz continuous with Lipschitz constant $\|Du_0\|_\infty$, then $|u(x, t) - u(x', t)| \leqslant \|Du_0\|_\infty |x - x'|$ for all $x, x' \in \mathbb{R}^n$ and $t > 0$.

Proof. We define the uniformly continuous function $U_0(x, z) := u_0(x) - z$ and the functions $V(x, z, t) = v(x, t) - z$, $W(x, z, t) = w(x, t) - z$. Then it is easy to check that V, W are respectively a viscosity sub and supersolution to (2.1) such that $V(x, z, 0) \leqslant U_0(x, z) \leqslant W(x, z, 0)$. Then by the comparison principle proved in [16] (and for general nonlocal geometric equations in [7]) we get that $V(x, y, t) \leqslant W(x, z, t)$ for all $x \in \mathbb{R}^n$, $z \in \mathbb{R}$, $t > 0$. This implies the result.

Moreover, again by the results proved in [7, 16], the system (2.1) admits a unique viscosity solution $U(x, z, t)$. If U_0 is Lipschitz continuous, then it is easy to check that also $U(x, z, t)$ is Lipschitz continuous in space, with Lipschitz constant less or equal to the Lipschitz constant of U_0, by using the comparison principle and the invariance by translation of the differential operator appearing in (2.1). By comparison and again using the fact that the operator is invariant by translation in space, for every $h \in \mathbb{R}$

$$U(x, z, t) + h = U(x, z + h, t).$$

Therefore, we conclude that $U(x, z, t) = z - u(x, t)$, where u is a viscosity solution to (2.4). $\square$

We now define the rescaled time variables as follows:

$$(2.5) \qquad \tau(t) := \frac{\log(t(s+1) + 1)}{s + 1} \qquad \text{that is} \qquad t = \frac{e^{(s+1)\tau} - 1}{s + 1}.$$

In the rescaled time variables, the evolution (1.1) becomes

$$(2.6) \qquad \partial_\tau \tilde{p} \cdot \tilde{\nu} = -\tilde{p} \cdot \tilde{\nu} - H_s(\tilde{p}, \tilde{E}_\tau).$$

We define the rescaled variables $e^\tau y = x$, with τ as in (2.5), and set

$$(2.7) \qquad \tilde{u}(y, \tau) := e^{-\tau} u \left(y e^\tau, \frac{e^{(s+1)\tau} - 1}{s + 1} \right).$$

Then $\tilde{u}$ solves the system

$$(2.8) \qquad \begin{cases} \tilde{u}_\tau + \tilde{u} - D\tilde{u} \cdot y + \sqrt{1 + |D\tilde{u}|^2} H_s((y, \tilde{u}(y, \tau)), \{(y', z) \mid z \leqslant \tilde{u}(y', t)\}) = 0 \\ \tilde{u}(y, 0) = u_0(y). \end{cases}$$

Clearly, the same existence, uniqueness and regularity results stated for (2.4) in Theorem 2.1 are valid for (2.8).

2.1. **Fractional curvature on graphs.** We recall an equivalent formulation of the fractional mean curvature H_s on graphical hypersurfaces, see [2, 19]. First of all observe that if $\Pi = \{(x', z'), \ | \ z' \geqslant u(x,t) + Du(x,t)(x'-x)\}$ then by symmetry

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^{n+1} \setminus B_\varepsilon(p)} \left(\chi_{\mathbb{R}^{n+1} \setminus \Pi}(y) - \chi_\Pi(y) \right) \frac{1}{|p-y|^{n+1+s}} \, dy = 0.$$

Therefore, for $p = (x, u(x,t))$ and $E_t := \{(x', z) \ | \ z \leqslant u(x',t)\}$, we get (intending the integrals in the principal value sense)

$$
\begin{aligned}
H_s(p, E_t) &= \int_{\mathbb{R}^{n+1}} \left(\frac{\chi_{\mathbb{R}^{n+1} \setminus E_t}(y) - \chi_\Pi(y)}{|p-y|^{n+1+s}} + \frac{\chi_{\mathbb{R}^{n+1} \setminus \Pi}(y) - \chi_{E_t}(y)}{|p-y|^{n+1+s}} \right) dy \\
&= 2 \int_{\mathbb{R}^n} \int_{u(x',t)}^{u(x,t)+Du(x,t)(x'-x)} \frac{1}{(|x'-x|^2 + |z'-u(x,t)|^2)^{(n+1+s)/2}} \, dz dx' \\
(2.9) \qquad &= 2 \int_{\mathbb{R}^n} \frac{1}{|x-x'|^{n+s}} \int_{\frac{u(x',t)-u(x,t)}{|x-x'|}}^{Du(x,t) \cdot \frac{x-x'}{|x-x'|}} \frac{1}{(1+w^2)^{(n+1+s)/2}} \, dw dx'.
\end{aligned}
$$

We now introduce the function

$$G_s(t) := \int_0^t \frac{1}{(1+w^2)^{(n+1+s)/2}} dw.$$

By (2.9) we get

$$
\begin{aligned}
H_s((x, u(x,t)), E_t) &= 2 \int_{\mathbb{R}^n} \frac{G_s \left(Du(x,t) \cdot \frac{x-x'}{|x-x'|} \right) - G_s \left(\frac{u(x',t)-u(x,t)}{|x-x'|} \right)}{|x-x'|^{n+s}} dx' \\
(2.10) \qquad &= 2 \int_{\mathbb{R}^n} \frac{G_s \left(Du(x,t) \cdot \frac{z}{|z|} \right) - G_s \left(\frac{u(x-z,t)-u(x,t)}{|z|} \right)}{|z|^{n+s}} dz.
\end{aligned}
$$

We observe that this formula holds also in the viscosity sense (that is it is verified at points where the graph of u can be touched with paraboloids).

Moreover, if we change variable from z to $-z$ in (2.10) and recalling that G_s is odd, we get

$$
\begin{aligned}
(2.11) \quad H_s((x, u(x,t)), E_t) &= -\int_{\mathbb{R}^n} \frac{G_s \left(\frac{u(x+z,t)-u(x,t)}{|z|} \right) + G_s \left(\frac{u(x-z,t)-u(x,t)}{|z|} \right)}{|z|^{n+s}} dz \\
&= -\int_{\mathbb{R}^n} \frac{G_s \left(\frac{u(x+z,t)-u(x,t)}{|z|} \right) - G_s \left(\frac{u(x,t)-u(x-z,t)}{|z|} \right)}{|z|^{n+s}} dz \\
&= -\int_{\mathbb{R}^n} A(x,z,u) \frac{u(x+z,t) + u(x-z,t) - 2u(x,t)}{|z|^{n+s+1}} dz.
\end{aligned}
$$

where

$$(2.12) \qquad A(x,z,u) := \int_0^1 G_s' \left(w \frac{u(x+z,t)-u(x,t)}{|z|} + (1-w) \frac{u(x,t)-u(x-z,t)}{|z|} \right) dw.$$

Observe that $A(x,z,u) = A(x,-z,u)$ and

$$\left(1 + 4\|Du_0\|_\infty^2 \right)^{-\frac{n+s+1}{2}} \leqslant A(x,z,u) \leqslant 1.$$

This implies that the differential operator $H_s((x, u(x,t)), E_t)$ is elliptic, see e.g. [20].

3. Regularity results

In this section, we provide some regularity results for the flow starting from a Lipschitz graph. These results are based on the fact that the fractional curvature for graphs is an elliptic fractional operator, and so it enjoys regularizing effects.

Proposition 3.1. *Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz continuous function. Then there exists $\alpha \in (0, s)$ depending on s and n such that the viscosity solution $u(x, t)$ to (2.4) is in $C^{1+\alpha}(\mathbb{R}^n)$ for every $t > 0$, with norm uniformly bounded in $\mathbb{R}^n \times [t_0, +\infty)$ by a constant only depending on $\|Du_0\|_\infty$ and t_0. In particular, there exists a constant $C > 0$ only depending on $\|Du_0\|_\infty$ and t_0 such that*

$$\|Du(\cdot, t)\|_{C^\alpha(\mathbb{R}^n)} \leqslant Ct^{-\frac{\alpha}{s+1}} \qquad for \ t \geqslant t_0.$$

This implies that, if $\tilde{u}$ is the rescaled function defined in (2.7), for every $\tau_0 > 0$, there exists a constant $C > 0$ only depending on $\|Du_0\|_\infty$ and τ_0 such that

$$\|\tilde{D}u(\cdot, \tau)\|_{C^\alpha(\mathbb{R}^n)} \leqslant C \left(\frac{1 - e^{-\tau_0(s+1)}}{s+1}\right)^{-\frac{\alpha}{s+1}} \qquad for \ \tau \geqslant \tau_0.$$

Proof. If u_0 is Lipschitz continuous, then by Theorem 2.1 the solution to (2.4) is Lipschitz continuous in x with Lipschitz constant bounded by $\|Du_0\|_\infty$. Then the differential operator H_s on graphical hypersurfaces is elliptic, see (2.11), see e.g. [20, 21] for the definition. Then, by applying the Hölder regularity theory to the incremental quotients of u, , see [20, Theorem 2.1, Theorem 2.2], we get that they are C^α for some $\alpha \in (0, 1)$, and norm bounded by the Lipschitz constant of u.

Finally observe that for every $r > 0$, there holds that $v_r(x, t) = r^{-1}u(rx, r^{1+s}t)$ is the viscosity solution to (2.4) with initial datum $v_0(x) = r^{-1}u_0(x)$. Then v_0 is Lipschitz continuous, with the same Lipschitz constant as u_0, and we may apply to v_r the same regularity results as for u. In. particular for every t_0 there exists a constant C depending on t_0 and on $\|Du_0\|_\infty$ such that for all $t \geqslant t_0$, $\|Dv_r(\cdot, t)\|_{C^\alpha} \leqslant C$. Rescaling back to u, for every $t \geqslant t_0 r^{s+1}$ we get

$$\|Du(\cdot, t)\|_{C^\alpha} \leqslant Cr^{-\alpha}.$$

We conclude by choosing $r = (t/t_0)^{\frac{1}{s+1}}$.

Finally we consider the rescaled solution $\tilde{u}$. Note that $D\tilde{u}(y, \tau) = Du\left(ye^\tau, \frac{e^{\tau(s+1)} - 1}{s+1}\right)$ and $\|D\tilde{u}(\cdot, t)\|_{C^\alpha(\mathbb{R}^n)} = e^{\tau\alpha}\|Du(\cdot, \frac{e^{\tau(s+1)} - 1}{s+1})\|_{C^\alpha(\mathbb{R}^n)}$. Then by the previous estimate, for every $\tau \geqslant \tau_0$,

$$\|D\tilde{u}(\cdot, \tau)\|_{C^\alpha(\mathbb{R}^n)} \leqslant e^{\alpha\tau} C \left(\frac{e^{\tau(s+1)} - 1}{s+1}\right)^{-\frac{\alpha}{s+1}} \leqslant C \left(\frac{1 - e^{-\tau_0(s+1)}}{s+1}\right)^{-\frac{\alpha}{s+1}}.$$

$\square$

Let now $u(x, t)$ be a $C^{2,1}$ solution to (2.4) with initial datum u_0, and let

$$w(x, t) := \sqrt{1 + |Du(x, t)|^2}H_s((x, u(x, t)), E_t).$$

Since $u_t(x,t) = -w(x,t)$, using (2.11) we compute

$$
\begin{aligned}
w_t &= -\frac{H_s((x,u(x,t)),E_t)}{\sqrt{1+|Du|^2}}Du(x,t)\cdot Dw(x,t) + \\
&\quad +2\sqrt{1+|Du|^2}\int_{\mathbb{R}^n}G_s'\left(\frac{u(x+z,t)-u(x,t)}{|z|}\right)\frac{w(x+z,t)-w(x,t)}{|z|^{n+s+1}}dz.
\end{aligned}
$$

Therefore w is a solution to

$$
\text{(3.1)} \qquad w_t + B(x,t)\cdot Dw(x,t) - 2\int_{\mathbb{R}^n}C(x,z,t)\frac{w(x+z,t)-w(x,t)}{|z|^{n+s+1}}dz = 0,
$$

where
(3.2)
$$
B(x,t) := \frac{H_s((x,u(x,t)),E_t)}{\sqrt{1+|Du(x,t)|^2}}Du(x,t) \qquad C(x,z,t) := \sqrt{1+|Du(x,t)|^2}G_s'\left(\frac{u(x+z,t)-u(x,t)}{|z|}\right).
$$

Lemma 3.2. *Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz continuous function such that $H_s((x,u_0(x)),E_0)$ is bounded, and let $u(x,t)$ be the viscosity solution to (2.4) with initial datum u_0.*
Then $C(x,z,t)$ is well defined for every $x,z \in \mathbb{R}^n$, $t > 0$, $C(\cdot,z,t) \in C^\alpha(\mathbb{R}^n)$ and

$$
0 < (1+\|Du_0\|_\infty^2)^{-\frac{n+s+1}{2}} \leqslant C(x,z,t) \leqslant \sqrt{1+\|Du_0\|_\infty^2}.
$$

Moreover, the following inequalities hold in the viscosity sense:

$$
-\frac{1}{2}C \leqslant \frac{H_s((x,u(x,t)),E_t)}{\sqrt{1+|Du(x,t)|^2}}|Du(x,t)| \leqslant \frac{1}{2}C,
$$

where $C = \|H_s((x,u_0(x)),E_0)\|_\infty(1+\|Du_0\|_\infty)$.

Proof. First of all we observe that by Theorem 2.1, $u(\cdot,t)$ is Lipschitz with $\|Du(x,t)\|_\infty \leqslant \|Du_0\|_\infty$. Moreover, by Proposition 3.1 we have that $u(\cdot,t) \in C^{1+\alpha}(\mathbb{R}^n)$ and $u(x,\cdot) \in C^{\frac{1+\alpha}{2}}(0,+\infty)$. Then the regularity and the bounds on $C(x,z,t)$ are a direct consequence of the definition of G_s.

Let $C := \|H_s((x,u_0(x)),E_0)\|_\infty(1+\|Du_0\|_\infty)$. Note that $u_0(x) \pm Ct$ are respectively a supersolution and a subsolution to (2.4), so that by comparison we get

$$
u_0(x) - Ct \leqslant u(x,t) \leqslant u_0(x) + Ct \qquad \text{for all } t \geqslant 0.
$$

Moreover for every $t \geqslant \tau > 0$, the functions $u(x,t) \pm \sup_x |u(x,\tau)-u_0(x)|$ are respectively a supersolution and a subsolution to (2.4) with initial datum $u(x,\tau)$, whence

$$
|u(x,t+\tau)-u(x,t)| \leqslant \sup_x |u(x,\tau)-u_0(x)| \leqslant C\tau.
$$

This implies that $u(x,\cdot)$ is Lipschitz continuous with $|u_t(x,t)| \leqslant C$, which in turns implies that, in the viscosity sense,

$$
-C \leqslant \sqrt{1+|Du(x,t)|^2}H_s((x,u(x,t)),E_t) \leqslant C
$$

for all $x \in \mathbb{R}^n$ and $t > 0$. We now conclude recalling that $u(\cdot,t) \in C^1$. $\qquad\square$

Theorem 3.3. *Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz continuous function such that $H_s((x,u_0(x)),E_0)$ is bounded. Let $u(x,t)$ be the viscosity solution to (2.4) with initial datum u_0.*
Then $u(\cdot,t) \in C^{2+\beta}(\mathbb{R}^n)$, for some $\beta > s$, $u(x,\cdot) \in C^{1+\alpha}(0,+\infty)$ for some $\alpha \in (0,1)$, with norms bounded in $\mathbb{R}^n \times [t_0,+\infty)$ by constants depending on t_0, $\|Du_0\|_\infty$ and $\|H_s((x,u_0(x)),E_0)\|_\infty$.
Finally, the map $t \mapsto \sup_{x\in\mathbb{R}^n}\sqrt{1+|Du(x,t)|^2}|H_s((x,u(x,t)),E_t)|$ is decreasing in t.

Proof. We first prove the result under the additional assumption that u_0 is in $C^{2+\alpha}$ with bounded norms. Then the general case will follow by the stability of viscosity solutions with respect to uniform convergence, recalling that all the estimates depend only on the Lipschitz constant of u_0 and on $\|H_s((x, u_0(x)), E_0)\|_\infty$.

The short time existence result in [17, Theorem 5.1] implies that, if $u_0 \in C^{2+\alpha}$ with bounded norms, then there exists a time $t > 0$, such that the system (2.4) admits a smooth solution $u(x, t)$.

Let $w(x, t)$ be the unique solution to (3.1) with initial datum $\sqrt{1 + |Du_0(x)|^2} H_s((x, u_0(x)), E_0)$. Since u is smooth, by the computations in (3.1) and (3.2), we get that

$$w(x, t) = \sqrt{1 + |Du(x, t)|^2} H_s((x, u(x, t)), E_t).$$

By comparison $w(x, t)$ is bounded by $\|w(x, 0)\|_\infty$ and $\sup_x |w(x, t+r)| \leqslant \sup_x |w(x, r)|$, for every $r \geqslant 0$ and $t > 0$.

Since w is a bounded viscosity solution of a linear integro-differential equation, with bounded drift and uniformly elliptic integro-differential operator, the regularity results obtained in [21, Theorem 8.1] apply. Hence there exists $\alpha \in (0, s)$ such that $w(\cdot, t) \in C^{1+\alpha}(\mathbb{R}^n)$, $w(x, \cdot) \in C^{\frac{1+\alpha}{2}}(0, +\infty)$ with

$$\sup_{t \in (0,T)} \|w(\cdot, t)\|_{C^{1+\alpha}} + \sup_{x \in \mathbb{R}^n} \|w(x, \cdot)\|_{C^{\frac{1+\alpha}{2}}} \leqslant C(\|Du_0\|_\infty, \|H_s((x, u_0(x)), E_0)\|_\infty, s).$$

This implies that $u_t(x, \cdot) = \sqrt{1 + |Du(x, \cdot)|^2} H_s((x, u(x, \cdot)), E_t) \in C^{\frac{1+\alpha}{2}}((0, +\infty))$.

Moreover since $\sqrt{1 + |Du(x, t)|^2} H_s((x, u(x, t), E_t) \in C^{1+\alpha}(\mathbb{R}^n)$ as a function of x, recalling that $u \in C^{1+\alpha}$ by Proposition 3.1, with norm bounded only by $\|Du_0\|_\infty$, by the bootstrap argument in [2, Theorem 6] we get that $u(\cdot, t) \in C^{1+s+\alpha+\beta}(\mathbb{R}^n)$ for all $\beta < 1$ and all $t > 0$, with norm bounded only on $\|Du_0\|_\infty, \|H_s((x, u_0(x)), E_0)\|_\infty$. $\square$

4. HOMOTHETICALLY EXPANDING GRAPHICAL SOLUTIONS

We shall provide a complete characterization of graphical homothetically expanding solutions to (1.1).

Theorem 4.1. *Let $\bar{u} : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz continuous and positively 1-homogeneous function, that is,*

(4.1) $\exists C > 0 \quad |\bar{u}(x) - \bar{u}(y)| \leqslant C|x - y| \qquad \bar{u}(rx) = r\bar{u}(x) \qquad \forall r > 0, x, y \in \mathbb{R}^n.$

Then, for every $t > 0$ the subgraph $\bar{E}_t$ of the viscosity solution $\bar{u}(x, t)$ to (2.4), with initial datum $\bar{u}$, satisfies for $p \in \partial \bar{E}_t$

(4.2) $p \cdot \nu = -t(s + 1) H_s(p, \bar{E}_t),$

that is, for every $t > 0$ $\bar{E}_t$ satisfies (1.2) with $c^{-1} = t(s + 1)$, and the flow starting from E_t is an expanding homothetic solution to (1.1) according to Definition 1.1.

Moreover $\bar{u}(x, t)$ is in $C^\infty(\mathbb{R}^n \times (0, +\infty))$, and for every $T > 0$,

$$\lim_{t \to +\infty} |\bar{u}(x, t + T) - \bar{u}(x, t)| = 0 \qquad \text{locally uniformly in } \mathbb{R}^n.$$

Proof. By the fact that the differential operator is invariant under translations and by uniqueness of solutions, see Theorem 2.1, we get that for all $r \neq 0$, it holds

$$\bar{u}(x, t) = \frac{1}{r} \bar{u}(rx, r^{s+1} t).$$

Letting $r := t^{-\frac{1}{s+1}}$ for $t > 0$, we get

$$(4.3) \qquad\qquad \bar{u}(x,t) = t^{\frac{1}{s+1}} \bar{u}(xt^{-\frac{1}{s+1}}, 1)$$

which is exactly (4.7), with $\lambda(t) = t^{\frac{1}{s+1}}$. This implies that, if $p \in \partial \bar{E}_1$ then $pt^{\frac{1}{s+1}} \in \bar{E}_t$ and

$$(4.4) \qquad\qquad H_s(pt^{\frac{1}{s+1}}, \bar{E}_t) = t^{-\frac{s}{s+1}} H_s(p, \bar{E}_1).$$

Substituting in (1.1) we get that $\bar{E}_1$ solves (1.2) with $c^{-1} = s + 1$. The same argument holds substituting $t = 1$ with another positive time t.

By the uniform $C^{1,\alpha}$ estimate in Proposition 3.1 we know that $\bar{u}(x,1)$ is in $C^{1,\alpha}$ and it is a stationary solution to (2.8). Therefore, since $\sqrt{1 + |D\bar{u}(x,1)^2}H_s(x, \bar{u}(x,1))$ is in C^{α}, with norm locally bounded by the Lipschitz constant of $\bar{u}$, we can apply the bootstrap argument in [2, Theorem 6] and get that $\bar{u}(x,1)$ is in C^{∞}. Finally, since $\bar{u}(x,t) = t^{\frac{1}{s+1}} \bar{u}(xt^{-\frac{1}{s+1}}, 1)$ for every $t > 0$, we conclude that $\bar{u}$ is in $C^{\infty}(\mathbb{R}^n \times (0, +\infty))$.

Now, observe that by rescaling properties (4.3), for every $T > 0$ and $t > 0$ and by the fact that $\bar{u}(x,t)$ is Lipschitz continuous with the same Lipschitz constant as $\bar{u}$,

$$
\begin{aligned}
|\bar{u}(x, t+T) - \bar{u}(x,t)| &= |(t+T)^{\frac{1}{s+1}} \bar{u}(x(t+T)^{-\frac{1}{s+1}}, 1) - t^{\frac{1}{s+1}} \bar{u}(xt^{-\frac{1}{s+1}}, 1)| \\
&\leqslant (t+T)^{\frac{1}{s+1}} |\bar{u}(x(t+r)^{-\frac{1}{s+1}}, 1) - \bar{u}(xt^{-\frac{1}{s+1}}, 1)| \\
&\quad + [(t+r)^{\frac{1}{s+1}} - t^{\frac{1}{s+1}}] |\bar{u}(xt^{-\frac{1}{s+1}}, 1)| \\
&\leqslant C(t+T)^{\frac{1}{s+1}} |x| |(t+r)^{-\frac{1}{s+1}} - t^{-\frac{1}{s+1}}| + C[(t+T)^{\frac{1}{s+1}} - t^{\frac{1}{s+1}}] |x| t^{-\frac{1}{s+1}} \\
&\leqslant C|x| \left(\left(1 + \frac{T}{t}\right)^{\frac{1}{s+1}} - \left(1 - \frac{T}{t}\right)^{\frac{1}{s+1}} \right).
\end{aligned}
$$

Sending $t \to +\infty$, we get the result.

$\square$

By using the properties of homothetically expanding solutions, we show the following result about uniform continuity of solutions to (2.4).

Proposition 4.2. *Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz continuous function. Then the viscosity solution to (2.4) with initial datum u_0 satisfies for all $x, y \in \mathbb{R}^n, t, s \geqslant 0$*

$$|u(x,t) - u(y,r)| \leqslant \|Du_0\|_{\infty} |x - y| + K|t - r|^{\frac{1}{s+1}}$$

for some constant $K > 0$ which depends only on $\|Du_0\|_{\infty}$.

Proof. We prove just the Hölder continuity in time, since the Lipschitz continuity has already been proved in Theorem 2.1. Let $C = \|Du_0\|_{\infty}$.

Fix $x_0 \in \mathbb{R}^n$ and consider $v_0(x - x_0) = C|x - x_0|$. Then, in the translated variables $y = x - x_0$, v_0 satisfies (4.1), so that the solution $v_C(y,t)$ with initial datum v_0 is a homothetically expanding solution to (2.4). Moreover, since $H_s((y, C|y|), \{(x', z'), \ | \ z' \leqslant C|x'|\}) \leqslant 0$ in the viscosity sense at every $y \in \mathbb{R}^n$, $v_0(y)$ is a stationary subsolution to (2.4), which implies by comparison that $v_C(y,t) \geqslant v_0(y)$ for every $t > 0$, and then, again by comparison that $v_C(y,t) \geqslant v_C(y,r)$ if $0 < r < t$.

Therefore, since by Lipschitz continuity we get that $u_0(x) \leqslant v_0(x - x_0) + u_0(x_0)$, we conclude that $u(x,t) \leqslant v_C(x - x_0, t) + u_0(x_0)$, for all x. If we compute the previous inequality in x_0,

using also (4.7) we get

$$u(x_0, t) - u_0(x_0) \leqslant v_C(0, t) = v_C(0, 1)t^{\frac{1}{s+1}} \qquad \forall x_0 \in \mathbb{R}^n.$$

By comparison this implies that

$$u(x_0, t + r) - u(x_0, t) \leqslant v_C(0, 1)r^{\frac{1}{s+1}} \qquad \forall x_0 \in \mathbb{R}^n.$$

The other inequality is obtained with an analogous argument by considering $-C|x - x_0|$. $\square$

Remark 4.3. We observe that if $\bar{u}$ is a convex function, then we may also prove that there exists $C > 0$ such that

$$(4.5) \qquad \sup_{x \in \mathbb{R}^n} |H_s(x, \bar{u}(x, t))| \leqslant Ct^{-\frac{s}{s+1}} \qquad \forall t > 0.$$

We conjecture that this property is actually true also for nonconvex cones $\bar{u}$, as in the local setting. Observe that due to (4.4) it is sufficient to show that that there exists $C > 0$ such that

$$\sup_{x \in \mathbb{R}^n} |H_s(x, \bar{u}(x, 1))| \leqslant C,$$

which is in turn equivalent to show that

$$\sup_{x \in \mathbb{R}^n} |\bar{u}(x, 1) - D\bar{u}(x, 1) \cdot x| \leqslant C$$

since $\bar{u}(x, 1)$ is a stationary solution to (2.8).

Note that $\bar{u}(x, 1)$ is a convex function, since convexity is preserved by the fractional flow (1.1), see [8]. Moreover $\bar{u}(x, 1) \geqslant \bar{u}(x)$ by comparison (since $\bar{u}(x)$ is a stationary subsolution to (2.4)). For $\lambda > 0$, and $x \in \mathbb{R}^n$ fixed, we define $v(\lambda) = \bar{u}(\lambda x, 1)$. Note that this function is convex. By convexity we get that

$$\bar{u}(0, 1) = v(0) \geqslant v(1) - v'(1) = \bar{u}(x, 1) - D\bar{u}(x, 1) \cdot x.$$

Recalling Proposition 4.2 we get that there exists $K > 0$ depending only on the Lipschitz constant of $\bar{u}(x)$ such that $\bar{u}(\lambda x) + K \geqslant \bar{u}(\lambda x, 1)$ for every $\lambda \in \mathbb{R}$. So using again convexity of v we get for $\lambda \geqslant 0$,

$$\lambda \bar{u}(x) + K = \bar{u}(\lambda x) + K \geqslant \bar{u}(\lambda x, 1) = v(\lambda) \geqslant v(1) + v'(1)(\lambda - 1) = \bar{u}(x, 1) + (\lambda - 1)D\bar{u}(x, 1) \cdot x.$$

This implies, sending $\lambda \to +\infty$, that $\bar{u}(x) \geqslant D\bar{u}(x, 1) \cdot x$ and so in turn

$$\bar{u}(x, 1) - D\bar{u}(x, 1) \cdot x \geqslant \bar{u}(x, 1) - \bar{u}(x) \geqslant 0.$$

So, we proved that

$$0 \leqslant \bar{u}(x, 1) - D\bar{u}(x, 1) \cdot x \leqslant \bar{u}(0, 1)$$

which gives the result.

Remark 4.4. Observe that if (4.5) is satisfied, then we may strengthened the convergence result in Theorem 4.1, that is for every $T > 0$ it holds that

$$\lim_{t \to +\infty} |\bar{u}(x, t + T) - \bar{u}(x, t)| = 0 \qquad \text{uniformly in } \mathbb{R}^n.$$

Indeed substituting in the equation (2.4) and recalling that $\bar{u}(x, t)$ is uniformly Lipschitz, we get that $|u_t(x, t)| \leqslant C't^{-\frac{s}{s+1}}$, for some $C' > 0$ depending on C and on the Lipschitz norm of u_0. So, integrating we get, for all $T > 0$

$$|u(x, t + T) - u(x, t)| \leqslant C\left((t + T)^{\frac{1}{s+1}} - t^{\frac{1}{s+1}}\right)$$

and so for every $T > 0$,

$$\lim_{t \to +\infty} |\bar{u}(x, t+T) - \bar{u}(x,t)| = 0 \qquad \text{uniformly in } \mathbb{R}^n.$$

On the other hand, every homothetically expanding graphical solution to (1.1) is obtained as in Theorem 4.1.

Proposition 4.5. *Assume that E_1 is a solution to (1.2) and that is the subgraph of a Lipschitz continuous function $u_1 : \mathbb{R}^n \to \mathbb{R}$.*

Then the solution $u(x,t)$ to (2.4) with initial datum $u(x,1) = u_1(x)$ is defined in $\mathbb{R}^n \times (t_0, +\infty)$ where $t_0 = 1 - \frac{1}{c(s+1)}$ and satisfies

(4.6)
$$\lim_{t \to t_0^+} u(x,t) = \bar{u}(x) \qquad \text{locally uniformly in } x,$$

where $\bar{u} : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous and 1-homogeneous as in (4.1).

Proof. According to Definition 1.1,

$$E_t = [c(s+1)(t-1) + 1]^{\frac{1}{s+1}} E_1 = \lambda(t) E_1$$

is a solution to (1.1), which means in particular that the solution u to (2.4) with initial datum $u(x,1) = u_1(x)$ can be obtained as

(4.7) $\quad u(x,t) := \lambda(t) u_1(x \lambda^{1-}(t)) = [c(s+1)(t-1) + 1]^{\frac{1}{s+1}} u_1 \left(x [c(s+1)(t-1) + 1]^{-\frac{1}{s+1}} \right).$

This implies immediately that u is well defined in $\mathbb{R}^n \times (t_0, +\infty)$, where $t_0 = 1 - \frac{1}{c(s+1)}$.

Since u_1 is Lipschitz continuous, we get that $v_r(x) = r u_1(x/r)$ are equilipschitz, and moreover $|v_r(x)| \leqslant r|u_1(0)| + \|Du_1\|_\infty |x|$. Then, by Ascoli Arzelà theorem, up to subsequences there exist the limits $\lim_{r_n \to 0^+} v_{r_n}(x)$, locally uniformly in x. We claim that actually the limit is unique, that is $\bar{u}(x) := \lim_{r \to 0^+} r u_1(x/r)$ locally uniformly in x. If the claim is true, then it is easy to check that $\bar{u}(x)$ satisfies (4.1) and moreover by (4.7), $\lim_{t \to t_0^+} u(x,t) = \lim_{t \to t_0^+} \lambda(t) u_1(x \lambda^{-1}(t) x) = \lim_{r \to 0^+} r u_1(x/r) = \bar{u}(x)$.

To prove the claim we observe that by (4.7) $v_r(x) = u_1(x/r) = u(x, \lambda^{-1}(r))$ for every $r > 0$. Let $r_n \to 0$ such that $\lim_{r_n \to 0} v_{r_n}(x) = \bar{v}(x)$. Then $u_n(x,t) := u(x, t + \lambda^{-1}(r_n))$ is the viscosity solution to (2.4) with $u_n(x,0) = v_{r_n}(x)$. By stability of viscosity solutions with respect to uniform convergence, since $v_{r_n} \to \bar{v}$, we get that $u_n(x,t) \to \bar{v}(x,t)$ locally uniformly, where $\bar{v}(x,t)$ is the solution to (2.4) with initial datum $\bar{v}$. But actually $\bar{v}(x,t) = u(x, t + t_0)$ for every $t > 0$, and then the limit $\bar{v}$ is unique and independent of the subsequence. $\square$

5. Convergence to self-similar solutions

We show that homotetically expanding solutions are the long-time attractors for the flow of Lipschitz graphs, when the initial data is a sublinear perturbation of a 1-homogeneous function.

We now generalize to the fractional curvature flows the result obtained in [12] for the mean curvature flow, under the assumption that the initial datum u_0 is Lipschitz continuous and is straight at infinity in the following sense: There exists $K > 0$ and $\delta > 0$ such that

$$|u_0(x) - Du_0(x) \cdot x| \leqslant K(1 + |x|)^{1-\delta} \qquad \forall x \in \mathbb{R}^n.$$

This condition is equivalent to the following one, that we are going to assume: There exist a function $\bar{u}$ which satisfies (4.1) and constants $K > 0$, $\delta > 0$ such that

$$(5.1) \qquad |u_0(x) - \bar{u}(x)| \leqslant K(1 + |x|)^{1-\delta}.$$

Note that, if we define $\phi(r, x) := ru_0\left(\frac{x}{r}\right)$ for all $x \in \mathbb{R}^n$, then the straight at infinity condition reads, for $r \leqslant 1$,

$$\left|\frac{\partial}{\partial r}\phi(r, x)\right| = \left|u_0\left(\frac{x}{r}\right) - \frac{x}{r} \cdot Du_0\left(\frac{x}{r}\right)\right| \leqslant K(r + |x|)^{1-\delta}r^{\delta-1} \leqslant K(1 + |x|)^{1-\delta}r^{\delta-1}.$$

Then, for all $0 < r_1 < r_2 \leqslant 1$ we get

$$\left|r_2u_0\left(\frac{x}{r_2}\right) - r_1u_0\left(\frac{x}{r_1}\right)\right| = \left|\int_{r_1}^{r_2}\frac{\partial}{\partial r}\phi(r, x)dr\right| \leqslant \frac{K}{\delta}(1 + |x|)^{1-\delta}(r_2^\delta - r_1^\delta).$$

Observe that, since u_0 is Lipschitz continuous, up to a subsequence there exists the limit $\lim_{r \to 0^+} ru_0\left(\frac{x}{r}\right)$, which is locally uniform in x. By the previous inequality, we conclude that the limit is unique, so that the limit is a function $\bar{u}$ which satisfies (4.1) and finally

$$\left|\int_0^1\frac{\partial}{\partial r}\phi(r, x)dr\right| = |u_0(x) - \bar{u}(x)| \leqslant \frac{K}{\delta}(1 + |x|)^{1-\delta}.$$

Actually, the convergence result proved in [12] is stronger than ours, since they provide exponential in time convergence of the flows.

Theorem 5.1. *Let u_0 be a Lipschitz continuous function, such that there exist $\bar{u}$ which satisfies (4.1), and constants $K > 0$, $\delta \in (0, 1)$ for which (5.1) holds. Let u and $\tilde{u}$ be respectively the solutions to (2.4) and (2.8) with initial datum u_0, and $\bar{u}(x, t)$ be the solution to (2.4) with initial datum $\bar{u}(x)$. Then*

$$\lim_{\tau \to +\infty} \tilde{u}(y, \tau) = \bar{u}\left(y, \frac{1}{s+1}\right) \qquad \text{locally uniformly in } C^1(\mathbb{R}^n).$$

In particular, the rescaled flow

$$(5.2) \qquad \frac{1}{[(s+1)t+1]^{1/(s+1)}}E_t$$

where E_t the subgraph of $u(\cdot, t)$, converges as $\tau \to +\infty$ to a graphical hypersurface $\tilde{E}$ which satisfies (1.2) (with $c = 1$).

Proof. Let $\psi : (0, +\infty) \to (0, +\infty)$ be a smooth function such that $\psi(k) \equiv 0$ if $k < 1$ and $\psi(k) \equiv 1$ if $k > 2$. Define for $r > 1$

$$u_0^r(x) = \bar{u}(x) + \psi\left(\frac{|x|}{r}\right)(u_0(x) - \bar{u}(x)).$$

Then our assumption implies that

$$|u_0(x) - u_0^r(x)| \leqslant K(1 + 2r)^{1-\delta} \qquad \forall x \in \mathbb{R}^n.$$

By the comparison principle we get that, if $\tilde{u}$ and $\tilde{u}^r$ are respectively the solutions to (2.8) with initial datum u_0 and u_0^r,

$$(5.3) \quad \tilde{u}^r(y, \tau) - K(1 + 2r)^{1-\delta}e^{-\tau} \leqslant \tilde{u}(y, \tau) \leqslant \tilde{u}^r(y, \tau) + K(1 + 2r)^{1-\delta}e^{-\tau} \qquad \forall y \in \mathbb{R}^n, \tau > 0.$$

On the other hand,

$$\bar{u}(x) - \frac{2K}{r^\delta}|x| \leqslant u_0^r(x) \leqslant \bar{u}(x) + \frac{2K}{r^\delta}|x|.$$

Let $\bar{u}_{\pm r}$ the solution to (2.4) with initial datum respectively $\bar{u}(x) \pm \frac{2K}{r^\delta}|x|$. Note that $\bar{u}(x) \pm \frac{2K}{r^\delta}|x|$ satisfy (4.1) with Lipschitz constant $\|D\bar{u}\|_\infty + \frac{2K}{r^\delta} \leqslant \|D\bar{u}\|_\infty + 2K$. By rescaling properties of $\bar{u}(x) \pm \frac{2K}{r^\delta}|x|$, see 4.3, and by formula (2.7), we get that $\bar{u}_{\pm r}\left(y, \frac{1-e^{-(s+1)\tau}}{s+1}\right)$ is the solution to (2.8) with initial datum $\bar{u}(x) \pm \frac{2K}{r^\delta}|x|$. Then by comparison principle

$$\bar{u}_{-r}\left(y, \frac{1-e^{-(s+1)\tau}}{s+1}\right) \leqslant \tilde{u}^r(y, \tau) \leqslant \bar{u}_{+r}\left(y, \frac{1-e^{-(s+1)\tau}}{s+1}\right).$$

By Proposition 4.2, recalling that $\bar{u}(x) \pm \frac{2K}{r^\delta}|x|$ are Lipschitz functions with Lipschitz constant less than $\|D\bar{u}\|_\infty + 2K$, we get that there exists B depending only on $\|D\bar{u}\|_\infty$ and K such that

$$\bar{u}_{-r}\left(y, \frac{1}{s+1}\right) - Be^{-\tau} \leqslant \tilde{u}^r(y, \tau) \leqslant \bar{u}_{+r}\left(y, \frac{1}{s+1}\right) + Be^{-\tau}.$$

Therefore by (5.3) we conclude that for all $y \in \mathbb{R}^n$, $\tau > 0$, and all $r >> 1$,

$$\bar{u}_{-r}\left(y, \frac{1}{s+1}\right) - (B+K(1+2r)^{1-\delta})e^{-\tau} \leqslant \tilde{u}(y, \tau) \leqslant \bar{u}_{+r}\left(y, \frac{1}{s+1}\right) + (B+K(1+2r)^{1-\delta})e^{-\tau}.$$

Note that as $r \to +\infty$, $\bar{u}_{\pm r}\left(y, \frac{1}{s+1}\right) \to \bar{u}_C\left(y, \frac{1}{s+1}\right)$ locally uniformly in y by stability of viscosity solutions, since $\bar{u}(x) \pm \frac{2K}{r^\delta}|x| \to \bar{u}(x)$ locally uniformly. So taking $r = e^\tau$ in the previous inequality and sending $\tau \to +\infty$, we get the local uniform convergence of $\tilde{u}$. Finally, since by Proposition 3.1, $\tilde{u}$ has uniform $C^{1,\alpha}$ norm in $\mathbb{R}^n \times [t_0, +\infty)$, for every $t > 0$, we conclude that the locally uniform convergence holds in $C^{1,\alpha}$ sense. $\square$

Remark 5.2. If condition (5.1) is violated, then in general we cannot expect the asymptotic convergence result proved in Theorem 5.1. Indeed, observe that if $\tilde{u}$ is the solution to (2.8) with initial datum u_0, then reasoning as in Proposition 4.2 we get that

$$|\tilde{u}(y, \tau) - e^{-\tau}u_0(ye^\tau)| \leqslant K\left(\frac{1-e^{-(s+1)\tau}}{s+1}\right)^{\frac{1}{s+1}} \leqslant K\left(\frac{1}{s+1}\right)^{\frac{1}{s+1}} =: K'.$$

In particular, if the convergence takes place, then for every compact set $B \subset \mathbb{R}^n$ there exists $\tau_B > 0$ such that

$$|e^{-\tau_1}u_0(ye^{\tau_1}) - e^{-\tau_2}u_0(ye^{\tau_2})| \leqslant 3K' \qquad \text{for all } \tau_1, \tau_2 \geqslant \tau_B.$$

For instance, this condition is not satisfied by initial data oscillating at infinity between different positively homogeneous functions. We refer to [12, Proposition 6.1] for an explicit example.

6. CONVERGENCE OF THE UNRESCALED FLOW

In this section we will consider some cases in which convergence of the unrescaled flow holds. To get stability without rescaling, we have to impose some decay or periodicity condition of the initial datum.

6.1. **Stability of hyperplanes.** We show that hyperplanes are stable with respect to the flow (1.1), that is, if the initial datum is flat at infinity (resp. periodic), then the solution stabilizes to the hyperplane at which the initial datum is (resp. stabilizes to a constant).

We remark that the behavior of the solution to (2.4) for these families of initial data is analogous to the behaviour of solutions to the fractional heat equation $u_t + (-\Delta)^{\frac{s+1}{2}} u = 0$, with the same initial data. Analogous results for the local mean curvature flow of graphs have been obtained in [11, 13, 18], with different approaches: either comparison with large balls as in our case (even if in the local case the argument is more involved), or reduction to stabilization of solutions to the heat equation.

We start with a result about periodic initial data, showing that the solution stabilizes to a constant. For a particular class of periodic initial datum we may show that actually this constant is given by the mean value of the initial datum.

Proposition 6.1. *Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function which is $\mathbb{Z}^n$ periodic. Then, there exists a constant $c \in (\min u_0, \max u_0)$ such that the solution u to (2.4) with initial datum u_0 satisfies*

$$\lim_{t \to +\infty} u(x,t) = c \qquad \text{uniformly in } C^1(\mathbb{R}^n).$$

If moreover u_0 has also the property that

(6.1)   *there exists $v \in \mathbb{R}^n$ such that for all $x \in \mathbb{R}^n$, $u_0(x + v) = -u_0(x)$*

then

$$\lim_{t \to +\infty} u(x,t) = 0 = \int_{[0,1]^n} u_0(x)dx \qquad \text{uniformly in } C^1(\mathbb{R}^n).$$

Proof. We observe that by uniqueness the solution $u(x,t)$ is $\mathbb{Z}^n$ periodic in the x variable. We define $M(t) = \max_x u(x,t)$ and $m(t) = \min_x u(x,t)$. Note that by Proposition 4.2, $M(t)$ and $m(t)$ are Hölder continuous functions, and moreover by comparison, we have that $\min_x u_0(x) \leqslant m(t) \leqslant M(t) \leqslant \max_x u_0(x)$, and that $M(t)$ is decreasing and $m(t)$ is increasing. Therefore $M(t)$ and $m(t)$ are differentiable a.e. We want to prove that $\lim_{t \to +\infty} M(t) - m(t) = 0$. If this holds, then the result follows, recalling that $M(t)$ is decreasing and $m(t)$ is increasing. The C^1 convergence is a consequence of the uniform estimates in Proposition 3.1. Assume by contradiction that $\lim_{t \to +\infty} M(t) - m(t) = \bar{C} > 0$. We fix $t > 0$ and $x_M \in \text{argmax } u(\cdot, t) \cap [0,1]^n$ and $x_m \in \text{argmin } u(\cdot, t) \cap [0,1]^n$. Let $C(t) = M(t) - m(t) \geqslant \bar{C}$. We recall that $u(\cdot, t)$ is Lipschitz continuous with Lipschitz constant less than $\|Du_0\|_\infty$ and we fix $\delta > 0$ such that $\delta\|Du_0\|_\infty \leqslant \frac{\bar{C}}{2}$. It is immediate to check that

$$u(x_M, t) - u(x, t) \geqslant u(x_M, t) - u(x_m, t) - \delta\|Du_0\|_\infty = C(t) - \frac{\bar{C}}{2} \geqslant \frac{C(t)}{2} \quad \forall x \in B(x_m, \delta),$$

and analogously

$$u(x, t) - u(x_m, t) \geqslant \frac{C(t)}{2} \quad \forall x \in B(x_M, \delta).$$

Then, at every point of differentiability t, the functions M, m satisfy

$$M'(t) = u_t(x_M, t), \; m'(t) = u_t(x_m, t), \qquad \text{for all } x_M \in \text{argmax } u(\cdot, t), x_m \in \text{argmin } u(\cdot, t).$$

Using the equation we get

$$
\begin{aligned}
M'(t) = u_t(x_M, t) &= -H_s(x_m, u(x_M, t)) \leqslant \left(1 + \|Du_0\|_\infty^2\right)^{-\frac{n+s+1}{2}} \int_{\mathbb{R}^n} \frac{u(y,t) - u(x_M, t)}{|y - x_M|^{n+s+1}} dz \\
&\leqslant -\left(1 + \|Du_0\|_\infty^2\right)^{-\frac{n+s+1}{2}} \frac{C(t)}{2} \int_{B(x_m, \delta)} \frac{1}{|y - x_M|^{n+s+1}} dz \\
&= -\left(1 + \|Du_0\|_\infty^2\right)^{-\frac{n+s+1}{2}} \frac{C(t)}{2} \frac{\omega_n \delta^n}{(\delta + 1)^{n+s+1}} < 0
\end{aligned}
$$

and

$$
\begin{aligned}
m'(t) = u_t(x_m, t) &= -H_s(x_m, u(x_m, t)) \geqslant \left(1 + \|Du_0\|_\infty^2\right)^{-\frac{n+s+1}{2}} \int_{\mathbb{R}^n} \frac{u(y,t) - u(x_m, t)}{|y - x_m|^{n+s+1}} dz \\
&\geqslant \left(1 + \|Du_0\|_\infty^2\right)^{-\frac{n+s+1}{2}} \frac{C(t)}{2} \frac{\omega_n \delta^n}{(\delta + 1)^{n+s+1}} > 0.
\end{aligned}
$$

These two inequalities imply that $M(t)$ is strictly decreasing, $m(t)$ is strictly increasing and $C'(t) \leqslant -KC(t)$ for a constant K depending only on $\bar{C}$ and $\|Du_0\|_\infty$. Therefore $\lim_{t \to +\infty} C(t) = 0$, which is in contradiction with our assumption.

Finally, observe that if u_0 satisfies (6.1), then by uniqueness, $-u(x, t) = u(x + v, t)$. This implies that necessarily $\lim_{t \to +\infty} u(x, t) = -\lim_{t \to +\infty} u(x, t)$ and then the limit is 0. $\qquad\square$

We first of all prove stability of constant functions in $\mathbb{R}^n$.

Theorem 6.2. *Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function such that*

$$
\lim_{|x| \to +\infty} u_0(x) = 0.
$$

Then, the solution u to (2.4) with initial datum u_0 satisfies

$$
\lim_{t \to +\infty} u(x, t) = 0 \qquad \text{uniformly in } C^1(\mathbb{R}^n).
$$

Proof. First of all we observe that it is sufficient to prove the result for initial data which are nonnegative everywhere or nonpositive everywhere. Indeed the general case is easily obtained by using as barriers the solutions with initial data $u_0^+ = \max(u_0, 0)$ and $u_0^- = \min(u_0, 0)$.

So, we prove the result only for the case $u_0 \geqslant 0$, since the other case $u_0(x) \leqslant 0$ is completely analogous. Note that by comparison, since the constant are stationary solutions to (2.4), $0 \leqslant u(x, t) \leqslant \max_y u_0(y)$ for all $x \in \mathbb{R}^n, t > 0$.

We claim now that all $t > 0$,

$$
\inf_x u(x, t) = 0 \text{ and that } M(t) := \max_x u(x, t) \text{ is decreasing in } t.
$$

Indeed for every $\varepsilon > 0$, let us fix $R > 0$ such that $|u_0(x)| \leqslant \varepsilon$ for all $|x| \geqslant R$. For every $|x| > R$, fix $K = |x| - R > 0$ and observe that the ball $B((x, K + \varepsilon), K)$ of center $(x, K + \varepsilon)$ and radius K, is contained in $\mathbb{R}^{n+1} \setminus E_0$. By monotoniciy of the flow (1.1). with respect to inclusions, see [7], there holds that

$$
B((x, K + \varepsilon), K(t)) \subseteq \mathbb{R}^{n+1} \setminus E_t,
$$

where $K(t) = (K^{s+1} - (s+1)\bar{c}t)^{\frac{1}{s+1}}$, and $\bar{c}$ is the fractional mean curvature of the unit ball in R^{n+1}. Therefore, we get that and $t \leqslant T \leqslant \frac{K^{s+1}}{2(s+1)\bar{c}}$

$$u(x,t) \leqslant \varepsilon + K - K(t) \leqslant \varepsilon - K'(T)t = \varepsilon + \frac{\bar{c}}{(K^{s+1} - (s+1)\bar{c}T)^{\frac{s}{s+1}}}t \leqslant \varepsilon + \frac{\bar{c}2^{\frac{s}{s+1}}}{K^s}t.$$

This implies that for all $\varepsilon > 0$, there exists $R > 0$ such that for $|x| > R$,

(6.2) $$0 \leqslant u(x,t) \leqslant \varepsilon + \frac{\bar{c}2^{\frac{s}{s+1}}}{(|x| - R)^s}t \qquad \text{for all } t \leqslant \frac{(|x| - R)^{s+1}}{2(s+1)\bar{c}}.$$

This implies that $\inf_x u(x,t) = 0$ for all $t > 0$ and moreover that $\sup_x u(x,t) = \max_x u(x,t)$. The fact that $\max_x u(x,t)$ is decreasing in t is a consequence of comparison with stationary solutions.

Now, since $M(t)$ is decreasing, let $\bar{M} = \lim_{t \to +\infty} M(t) = \inf_t M(t)$. We claim that $\bar{M} = 0$. If the claim holds, then we get the conclusion. The C^1 convergence is a consequence of the uniform estimates in Proposition 3.1.

Assume by contradiction that $\bar{M} > 0$. We fix $0 < \varepsilon < \frac{\bar{M}}{2}$ and $\bar{t} > 0$ such that $M(\bar{t}) \leqslant \bar{M} + \varepsilon$. We fix also $R = R(\bar{t})$ such that $u(x, \bar{t}) < \frac{\bar{M}}{2}$ for all $|x| > R$.

Now we aim to get a contradiction by constructing a periodic barrier which satisfies (up to suitable vertical translation) a condition like (6.1). We fix a Lipschitz continuous function $\phi : [-R, 2R] \to \mathbb{R}$, such that ϕ is non increasing, $\phi(z) = \bar{M} + \varepsilon$ for $z \in [-R, R]$, and $\phi(2R) = \frac{\bar{M}}{2}$. Now we extend it to a function $\phi : [-R, 5R] \to \mathbb{R}$ by putting $\phi(z) = \frac{3}{2}\bar{M} + \varepsilon - \phi(z - 3R)$ for all $2R \leqslant z \leqslant 5R$. Finally, we extend it by periodicity to be a $6R\mathbb{Z}$ periodic function. Then the function $v_0(x) = \phi(x \cdot e_1)$, is a $6R\mathbb{Z}^n$ periodic function, which is Lipschitz continuous, and satisfies $v_0(x + 3Re_1) = \frac{3}{2}\bar{M} + \varepsilon - v_0(x)$.

Note that by construction, $u(x, \bar{t}) \leqslant v_0(x)$ for all $x \in \mathbb{R}^n$ and then by comparison

$$u(x, t + \bar{t}) \leqslant v(x,t), \qquad \text{and in particular } \limsup_{t \to +\infty} u(x,t) \leqslant \lim_{t \to +\infty} v(x,t)$$

where $v(x,t)$ is the solution to (2.4) with initial datum v_0. Now by Proposition 6.1 we get that $\lim_{t \to +\infty} v(x,t) = c$ uniformly in C^1, and moreover, since $v_0(x + 3Re_1) = \frac{3}{2}\bar{M} + \varepsilon - v_0(x)$ there holds that $c = \frac{3}{2}\bar{M} + \varepsilon - c$, and so $c = \frac{3}{4}\bar{M} + \frac{\varepsilon}{2} < \bar{M}$, recalling our choice of ε. But then we get that $\limsup_{t \to +\infty} u(x,t) < \bar{M}$, in contradiction with the definition of $\bar{M}$. $\qquad \square$

Remark 6.3. Let u_0^λ be a family of Lipschitz continuous functions which fulfills uniformly in λ the condition in Theorem 6.2, in the sense that

$$\sup_\lambda \sup_{|x| > R} |u_0^\lambda(x)| \to 0 \qquad \text{as } R \to +\infty.$$

Then it is easy to check that the convergence is uniform in λ in the sense that

$$\sup_\lambda \|u^\lambda(x,t)\|_{C^1} \to 0 \qquad \text{as } t \to +\infty$$

where u^λ is the solution to (2.4) starting from u_0^λ.

Finally we give the general result about stability of hyperplanes. We denote with $d(A, B)$ the Hausdorff distance between the sets A, B.

Corollary 6.4. *Let $E_0 \subseteq \mathbb{R}^{n+1}$ be a set such that ∂E_0 is a Lipschitz surface and that there exists a half-space H for which*

$$\lim_{R \to +\infty} d(E_0 \setminus B(0, R), H \setminus B(0, R)) = 0.$$

Then the outer and inner level set flows $E^+(t), E^-(t)$ defined in (2.3) satisfy

$$\lim_{t \to +\infty} d(E^+(t), H) = 0 = \lim_{t \to +\infty} d(E^-(t), H).$$

Proof. Since the fractional mean curvature is invariant by rotations and translations, we may assume without loss of generality that $H = \{(x, z) \in \mathbb{R}^n \times \mathbb{R} \mid z \leqslant 0\}$. Moreover, by the assumption that $\lim_{R \to +\infty} d(E_0 \setminus B(0, R), H \setminus B(0, R)) = 0$, there exist two Lipschitz functions $u_0, v_0 : \mathbb{R}^n \to \mathbb{R}$ such that $\lim_{|x| \to +\infty} u_0(x) = 0 = \lim_{|x| \to +\infty} v_0(x)$ and $\{(x, z) \in \mathbb{R}^n \times \mathbb{R} \mid z \leqslant u_0(x)\} \subseteq E_0 \subseteq \{(x, z) \in \mathbb{R}^n \times \mathbb{R} \mid z \leqslant v_0(x)\}$. By comparison we get that $\{(x, z) \in \mathbb{R}^n \times \mathbb{R} \mid z \leqslant u(x, t)\} \subseteq E_t^- \subseteq E_t^+ \subseteq \{(x, z) \in \mathbb{R}^n \times \mathbb{R} \mid z \leqslant v(x, t)\}$, where $u(x, t), v(x, t)$ are the solutions to (2.4) with initial datum u_0, v_0. By Theorem 6.2 $\lim_{t \to +\infty} u(x, t) = \lim_{t \to +\infty} v(x, t) = 0$ uniformly in $\mathbb{R}^n$, and this gives the conclusion. $\square$

6.2. **Stability of convex cones.** In this section we provide the convergence of the unrescaled flow in the case the initial data is decaying at infinity to a H_s-mean convex cone, staying above it. The result can be strenghtened if the initial cone is convex, by using the stability of hyperplanes.

Proposition 6.5. *Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz continuous function. Assume there exists a non linear function $\bar{u}$ which satisfies (4.1), and*

$$(6.3) \qquad\qquad H_s(x, \bar{u}(x)) \leqslant 0 \text{ in the viscosity sense,}$$

such that

$$u_0(x) \geqslant \bar{u}(x) \qquad and \qquad \lim_{|x| \to +\infty} u_0(x) - \bar{u}(x) = 0.$$

Then, if u the solution to (2.4) with initial datum u_0, it holds

$$\lim_{t \to +\infty} u(x, t) - \bar{u}(x, t) = 0 \qquad locally \text{ uniformly in } C^1(\mathbb{R}^n).$$

Proof. Observe that by (6.3), $\bar{u}(x)$ is a stationary viscosity subsolution to (2.4), therefore $\bar{u}(x, t) \geqslant \bar{u}(x)$ and so in particular $\bar{u}(0, t) \geqslant \bar{u}(0)$. Observe that, if $\bar{u}$ is a homogeneous Lipschitz function, then either it is linear or it is singular at 0, in the sense that the curvature in a neighborhood of $x = 0$ is not bounded. Therefore, since we assumed that $\bar{u}$ is non linear, then $\bar{u}(0, t) > 0$ since $\bar{u}$ is smooth by Theorem 4.1. Again by comparison, we get that also $\bar{u}(x, t + r) \geqslant \bar{u}(x, t)$ for all $t \geqslant 0$, $r > 0$, $x \in \mathbb{R}^n$.

Fix $\varepsilon > 0$ and $R > 0$ such that $u_0(x) \leqslant \bar{u}(x) + \varepsilon$ for all $|x| > R$. Therefore we get that for all $T > 0$,

$$(6.4) \qquad\qquad u_0(x) \leqslant \bar{u}(x, T) + \varepsilon \qquad \text{for all } |x| > R.$$

Observe now that by (4.3) and Lipschitz continuity

$$\bar{u}(x, t) \geqslant \bar{u}(0, t) - C|x| = t^{\frac{1}{s+1}} \bar{u}(0, 1) - C|x|.$$

Since $\bar{u}(0, 1) > 0$ there exists $T > 0$ sufficiently large such that

$$(6.5) \qquad \bar{u}(x, T) \geqslant T^{\frac{1}{s+1}} \bar{u}(0, 1) - C|x| \geqslant u_0(x) \qquad \text{for all } |x| \leqslant R.$$

Therefore, by (6.4), (6.5), and by comparison we get that for some $T > 0$ sufficiently large

$$u(x,t) \leqslant \bar{u}(x, T+t) + \varepsilon \qquad \forall t \geqslant 0, x \in \mathbb{R}^n.$$

Note that since $u_0 \geqslant \bar{u}$, by comparison $u(x,t) \geqslant \bar{u}(x,t)$ for all x, t. Then we get, for $\varepsilon > 0$ and $T > 0$ fixed (and depending on ε),

$$0 \leqslant u(x,t) - \bar{u}(x,t) \leqslant \bar{u}(x, T+t) - \bar{u}(x,t) + \varepsilon.$$

We conclude by letting $t \to +\infty$ and recalling that, by Theorem 4.1, $\bar{u}(x, T+t) - \bar{u}(x,t) \to 0$ as $t \to +\infty$ uniformly in x, for all fixed T. $\qquad\square$

Theorem 6.6. *Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz continuous function. Assume there exists a convex function $\bar{u}$ which satisfies (4.1) and such that*

$$\lim_{|x| \to +\infty} u_0(x) - \bar{u}(x) = 0.$$

Then, if u the solution to (2.4) with initial datum u_0,

$$\lim_{t \to +\infty} u(x,t) - \bar{u}(x,t) = 0 \qquad uniformly \ in \ C^1(\mathbb{R}^n).$$

Proof. We divide the proof in several steps.
Step 1: for every $\varepsilon > 0$ there exists $T = T(\varepsilon)$ such that

(6.6) $$u(x,t) \leqslant \bar{u}(x, T(\varepsilon) + t) + \varepsilon \qquad \forall t \geqslant 0, x \in \mathbb{R}^n.$$

Since $\bar{u}$ is convex, then it also satisfies (6.3). We proceed as in Proposition 6.5. So for every $\varepsilon > 0$, there exists $R = R(\varepsilon) > 0$ such that $u_0(x) \leqslant \bar{u}(x) + \varepsilon \leqslant \bar{u}(x,t) + \varepsilon$ for all $|x| > R$ and moreover, arguing as in the proof of (6.5), we get that there exists $T > 0$ sufficiently large such that

$$\bar{u}(x,T) \geqslant u_0(x) \qquad \text{for all } |x| \leqslant R.$$

Therefore, since $u_0(x) \leqslant \bar{u}(x,T) + \varepsilon$, we conclude by comparison.
Step 2: for every $\delta > 0$ there exist $T(\delta) > 0$ such that

(6.7) $$u(x,t) \geqslant \bar{u}(x) - \delta \qquad \forall t \geqslant T(\delta).$$

Since $\bar{u}$ is convex and is positively 1-homogeneous, we get that for all $\nu \in \mathbb{S}^n$ there exists $p_\nu \in \mathbb{R}^n$, such that $\bar{u}(x) \geqslant p_\nu \cdot x$, with equality at every $x = \lambda \nu$ with $\lambda \geqslant 0$.

We define the family of functions $u_0^\nu(x) = \inf(u_0(x), p_\nu \cdot x)$ and observe that by the assumption there holds that

$$\lim_{R \to +\infty} \sup_{\nu \in \mathbb{S}^n} \sup_{|x| > R} u_0^\nu(x) - p_\nu \cdot x = 0.$$

So, by Corollary 6.4, and arguing as in Remark 6.3, we get that

$$\lim_{t \to +\infty} \sup_{\nu \in \mathbb{S}^n} d(E_t^\nu, H^\nu) = 0$$

where H_ν is the halfspace with normal $(-p_\nu, 1)$. This in particular implies that $\lim_{t \to +\infty} u^\nu(x,t) - p_\nu \cdot x = 0$ uniformly in ν, which in turns gives that $\liminf_{t \to +\infty} u(x,t) - p_\nu \cdot x \geqslant \lim_{t \to +\infty} u^\nu(x,t) - p_\nu \cdot x = 0$ uniformly in ν, and so in particular $\liminf_{t \to +\infty} u(x,t) \geqslant \bar{u}(x)$. This permits to conclude.
Step 3: conclusion.

Observe that by Step 2, and comparison principle, for $\delta > 0$ fixed, there holds that $u(x, t + T(\delta)) \geqslant \bar{u}(x, t) - \delta$. So, for every $\varepsilon > 0$ and $t \geqslant T(\delta)$, we get by Step 1 and the previous observation that

$$\bar{u}(x, t - T(\delta)) - \bar{u}(x, t) - \delta \leqslant u(x, t) - \bar{u}(x, t) \leqslant \bar{u}(x, t + T(\varepsilon)) - \bar{u}(x, t) + \varepsilon.$$

Now we conclude by arbitrariness of ε, δ and by Remark 4.4, letting $t \to +\infty$.

$\square$

7. Ancient and homothetically shrinking solutions

Finally we consider homothetically shrinking solutions in the graphical case, and we show that they are necessarily hyperplanes.

Definition 7.1. An ancient solution to (1.1) is a solution to (1.1) defined for all $t \in (-\infty, 0)$.

We recall the following Liouville theorem for ancient solutions of parabolic fractional equations with rough kernels, proved in [20, Theorem 3.1]. We state it in the setting we are going to apply it.

Theorem 7.2. [20, Theorem 3.1] *Let I be a translation invariant operator, elliptic with fractional order $1+s$, with $I(0) = 0$ and $u \in C(\mathbb{R}^n \times (-\infty, 0])$ be a viscosity solution to $u_t - I(u) = 0$ in $\mathbb{R}^n \times (-\infty, 0]$. Assume there exists $C > 0$ such that for all $R \geqslant 1$ there holds*

$$\sup_{|x| \leqslant R, -R^{1+s} \leqslant t \leqslant 0} |u(x, t)| \leqslant CR.$$

Then there exists $a \in \mathbb{R}^n, b \in \mathbb{R}$ such that $u(x, t) = a \cdot x + b$.

Theorem 7.3. *The only graphical Lipschitz solutions to (1.3) are hyperplanes (with $c = 0$).*
Moreover the only graphical uniformly Lipschitz ancient solutions to (1.1) are hyperplanes.

Proof. The result is a consequence of Theorem 7.2.

Let E be a graphical Lipschitz solution to (1.3), that is let $u_1 : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz continuous function such that $E = \{(x, z) \, | z \leqslant u_1(x)\}$ is a solution to (1.3). Then arguing as in Proposition 4.5 we may connstruct a solution to (2.4) in $(-\infty, 0)$ with $u_1(x, t) = u_1(x)$. Indeed let

$$E_t := [-c(s+1)t + 1]^{\frac{1}{s+1}} E = \lambda(t) E \qquad \text{for } t < 0.$$

It is easy to check, using the fact that E is a solution to (1.3), that E_t is a solution to (1.1). Therefore the function

$$(7.1) \qquad u_1(x, t) := \lambda(t) u_1(x \lambda^{1-}(t)) = [-c(s+1)t + 1]^{\frac{1}{s+1}} u_1\left(x \left[-c(s+1)t + 1\right]^{-\frac{1}{s+1}}\right)$$

is a solution to (2.4) in $(-\infty, 0)$ and satisfies $u_1(x, 0) = u_1(x)$. Since u_1 is Lipschitz continuous we get that

$$|u_1(x, t)| \leqslant \lambda(t) |u_1(0)| + \|Du_1\|_\infty |x|.$$

This implies that there exists $K > 0$ depending on $c, s, \|Du_1\|_\infty, u_1(0)$, such that for all $R > 1$,

$$\max_{|x| \leqslant R, t \in [-R^{s+1}, 0]} |u_1(x, t)| \leqslant KR.$$

Recalling the formula for H_s (2.11), we get that u_1 is a viscosity solution to

$$u_t - I(u) = 0, \qquad t \in (-\infty, 0),$$

where I is a translation invariant operator, elliptic with fractional order $s + 1$, and $I(0) = 0$. Then, by Theorem 7.2 we conclude that there exist $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $u_1(x, t) = a \cdot x + b$ for all $t \leqslant 0$ and $x \in \mathbb{R}^n$. This implies that E is a hyperplane and $c = 0$.

Finally, if E_t is a graphical uniformly Lipschitz and ancient solution to (1.1), then $u(x, t)$ is a continuous viscosity solution to $u_t - I(u) = 0$ for $t \in (-\infty, 0)$ and moreover, since $|Du(x, t)| \leqslant C$, arguing as in Proposition 4.2 we obtain that there exists a constant K only depending on C such that $|u(x, t) - u(x, t + h)| \leqslant K|h|^{\frac{1}{1+s}}$. So, again by Theorem 7.2 we conclude that u is affine and does not depend on t. $\qquad \square$

Remark 7.4. In the case of classical mean curvature flow, see for instance [14] and references therein, there are translating, hence eternal, solutions which are smooth but not Lipschitz. We expect that such solutions, with polynomial growth depending on s, exist also for the graphical fractional mean curvature flow (2.4).

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